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Alpha Power Extended Inverse Weibull Poisson Distribution: Properties, Inference, and Applications to lifetime data

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Abstract

In this paper, a new four-parameter extended inverse Weibull distribution called Alpha power Extended Inverse Weibull Poisson distribution is introduced using the alpha power Poisson generator. This method adds two shape parameters to a baseline distribution thereby increasing its flexibility and applicability in modeling lifetime data. We study the structural properties of the new distribution such as the mean, variance, quantile function, median, ordinary and incomplete moments, reliability analysis, Lorenz and Bonferroni curves, Renyi entropy, mean waiting time, mean residual life, and order statistics. We use the method of maximum likelihood technique for estimating the model parameters of Alpha power extended inverse Weibull distribution and the corresponding confidence intervals are obtained. The simulation method is carried out to evaluate the performance of the maximum likelihood estimate in terms of their Absolute Bias and Mean Square Error using simulated data. Two lifetime data sets are presented to demonstrate the applicability of the new model and it is found that the new model has superior modeling power when compare to Inverse Weibull distribution, Alpha Power Poisson inverse exponential distribution, Alpha Power Extended Inverse Weibull distribution, and Alpha Power Extended Inverse Exponential distribution.

Keywords: reliability analysis, Lorenz and Bonferroni and curves, order statistics, moments, maximum likelihood estimation

1. Introduction

Adding an extra shape parameter to a classical (conventional) distribution is very common in statistical distribution theory. Often introducing an extra parameter(s) brings more flexibility to a class of distribution functions essentially for data analysis purposes to improve the modeling potential of the classical distribution. For example, Azzalini (1985) introduced the skew-normal distribution by introducing an extra parameter to the normal distribution to induce more flexibility into the normal distribution. Mudholkar and Srivastava (1993) proposed a method that introduced an extra parameter to a two-parameter Weibull distribution and called it exponentiated Weibull model which has two shape parameters and one scale parameter. Marshall and Olkin (1997) introduced another method that adds a parameter to any distribution function; two special cases were considered namely when X follows exponential or Weibull distribution and derived many properties of this proposed model. The well-known generators are the following: the beta-G family of distribution which was developed and studied by Eugene et al. (2002), Cordeiro and de Castro (2011) developed the Kumaraswamy-G family of distribution, exponentiated generalized-G family of distribution was proposed and studied by Cordeiro et al. (2013), Nofal et al. (2017) developed the generalized transmuted-G family of distribution, transmuted exponentiated generalized-G family of distribution was proposed and studied by Yousof et al. (2015), transmuted geometric-G family of distribution was developed and studied by Afify et al. (2016) , Kumaraswamy transmuted-G family of distribution was studied by Afify et al. (2016b). Alizadeh et al. (2017) developed the generalized odd generalized exponential family of distribution, exponentiated Weibull-H family of distribution was proposed and developed by Cordeiro et al. (2017), exponentiated generalized-G Poisson family of distribution was developed and studied by Aryal and Yousof (2017), Alizadeh et al. (2018) proposed and studied transmuted Weibull-G family of distribution, Marshall-Olkin generalized-G Poisson family of distribution was developed and studied by Korkmaz et al. (2018). Oluyede, et al. (2018) introduced the gamma Weibull-G family of distributions by combining the gamma generator with the Weibull-G family of distributions which was defined by Bourguignon et al. (2014) and odd Lomax-G family of distribution was studied by Cordeiro et al. (2019) Recently, the alpha power transformation was proposed and studied by Mahdavi and Kundu (2017).

Let $\overline{H}$ be the CDF of any continuous random variable X, then the CDF of Alpha Power Transformed (APT) family is given by

$$G(x) = \begin{cases} \frac{\alpha^{\bar{H}(x)}-1}{\alpha-1}, & if\ \alpha > 0, \alpha \neq 1 \\ \bar{H}(x), & \alpha = 0 \end{cases} \tag{1}$$

And the associated pdf is given by

$$g(x) = \begin{cases} \frac{log\alpha}{\alpha-1} \bar{h}(x)\alpha^{\bar{H}(x)}, & if\ \alpha > 0, \alpha \neq 1 \\ \bar{H}(x), & if\ \ \alpha = 0 \end{cases} \tag{2}$$

The transformation has been used by several researchers to obtain alpha transformed distributions. Namely, Dey et al. (2017a, 2017b, 2018, 2019) studied the properties of the new extensions of generalized exponential distribution with an application to ozone data, a new extension of Weibull distribution with application to real-life data, extended Weibull distribution with application to real-life data, alpha transformed inverse Lindley distribution which exhibits upside-down bathtub shape failure rate, and alpha power transformed Lindley distribution with applications to earthquake data. Hassan et al. (2018) investigate the properties of alpha power transformed extended exponential distribution, alpha power Weibull distribution was studied by Nasser et al. (2017). Ogunde et al. (2020a, 2020b) studied the properties of alpha power extended Bur II distribution and alpha power extended inverted Weibull distribution respectively.

Motivated by the advantages offered by a generalized distribution which makes it more relevant in modeling lifetime data that are non-monotonic, exhibiting different shapes of the hazard function ranges from increasing, decreasing, and bathtub shapes, as well as the versatility of compounding alpha power Inverse Weibull and Poisson distribution in modeling real-life data. We study a new generalization called the Alpha power extended Inverse Weibull Poisson (APEIWP) distribution which possesses these properties.

We are also motivated to study the APEIWP distribution because of its simplicity and extensive usage of IW distribution in modeling lifetime events. Also, the current generalization provides a wider application even to complex situations that involve different shapes of the hazard function.

2. The Model, Sub-Models, and Properties of Alpha Power Extended Inverse Weibull Poisson (APEIWP) Distribution

The probability density function (PDF) and the associated distribution function (CDF) of the two-parameter inverse Weibull (IW) distribution is given by

$$\bar{h}(x;\eta,\omega) = \eta\omega x^{-\omega-1} e^{-\eta x^{-\omega-1}}, \quad x > 0 \tag{3}$$

and

$$\bar{H}(x;\eta,\omega) = e^{-\eta x^{-\omega}}, \quad x > 0 \tag{4}$$

where η is a positive scale parameter $(\eta > 0)$ and ω is a positive shape parameter $(\omega > 0)$, respectively. Keller et al. (1982) used the IW distribution to describe the wear and tear phenomena of some mechanical components such as crankshaft and pistons of diesel engines. In addition, the IW model has many important applications in Insurance, reliability engineering, useful life, wear-out periods, service records, and life testing, see Khan and King (2012).

Several generalizations of the Inverse Weibull distribution have been proposed and studied, see, for example, beta Inverse Weibull by Khan (2010), generalized Inverse Weibull was studied by de Gusmao et al. (2011), modified Inverse Weibull by Khan and King (2012), Pararai et al. (2014) studied the properties of gamma Inverse Weibull, Kumaraswamy generalized Inverse Weibull by Oluyede and Yang (2014), Aryal and Elbatal (2015) investigated the properties of Kumaraswamy modified Inverse Weibull distribution, the properties of Marshall-Olkin Inverse Weibull was investigated by Okasha et al. (2017), alpha power Inverse Weibull was studied by Basheer (2019), and the extended Inverse Weibull distribution was developed and studied by Said Alkarni et al. (2020).

Given that $\bar{H}(x)$ is the CDF of a distribution given in (4), then inserting (4) in (1) given another distribution called Alpha power extended Inverse Weibull distribution (APEIW) which CDF is given by

$$G(x) = \begin{cases} \dfrac{\alpha^{e^{-\eta x^{-\omega}}} - 1}{\alpha - 1}, & if\ \alpha > 0, \alpha \neq 1 \\ \alpha^{e^{-\eta x^{-\omega}}}, & \alpha = 0 \end{cases} \tag{5}$$

And the corresponding PDF is given by

$$g(x) = \begin{cases} \frac{log\alpha}{\alpha - 1}\eta\omega x^{-\omega-1}e^{-\eta x^{-\omega}}\alpha^{e^{-\eta x^{-\omega}}}, & if\ \alpha, > 0, \alpha \neq 1 \\ \eta\omega x^{-\omega-1}e^{-\eta x^{-\omega}}, & if\ \alpha = 0 \end{cases} \tag{6}$$

Suppose that X has at the Alpha power extended Inverse Weibull distribution where its PDF and CDF are given in (5) and (6) respectively. Given N, let $X_1, \dots, X_N$ be independent and identically distributed random variables from $APEIW$ distribution. Let N be distributed according to the zero truncated Poisson distribution with pdf

$$P(N = n) = \frac{\lambda^n e^{-\lambda}}{n!\,(1 - e^{-\lambda})}, \qquad n = 1,2,3, \dots, \lambda > 0$$

Let $X = max(Z_1, \dots, Z_N)$, then the CDF of $X/N = n$ is given by

$$F_{X/N=n}(x) = \left(\frac{\alpha^{e^{-\eta x^{-\omega}}} - 1}{\alpha - 1}\right)^n,$$

Which is the exponentiated alpha power extended Inverse Weibull distribution. The Alpha power Extended Inverse Weibull Poisson distribution is the marginal CDF of X, given by

$$F(x; \eta, \omega, \lambda) = \frac{1 - exp\left[-\lambda\left(\frac{\alpha^{e^{-\eta x^{-\omega}}} - 1}{\alpha - 1}\right)\right]}{e^{-\lambda} - 1}, \qquad x > 0 \tag{7}$$

Where ω, and λ are positive shape parameters and η is a positive scale parameter respectively. The corresponding $APEIWP$ density function is given by

$$f(x) = \frac{\eta\omega\lambda x^{-\omega-1}log\alpha e^{-\eta x^{-\omega}}\alpha^{e^{-\eta x^{-\omega}}} exp\left[-\lambda\left(\frac{\alpha^{e^{-\eta x^{-\omega}}} - 1}{\alpha - 1}\right)\right]}{(\alpha - 1)(e^{-\lambda} - 1)}, x > 0 \tag{8}$$

Where ω, and λ are positive shape parameters and η is a positive scale parameter respectively. The graph of the CDF and PDF are respectively drawn below in figure (1) and (2) for various values of the parameters of APEIWP distribution.

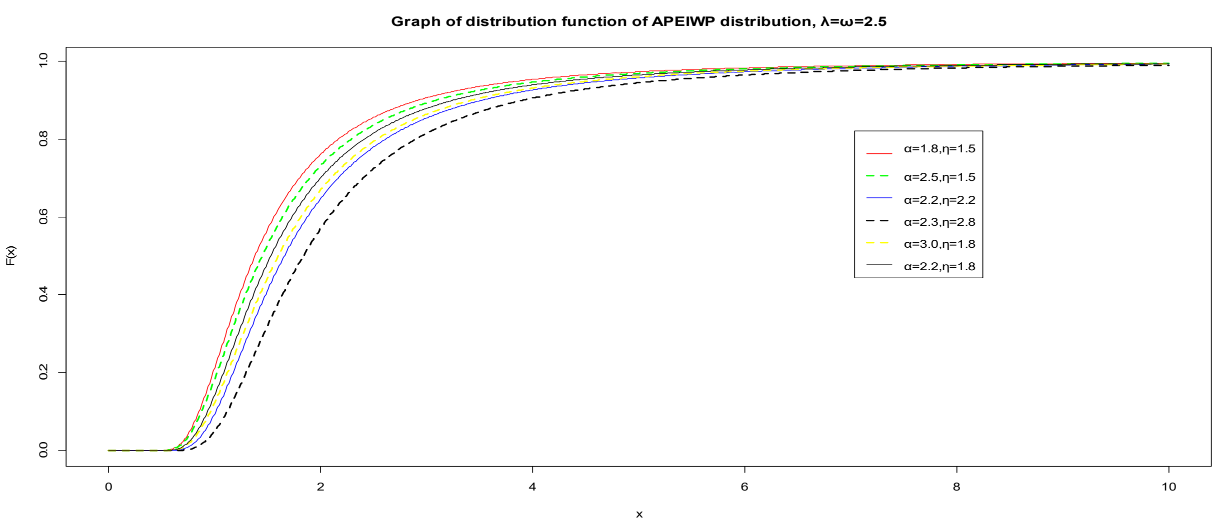

Figure 1. The graph of the CDF of $APEIWP$ distribution

- ✓ Figure 1 indicates that the $APEIWP$ distribution has a proper density function.

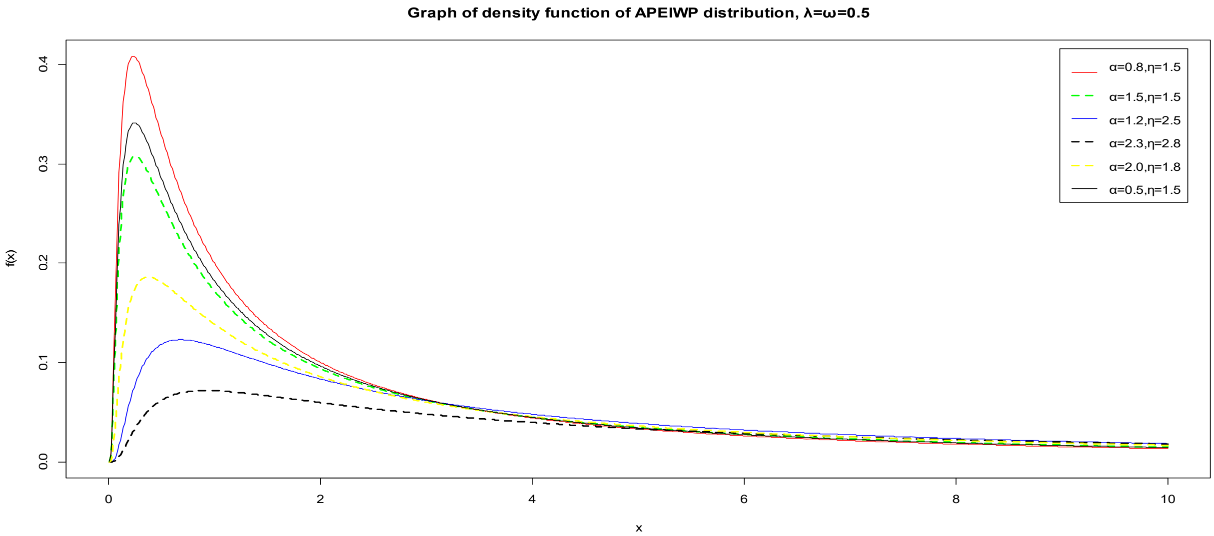

Figure 2. The graph of the PDF of $APEIWP$ distribution

- ✓ Figure 2 indicates that the graph of $APEIWP$ distribution is unimodal

The survival function $(S(x))$ is obtained by using the relation,

$$S(x) = 1 - F(x) \tag{9}$$

$$= 1 - \frac{1 - exp\left[-\lambda\left(\frac{\lambda\alpha^{e^{-\eta x^{-\omega}}} - 1}{\alpha - 1}\right)\right]}{e^{-\lambda} - 1}$$

And the hazard function is given as

$$h(x) = \frac{f(x)}{S(x)} \tag{10}$$

$$= \frac{\eta\omega\lambda x^{-\omega-1} log\alpha e^{-\eta x^{-\omega}} \alpha^{e^{-x^{-\omega}}} exp\left[-\lambda\left(\frac{\lambda\alpha^{e^{-\eta x^{-\omega}}} - 1}{\alpha - 1}\right)\right]}{(\alpha - 1)(e^{-\lambda} - 1)\left(\mathbf{1} - \frac{1 - exp\left[-\lambda\left(\frac{\lambda\alpha^{e^{-\eta x^{-\omega}}} - 1}{\alpha - 1}\right)\right]}{e^{-\lambda} - 1}\right)}$$

Figures 3 and 4 are the graph of the hazard function of $APEIWP$ distribution for various values of the parameters. The graph shows that the hazard function of $APEIWP$ model exhibits the non-monotone failure rate or upside-down bathtub failure rate for the values of the parameters considered.

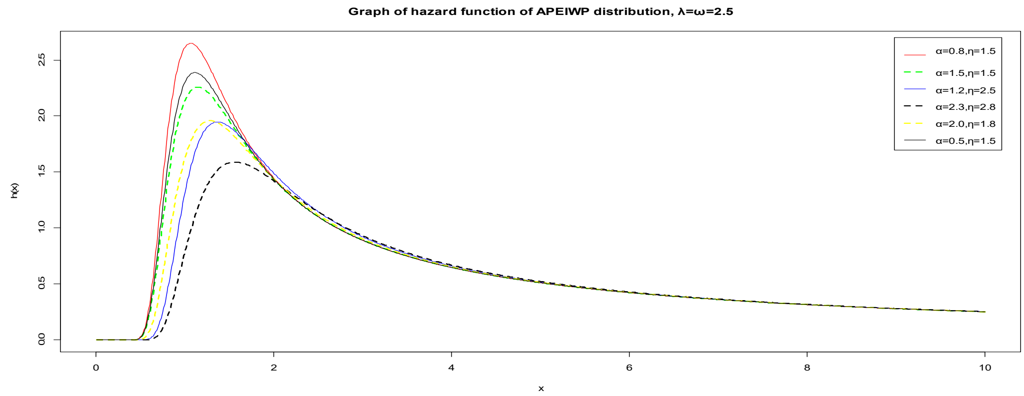

Figure 3. The graph of the hazard function of APEIWP distribution

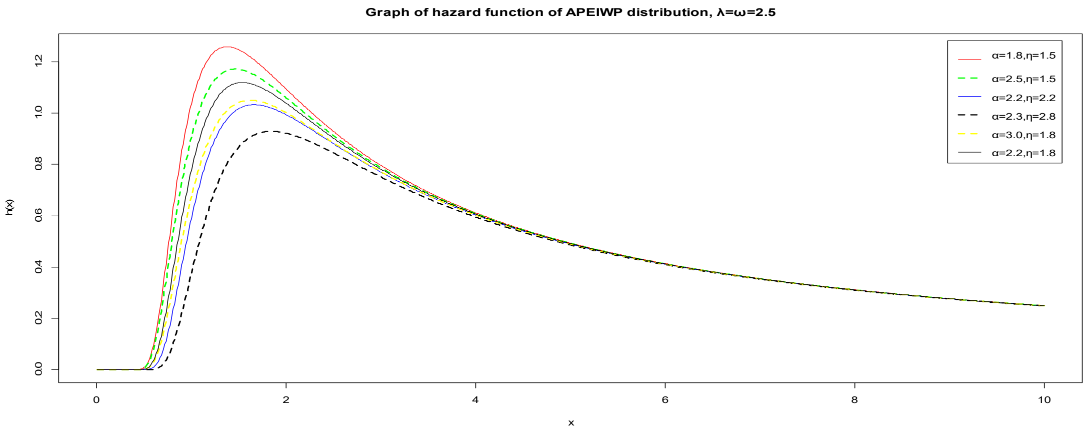

Figure 4. The graph of the hazard function of $APEIWP$ distribution

2.1 Quantile Function

Quantile function can be defined as an inverse of the distribution function. Consider the relation

$$F(X) = U \Rightarrow X = F^{-1}(U)$$

Where U follows standard Uniform distribution. The p^{th} quantile of $APEIWP$ distribution is given by

$$X_u = \left[-\frac{1}{\eta}\left\{\left(\frac{1}{log\alpha}\right) log\left[(\alpha-1)\left(1-\frac{1}{\lambda}log[(\alpha-1)\{1-u(e^{-\lambda}-1)\}]\right)\right]\right\}\right]^{-1/\omega} \tag{11}$$

The lower quartile, mean, and the upper quartile $APEIWP$ distribution can be obtained from (11) by setting the value of u to be 0.25, 0.5, and 0.75 respectively. An expression for the lower quartile, median, and upper quartile is given as

$$X_{0.25} = \left[-\frac{1}{\eta}\left\{\left(\frac{1}{log\alpha}\right) log\left[(\alpha-1)\left(1-\frac{1}{\lambda}log[(\alpha-1)\{1-0.25(e^{-\lambda}-1)\}]\right)\right]\right\}\right]^{-1/\omega} \tag{12}$$

$$X_{0.5} = \left[-\frac{1}{\eta}\left\{\left(\frac{1}{log\alpha}\right) log\left[(\alpha-1)\left(1-\frac{1}{\lambda}log[(\alpha-1)\{1-0.5(e^{-\lambda}-1)\}]\right)\right]\right\}\right]^{-1/\omega} \tag{13}$$

and

$$X_{0.75} = \left[-\frac{1}{\eta}\left\{\left(\frac{1}{log\alpha}\right) log\left[(\alpha-1)\left(1-\frac{1}{\lambda}log[(\alpha-1)\{1-0.75(e^{-\lambda}-1)\}]\right)\right]\right\}\right]^{-1/\omega} \tag{14}$$

Random numbers generation

Random numbers can be generated for the $APEIWP$ $(\alpha, \lambda, \eta, \omega)$ distribution, for this let, simulating values of random variable X with the CDF given in (7) and q denote a uniform random variable in (0, 1), then the simulated values of X are obtained by as,

$$X = \left[-\frac{1}{\eta}\left\{\left(\frac{1}{log\alpha}\right) log\left[(\alpha-1)\left(1-\frac{1}{\lambda}log\left[(\alpha-1)\left\{1-q\left(e^{-\lambda}-1\right)\right\}\right]\right)\right]\right\}\right]^{-1/\omega}$$

Skewness and Kurtosis

The quantile function can be used to determine Bowley's skewness (B_s), Kenny and keeping (1992) and Moor's kurtosis (M_k), Moor's (1988). These measures are obtained as

$$B_s = \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2Q\left(\frac{2}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}$$

and

$$M_k = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}$$

Table 1 drawn below gives the values of Bowley's skewness (B_s) and kurtosis (M_k) for various values of the parameters of $APEIW$ distributions taken $\eta = 1.3$ and $\omega = 1.2$.

	$\alpha = 2, \lambda = 0.25$	$\alpha = 1.8, \lambda = 5.2$	$\alpha = 1.8, \lambda = 4.2$	$\alpha = 2, \lambda = 3.2$
$Q\left(\frac{1}{4}\right)$	2.9862	3.0184	3.0817	8.4553
$Q\left(\frac{2}{4}\right)$	1.5227	2.4677	2.4073	5.0001
$Q\left(\frac{3}{4}\right)$	0.9554	2.2449	2.1499	3.7314
$Q\left(\frac{1}{8}\right)$	5.5389	3.0184	3.0817	14.6247
$Q\left(\frac{3}{8}\right)$	2.0364	2.6128	2.5794	6.1975
$Q\left(\frac{5}{8}\right)$	1.1922	2.3471	2.2669	4.2501
$Q\left(\frac{7}{8}\right)$	0.7716	2.1569	2.0404	3.3493
B_s	-0.4413	-0.4239	-0.4475	-0.4628
M_k	1.9318	0.7703	0.7821	1.9746

2.2 Mixture Representation for the Density Function

The mixture representation of the density function is a very useful tool used in deriving the statistical properties of generalized distribution. In this section, the mixture representation of the APEIWP density function is obtained. Using the following series representation:

$$e^y = \sum_{t=0}^{\infty} \frac{y^t}{t!} \tag{15}$$

$$(1-z)^y = \sum_{t=0}^{\infty} (-1)^t \binom{y}{t} z^t \tag{16}$$

$$\alpha^{v} = \sum_{t=0}^{\infty} \frac{(log\alpha)^t}{t!} v^t \tag{17}$$

Using the series expansion given in (15) in (8), we have

$$exp\left[-\lambda\left(\frac{\lambda\alpha^{e^{-\eta x^{-\omega}}}-1}{\alpha-1}\right)\right] = \sum_{i=0}^{\infty} \frac{(-\lambda)^i}{i!}\left\{\left(\frac{\alpha^{e^{-\eta x^{-\omega}}}-1}{\alpha-1}\right)\right\}^i \tag{18}$$

Also, using (16) in (18), we have

$$\left\{\left(\frac{\alpha^{e^{-\eta x^{-\omega}}}-1}{\alpha-1}\right)\right\}^i = (-1)^i\left(\frac{1}{\alpha-1}\right)^i\left(1-\alpha^{e^{-\eta x^{-\omega}}}\right)^i$$

$$= (-1)^i\left(\frac{1}{\alpha-1}\right)^i \sum_{j=0}^{\infty}\binom{i}{j}(-1)^j\left(\alpha^{e^{-\eta x^{-\omega}}}\right)^j$$

Consequently,

$$f(x) = \frac{\eta\omega\lambda x^{-\omega-1} log\alpha e^{-\eta x^{-\omega}}}{(\alpha-1)(e^{-\lambda}-1)}\left(\frac{1}{\alpha-1}\right)^i \sum_{i,j=0}^{\infty}\binom{i}{j}(-1)^{i+j}\frac{(-\lambda)^i}{i!}\left(\frac{1}{\alpha-1}\right)^i \alpha^{(1+j)e^{-\eta x^{-\omega}}} \tag{19}$$

applying (17) to (19), finally we have,

$$f(x) = \frac{\eta\omega\lambda}{(e^{-\lambda}-1)}\sum_{i,j,k=0}^{\infty}\binom{i}{j}(-1)^{i+j}\frac{(-\lambda)^i}{i!}\left(\frac{1}{\alpha-1}\right)^{i+1}(1+j)^k(log\alpha)^k x^{-\omega-1}e^{-(1+k)\eta x^{-\omega}} \tag{20}$$

The above expression is a density of inverse Weibull distribution with scale parameter $(1+k)\eta$ and shape parameter α

3. Ordinary and Incomplete Moment

The ordinary moments of distribution play a very important role in statistical applications. The r^{th} moment of a random variable X can be obtained using

$$E(X^r) = \mu_r' = \int_{-\infty}^{\infty} x^r f(x)dx \tag{21}$$

Putting (20) in (21), we have

$$\mu_r' = \frac{\eta\omega\lambda}{(e^{-\lambda}-1)}\sum_{i,j,k}^{\infty}\binom{i}{j}(-1)^{i+j}\frac{(-\lambda)^i}{i!}\left(\frac{1}{\alpha-1}\right)^{i+1}(1+j)^k(log\alpha)^k f^{\eta,\omega}$$

where

$$f^{\eta,\omega} = \int_{-\infty}^{\infty} x^{r-\omega-1}e^{-(1+k)\eta x^{-\omega}}\, dx \tag{22}$$

By letting $z = (1+k)\eta x^{-\omega}, x = z^{-\frac{1}{\omega}}\left((1+k)\eta\right)^{\frac{1}{\omega}}$ and putting it in (22), we have

$$f^{\eta,\omega} = \frac{1}{\omega}\left((1+k)\eta\right)^{\frac{r}{\omega}}\Gamma(1-{}^{r}/_{\omega})$$

Finally r^{th} moment of $APEIWP$ distribution is given by

$$\mu_r' = \frac{\eta\lambda}{(e^{-\lambda}-1)}\sum_{i,j,k}^{\infty} M_{i,j,k}^{\lambda}\left((1+k)\eta\right)^{\frac{r}{\omega}}\Gamma(1-{}^{r}/_{\omega}) \tag{23}$$

where

$$M_{i,j,k}^{\lambda} = \binom{i}{j}(-1)^{i+j}\frac{(-\lambda)^i}{i!}\left(\frac{1}{\alpha-1}\right)^{i+1}(1+j)^k(log\alpha)^k$$

$r < \omega$. Fo $r = 1,2,\ldots$ $\Gamma(.)$ is the gamma function. By taking $r = 1$, we obtain the mean of X that is, $\mu_1' = \mu$. The variance of X obtained by $\sigma^2 = E[(X-\mu)^2] = \mu_2' - \mu^2$. Also, we can determine the r^{th} central moment and r^{th} cumulant of X respectively defined by

$$\mu_r = E[(X-\mu)^r] = \sum_{m=0}^{r}\binom{r}{m}\mu_{r-m}'(-1)^m\mu^m, \qquad k_r = \mu_r' - \sum_{m=1}^{r-1}\binom{r-1}{m-1}k_m\mu_{r-m}',$$

Taking $k = \mu$, several measures of skewness and kurtosis based on the central moments (or cumulants) can be obtained.

Table 1 drawn below gives the first six moments and variance (σ^2) and coefficient of variation (CV) of $APEIWP$ distributions. The values for $CV = \frac{\sigma}{\mu} = \sqrt[2]{\frac{\mu_2'}{\mu^2} - 1}$

Table 2. First six moments and σ^2 and CV for $APEIWP$ distribution

$Moment$	$\alpha = 0.2, \lambda = 0.5$	$\alpha = 0.5, \lambda = 1.0$	$\alpha = 1.5, \lambda = 2.5$	$\alpha = 5.5, \lambda = 5.5$
μ_1'	1.0664	1.0867	1.0696	1.0364
μ_2'	1.1769	1.2268	1.1809	1.0917
μ_3'	1.3645	1.4623	1.3637	1.1733
μ_4'	1.7140	1.9029	1.6940	1.2978
μ_5'	2.5349	2.9491	2.4452	1.5175
μ_6'	6.5953	8.1991	6.0485	2.2605
σ^2	0.0397	0.0459	0.0369	0.0176
CV	0.1868	0.1972	0.1796	0.1280

An expression for an Incomplete moment is given by

$$\varphi_r(t) = \int_0^t x^r f(x)dx \tag{24}$$

Putting (20) in (24), we have

$$\boldsymbol{\varphi_r(t)} = \frac{\eta\omega\lambda}{(e^{-\lambda}-1)}\sum_{-\infty}^{\infty}\binom{i}{j}(-1)^{i+j}(-1)^{i+j}\frac{(-\lambda)^i}{i!}\left(\frac{1}{\alpha-1}\right)^{i+1}(1+j)^k(log\alpha)^k f^*$$

where

$$f^* = \int_0^t x^{r-\omega-1}e^{-(1+k)\eta x^{-\omega}}\,dx \tag{25}$$

Also**,** by letting $z = (1+k)\eta x^{-\omega}, x = z^{-\frac{1}{\omega}}\left((1+k)\eta\right)^{\frac{1}{\omega}}$ and putting it in (25), we have

$$f^{\eta,\omega} = \frac{1}{\omega}\left((1+k)\eta\right)^{\frac{r}{\omega}}\Gamma(1-{}^{r}/_{\omega}, (1+k)\eta t^{-\omega})$$

Finally the r^{th} incomplete moment of APEIWP distribution is given by

$$\varphi_r(t) = \frac{\eta\lambda}{(e^{-\lambda}-1)} \sum_{i,j,k}^{\infty} M_{i,j,k}^{\lambda} \left((1+k)\eta\right)^{\frac{r}{\omega}} \Gamma(1 - {}^{r}/_{\omega}, (1+k)\eta t^{-\omega}) \tag{26}$$

Where $M_{i,j,k}^{\lambda}$ is defined in (23), $\Gamma(m,n) = \int_n^{\infty} v^{m-1} e^{-v}\, dv$ is the complementary incomplete gamma function.. the first incomplete moment of APEIWP distribution is given as

$$\varphi_1(t) = \frac{\eta\lambda}{(e^{-\lambda}-1)} \sum_{i,j,k}^{\infty} M_{i,j,k}^{\lambda} \left((1+k)\eta\right)^{\frac{1}{\omega}} \Gamma\left(1 - {}^{1}/_{\omega}, (1+k)\eta t^{-\omega}\right) \tag{27}$$

The mean deviation, $\gamma_1(x)$ and median deviation, $\gamma_2(x)$, can be obtained by using the relation, $\gamma_1(x) = 2\mu F(\mu) - 2\gamma_1(\mu)$ and $\gamma_2(x) = \mu - 2\gamma_1(M)$. Where $\mu = E(X)$ and M is the median of the APEIWP random variable. Both the $\gamma_1(\mu)$ and $\gamma_1(M)$ are calculated from the first incomplete moment as given in (27)

4. Inequality Measures

Inequality measures can be applied in biomedical sciences, product quality control economics, insurance and demography, and many more. Here we consider the following inequality measures:

4.1 Mean Residual Life ($\boldsymbol{MRL}$)

Residual life is defined as the expected additional life length for a unit that is alive at age t, and it is represented mathematically by $m_x(t) = E(X - t/X > t),\ t > 0$.

The MRL of X can be obtained by using the formula:

$$m_x(t) = \frac{[1-\varphi_1(t)]}{S(t)} - t, \tag{28}$$

Where $S(t)$ is the survival function of X and $\varphi_1(t)$ as given in (27). Then we have

$$m_x(t) = \frac{1}{S(t)} \left(\frac{\eta\lambda}{(e^{-\lambda}-1)} \sum_{i,j,k}^{\infty} M_{i,j,k}^{\lambda} \left((1+k)\eta\right)^{\frac{1}{\omega}} \Gamma\left(1 - {}^{1}/_{\omega}, (1+k)\eta t^{-\omega}\right) \right) - t, \tag{29}$$

The mean inactivity time (MIT) (mean waiting time) is defined by $M_x(t) = E(tX/X \le t),\ t > 0$, and it can be obtained by the formula:

$$M_x(t) = t - \left[\frac{\varphi_1(t)}{F(t)}\right] \tag{30}$$

Also putting (27) in (30), we obtain an expression for MIT for $APEIWP$ distribution as

$$M_x(t) = t - \left[\frac{\frac{\eta\lambda}{(e^{-\lambda}-1)} \sum_{i,j,k}^{\infty} M_{i,j,k}^{\lambda} \left((1+k)\eta\right)^{\frac{1}{\omega}} \Gamma\left(1 - {}^{1}/_{\omega}, (1+k)\eta t^{-\omega}\right)}{F(t)} \right] \tag{31}$$

4.2 Bonferroni And Lorenz Curves

The Bonferroni and Lorenz curve of APEIWP distribution are respectively given by

$$\mathfrak{B}_F = \frac{1}{\mu F(t)} \int_0^t x^r f(x) dx \tag{32}$$

Since,

$$\int_0^t x^r f(x) dx = \frac{\eta\lambda}{(e^{-\lambda}-1)} \sum_{i,j,k}^{\infty} M_{i,j,k}^{\lambda} \left((1+k)\eta\right)^{\frac{1}{\omega}} \Gamma\left(1 - {}^{1}/_{\omega}, (1+k)\eta t^{-\omega}\right)$$

therefore

$$\mathfrak{B}_F(t) = \frac{1}{\mu F(t)} \frac{\eta\lambda}{(e^{-\lambda}-1)} \frac{\eta\lambda}{(e^{-\lambda}-1)} \sum_{i,j,k}^{\infty} M_{i,j,k}^{\lambda} \left((1+k)\eta\right)^{\frac{1}{\omega}} \Gamma\left(1 - {}^{1}/_{\omega}, (1+k)\eta t^{-\omega}\right) \tag{33}$$

And the Lorenz curve

$$L_F(t) = \frac{1}{\mu} \int_0^t x^r f(x) dx \tag{34}$$

$$= \frac{1}{\mu} \frac{\eta\lambda}{(e^{-\lambda}-1)} \sum_{i,j,k}^{\infty} M_{i,j,k}^{\lambda} \left((1+k)\eta\right)^{\frac{1}{\omega}} \Gamma\left(1 - {}^{1}/_{\omega}, (1+k)\eta t^{-\omega}\right)$$

4.3 Stress-Strenght Parameter

Suppose X_1 and X_2 be two continuous and independent random variables where $X_1 \sim APEIWP$ $(\alpha_1, \lambda_1, \eta, \omega)$ and $X_2 \sim APEIWP$ $(\alpha_2, \lambda_2, \eta, \omega)$, then the stress-strength parameter, say, , is defined as

$$\mathcal{S} = \int_{-\infty}^{\infty} f_1(x) F_2(x) dx \tag{35}$$

Using the CDF and the PDF of APEIWP in (35), the stress-strength parameters can be obtained as

$$\mathcal{S} = \frac{F_2(x)}{e^{-\lambda_1 - 1}} - \frac{1}{e^{-(\lambda_2+\lambda_1+2)}} \sum_{i,j,l,m,p,q}^{\infty} (-1)^{i+l} \binom{i}{j} \binom{l}{m} (\alpha_2 - 1)^{i+1} (log\alpha_2)^{j+1} (log\alpha_1)^p \frac{i^q l^p}{i!\, l!\, p!\, q!}$$
$$\times \left(\frac{1}{(q+1)\eta}\right) \tag{36}$$

5. Entropy

The Renyi entropy of APEIWP distribution can be obtained using a formula suggested by Renyi (1961) as

$$R_v = \frac{1}{v-1} \int_{-\infty}^{\infty} f^v(x) dx \tag{37}$$

Inserting (8) in (36), we have

$$R_v = \frac{1}{v-1} \int_{-\infty}^{\infty} \left\{ \frac{\eta\omega\lambda x^{-\omega-1} log\alpha e^{-\eta x^{-\omega}} \alpha^{e^{-x^{-\omega}}} exp\left[-\lambda\left(\frac{\alpha^{e^{-\eta x^{-\omega}}} - 1}{\alpha - 1}\right)\right]}{(\alpha-1)(e^{-\lambda}-1)} \right\}^v dx \tag{38}$$

Using Taylor series expansion in (15), (16), and (17), we have

$$\int_{-\infty}^{\infty} f^v(x) dx = \frac{\eta^v \omega^{v-1} \lambda^v}{(e^{-\lambda}-1)^v} \sum_{i,j,k}^{\infty} \frac{(v\lambda)^i (v+j)^k (-1)^{i+j}}{(\alpha-1)^{v+i} i!\, k!} \binom{i}{j} \int_{-\infty}^{\infty} x^{-v(\omega+1)} e^{-(k+v)\eta x^{-\omega}} dx$$

By letting the value of $z = (k+v)\eta x^{-\omega}$, $x = z^{-1/\omega}((k+v)\eta)^{-1/\omega}$,

$$\int_{-\infty}^{\infty} f^v(x) dx = \frac{\eta^v \omega^{v-1} \lambda^v}{(e^{-\lambda}-1)^v} \sum_{i,j,k}^{\infty} \frac{(v\lambda)^i (v+j)^k (-1)^{i+j}}{(\alpha-1)^{v+i} i!\, k!} \binom{i}{j} G_{\omega}^{*}$$

Where,

$$G_{\omega}^{*} = ((k+v)\eta)^{\frac{1-v(\omega+1)}{\omega}} \Gamma\left\{1 + \frac{(\omega+1)(v-1)}{\omega}\right\}$$

Finally, the Renyi entropy of APEIWP distribution is given by

$$R_v = \frac{1}{v-1} \frac{\eta^v \omega^{v-1} \lambda^v}{(e^{-\lambda}-1)^v} \sum_{i,j,k}^{\infty} \frac{(v\lambda)^i (v+j)^k (-1)^{i+j}}{(\alpha-1)^{v+i} i!\, k!} \binom{i}{j} G_{\omega}^{*} \tag{39}$$

5.1 Order Statistics

Suppose a random sample is drawn from the $APEIWP$ $(\boldsymbol{\alpha}, \eta, \omega, \lambda)$ denoted by $\underline{X}$ of size m have the following order statistics denoted by $X_{1:r} < X_{2:n} < \ldots < X_{r:n}$. Then, the PDF of the r^{th}order statistics is given by

$$f_{r:n}(x)=\frac{n!}{(r-1)!(n-r)!}F^{r-1}(x)[1-F(x)]^{n-r}f(x) \quad (40)$$

Using the series expansion (16) in (39), we have

$$f_{r:n}(x)=\frac{n!}{(r-1)!(n-r)!}\sum_{j=0}^{n-r}(-1)^j\binom{n-r}{j}F^{j+r-1}(x)f(x) \quad (41)$$

Considering, $F^{j+r-1}(x)f(x)$ and further applying the Taylor series given in (15) and (16), we have

$$F^{j+r-1}(x)f(x)=\frac{\eta\omega\lambda}{(e^{-\lambda}-1)^{j+r}}x^{-\omega-1}\sum_{l,m,p,q=0}^{\infty}\frac{(-1)^{l+m+p}}{m!\,q!\,(\alpha-1)^{m+1}}\big(-\lambda(l+1)\big)^m\binom{m}{p}(log\alpha)^{1+q}$$

$$\times\binom{j+r-1}{l}(p+1)^q e^{-(q+1)\eta x^{-\omega}} \quad (42)$$

Finally the r^{th}order statistics of $APEIWP$ distribution is given by

$$f_{r:n}(x)=\frac{n!\,\eta\omega\lambda}{(r-1)!\,(n-r)!}\sum_{j=0}^{\infty}\sum_{l,m,p,q=0}^{\infty}\frac{(-1)^{j+\_l+m+p}}{m!\,q!\,(\alpha-1)^{m+1}(e^{-\lambda}-1)^{j+r}}\big(-\lambda(l+1)\big)^m\binom{n-r}{j}$$

$$\times\binom{m}{p}\binom{j+r-1}{l}(log\alpha)^{1+q}(p+1)^q e^{-(q+1)\eta x^{-\omega}} \quad (43)$$

6. Simulation Study

In this section, we carry out t6he simulation study to ascertain the performance of MLEs of APEIWP distribution. The random number generation is obtained with its quantile function (qf) given in (11). We generated N=1000 sample sizes 50, 100, 200, 300, 400, and 500 from APEIWP distribution using its qf taking $\alpha=2.0, \lambda=0.5, \eta=1.3\ and\ \omega=1.2$. Then we calculated the empirical means, standard deviation (SD), variance (σ^2) absolute bias (AB). We observed that as the sample size increases, for each of the parameter estimates the mean square error approaches zero as expected.

Table 2. The empirical means, AB, SD, σ^2 and MSE for $APEIWP$ distribution parameters

	parameter	$mean$	AB	SD	δ^2	MSE
$n=50$	α	0.3226	1.6774	0.4206	0.1769	2.9906
	λ	0.9420	0.4420	1.7967	3.2281	3.4235
	η	3.5929	2.2929	0.8280	0.6856	5.9430
	ω	1.4026	0.2026	0.3593	0.1291	0.1702
$n=100$	α	0.2933	1.7067	0.2964	0.0879	3.0007
	λ	0.7827	0.2827	1.0925	1.1936	1.2735
	η	3.1679	1.8679	0.5142	0.2647	3.7538
	ω	1.3456	0.01456	0.2220	0.0493	0.0705
$n=200$	α	0.4834	1.5166	0.4982	0.2482	2.5483
	λ	1.0536	0.5536	0.9687	0.9384	1.2449
	η	3.1661	1.8661	0.4276	0.1828	3.6651
	ω	1.2840	0.0840	0.1964	0.0386	0.0457
$n=300$	α	0.4490	1.5510	0.1409	0.0196	2.4252
	λ	0.7217	0.5783	0.3242	0.1051	0.4395
	η	2.8109	1.5109	0.1583	0.0251	2.3079
	ω	1.2667	0.0667	0.0743	0.0055	0.0099
$n=400$	α	0.4707	1.5293	0.2466	0.0608	2.3996
	λ	0.8501	0.3501	0.5583	0.3117	0.4342
	η	2.9522	1.6522	0.2672	0.0714	2.8012
	ω	1.2558	0.0558	0.1235	0.0153	0.0184
$n=500$	α	0.4917	1.5083	0.2611	0.0682	2.3432
	λ	0.8693	0.3693	0.4945	0.2445	0.3809
	η	2.9023	1.6023	0.2397	0.0575	2.6248
	ω	1.2402	0.0402	0.1092	0.0119	0.0135

6.1 Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from APEIWP $(\alpha, \eta, \lambda, \omega)$ then the likelihood function is given by

$$\boldsymbol{L}(\underline{\boldsymbol{x}}, \alpha, \eta, \lambda, \omega) = \prod_{\boldsymbol{i=0}}^{\boldsymbol{n}} \frac{\eta\omega\lambda x^{-\omega-1} log\alpha e^{-\eta x^{-\omega}} \alpha^{e^{-\eta x^{-\omega}}} exp\left[-\lambda\left(\frac{\alpha^{e^{-\eta x^{-\omega}}}-1}{\alpha-1}\right)\right]}{(\alpha-1)(e^{-\lambda}-1)}, \tag{44}$$

Then, taking $z^{x_i} = e^{-\eta x^{-\omega}}$ the loglikelihood function ($logL = l$) is given by

$$l = nlog\left(\frac{\eta\omega\lambda}{(\alpha-1)(e^{-\lambda}-1)}\right) + nlog[\log(\alpha)] - \eta\sum_{i=1}^{n} x_i{}^{-\omega} + \log(\alpha)\sum_{i=1}^{n} z^{x_i}$$

$$-\frac{\lambda}{(\alpha-1)}\sum_{i=1}^{n}\left(\alpha^{z^{x_i}}-1\right) \tag{45}$$

We differentiate (45) with respect α, λ, η and ω, to obtain the element of the score vector $\left(V_\alpha = \frac{\partial l}{\partial \alpha}, V_\lambda = \frac{\partial l}{\partial \lambda}, V_\eta = \frac{\partial l}{\partial \eta}, V_\omega = \frac{\partial l}{\partial \omega}\right)^T$. The elements of the score vector are given by

$$V_\alpha = \frac{n}{\alpha-1} + \frac{n(\alpha-1-\alpha\log(\alpha))}{\alpha(\alpha-1)\log(\alpha)} + \frac{1}{\alpha}\sum_{i=1}^{n} z^{x_i} - \frac{\lambda}{(\alpha-1)^2}\sum_{i=1}^{n}\alpha^{z^{x_i}} + \frac{\lambda}{\alpha(\alpha-1)}\sum_{i=1}^{n}\alpha^{z^{x_i}} z^{x_i}$$

$$V_\lambda = \frac{n}{\lambda} - \frac{ne^{-\lambda}}{e^{-\lambda}-1} - \frac{1}{(\alpha-1)}\sum_{i=1}^{n}\left(\alpha^{z^{x_i}}-1\right),$$

$$V_\eta = \frac{n}{\eta} - \sum_{i=1}^{n} x_i{}^{-\omega} - \log(\alpha)\sum_{i=1}^{n} x_i^{-\omega} z^{x_i} - \frac{\lambda}{(\alpha-1)} log(\alpha)\sum_{i=1}^{n} x_i^{-\omega}\alpha^{z^{x_i}} z^{x_i}$$

$$V_\omega = \frac{n}{\omega} - \eta\sum_{i=1}^{n} x_i^{-\omega} log(\alpha) - \eta log(\alpha)\sum_{i=1}^{n} x_i^{-\omega} e^{-\eta x^{-\omega}} - \eta og(\alpha)\sum_{i=1}^{n} x_i^{-\omega}\log(x)\, z^{x_i}\alpha^{z^{x_i}}$$

By setting the non-linear system of equations $V_\alpha = V_\lambda = V_\eta = V_\omega = 0$ and obtaining a feasible solution by solving the simultaneously, the MLE of the parameters of the APEIWP model are obtained. However, these equations cannot be solved analytically, statistical software can be employed to solve them numerically by using iterative methods such as Newton-Raphson algorithms. To carry out interval estimation of the model parameters, we require the observed information matrix

$$H(\xi) = -\begin{bmatrix} V_{\alpha\alpha} & V_{\alpha\lambda} & V_{\alpha\eta} & V_{\alpha\omega} \\ V_{\lambda\alpha} & V_{\lambda\lambda} & V_{\lambda\eta} & V_{\lambda\omega} \\ V_{\eta\alpha} & V_{\eta\lambda} & V_{\eta\eta} & V_{\eta\omega} \\ V_{\omega\alpha} & V_{\omega\lambda} & V_{\omega\eta} & V_{\omega\omega} \end{bmatrix}$$

Under certain standard regularity conditions as $n \longrightarrow \infty$, the distribution of $\hat{\xi}$ can be approximated by a multivariate normal $N_4\left(0, H(\hat{\xi})^{-1}\right)$ distribution to construct approximate confidence intervals for the parameters. Here, $H(\hat{\xi})$ represent the total observed information matrix calculated at $\hat{\xi}$.

Asymptotic $(1-p)100\%$ confidence intervals for parameters can be obtained as

$$\hat{\alpha} \pm Z_{p/2}\sqrt{\Sigma_{11}}, \quad \hat{\lambda} \pm Z_{p/2}\sqrt{\Sigma_{11}}, \quad \hat{\eta} \pm Z_{p/2}\sqrt{\Sigma_{11}}, \quad \hat{\omega} \pm Z_{p/2}\sqrt{\Sigma_{11}}$$

6.2 Least Square Method (LSE)

Let $x_1, \ldots, x_n$ be a random sample from APEIWP distribution with parameters α, λ, η, and ω. By considering the corresponding order statistics $X_{1:n}, \ldots, X_{n:n}$, taking $E[F(X_{i:n})] = \frac{I}{n+1}$. The least square estimates can be obtained by minimizing the following expression

$$L^s(\xi) = \sum_{i=1}^{n} \left[F(x_i) - E[F(x_i)]\right]^2 = \sum_{i=1}^{n} \left[F(x_i) - \frac{i}{n+1}\right]^2.$$

Minimizing $L^s(\xi)$ with respect to α, λ, η and ω, we have the following system of non-linear equations:

$$\frac{\partial L^s(\xi)}{\partial \alpha} = 2\sum_{i=1}^{n} \left[F(x_i) - \frac{i}{n+1}\right] F'(x_i)_\alpha = 0,$$

$$\frac{\partial L^s(\xi)}{\partial \lambda} = 2\sum_{i=1}^{n} \left[F(x_i) - \frac{i}{n+1}\right] F'(x_i)_\lambda = 0$$

$$\frac{\partial L^s(\xi)}{\partial \eta} = 2\sum_{i=1}^{n} \left[F(x_i) - \frac{i}{n+1}\right] F'(x_i)_\eta = 0$$

$$\frac{\partial L^s(\xi)}{\partial \omega} = 2\sum_{i=1}^{n} \left[F(x_i) - \frac{i}{n+1}\right] F'(x_i)_\omega = 0$$

Where $F'(x_i)_\alpha = \frac{\partial y}{\partial \alpha} F(x_i)$, $F'(x_i)_\lambda = \frac{\partial y}{\partial \lambda} F(x_i)$ $F'(x_i)_\eta = \frac{\partial y}{\partial \eta} F(x_i)$ and $F'(x_i)_\omega = \frac{\partial y}{\partial \omega} F(x_i)$. These equations can be solved numerically by any software to obtain the estimates $\hat{\alpha}_{LSE}$, $\hat{\lambda}_{LSE}$, $\hat{\eta}_{LSE}$, and $\hat{\omega}_{LSE}$

6.3 Weighted Least Square (WLS)

Let $x_1, \dots, x_n$ be a random sample from APEIWP distribution with parameters $\alpha, \lambda, \eta,$ and ω. The likelihood function for a weighted least square estimate is given by

$$\boldsymbol{W}(\boldsymbol{\xi}) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_i) - \frac{i}{n+1}\right]^2.$$

Minimizing $W(\xi)$ with respect to α, λ, η and ω, we have the following system of non-linear equations:

$$\frac{\partial W(\xi)}{\partial \alpha} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_i) - \frac{i}{n+1}\right] F'(x_i)_\alpha = 0$$

$$\frac{\partial W(\xi)}{\partial \lambda} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_i) - \frac{i}{n+1}\right] F'(x_i)_\lambda = 0$$

$$\frac{\partial W(\xi)}{\partial \eta} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_i) - \frac{i}{n+1}\right] F'(x_i)_\eta = 0$$

$$\frac{\partial W(\xi)}{\partial \omega} = 2\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_i) - \frac{i}{n+1}\right] F'(x_i)_\omega = 0$$

This system of non-linear equations can be solved numerically by any software to obtain the estimates $\hat{\alpha}_{LSE}$, $\hat{\lambda}_{LSE}$, $\hat{\eta}_{LSE}$, and $\hat{\omega}_{LSE}$

6.4 Cramer Von Mises (CVM)

Crammer von Mises is a type of minimum distance estimators. Let $x_1, \dots, x_n$ be a random sample from APEIWP distribution with parameters $\alpha, \lambda, \eta,$ and ω. The likelihood function for Crammer von Mises estimate is given by

$$C(\xi) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_i) - \frac{2i-1}{2n} \right]^2.$$

Minimizing $C(\xi)$ with respect to α, λ, η and ω, we have the following system of non-linear equations:

$$\frac{\partial C(\xi)}{\partial \alpha} = 2 \sum_{i=1}^{n} \left[F(x_i) - \frac{2i-1}{2n} \right] F'(x_i)_\alpha = 0,$$

$$\frac{\partial W(\xi)}{\partial \lambda} = 2 \sum_{i=1}^{n} \left[F(x_i) - \frac{2i-1}{2n} \right] F'(x_i)_\lambda = 0$$

$$\frac{\partial W(\xi)}{\partial \eta} = 2 \sum_{i=1}^{n} \left[F(x_i) - \frac{2i-1}{2n} \right] F'(x_i)_\eta = 0$$

$$\frac{\partial L^s(\xi)}{\partial \omega} = 2 \sum_{i=1}^{n} \left[F(x_i) - \frac{2i-1}{2n} \right] F'(x_i)_\omega = 0$$

These equations can be solved numerically by any software to obtain the estimates $\hat{\alpha}_{LSE}$, $\hat{\lambda}_{LSE}$, $\hat{\eta}_{LSE}$, and $\hat{\omega}_{LSE}$

6.5 Practical Applications

In this subsection, we evaluate the performance of the APEIWP distributions with the other four competing models to two reliability data sats. The data sets are described as follows:

The data set (data set 1). The data set was presented by Murthy et al. (2004) on the failure times (in weeks) of 50 components. The data set are: 0.013, 0.065,0.111, 0.111,0.163, 0.309, 0.426, 0.535, 0.684, 0.747, 0.997, 1.284, 1.304, 1.647, 1.829, 2.336, 2.838, 3.269, 3.977, 3.981, 4.520,4.789, 4.849, 5.202, 5.291, 5.349, 5.911, 6.018, 6.427, 6.456, 6.572, 7.023, 7.087, 7.291, 7.787, 8.596, 9.388, 10.261, 10.713, 11.658, 13.006, 13.388, 13.842, 17.152, 17.283, 19.418, 23.471, 24.777, 32.795, 48.105.

The data set (data set 2). The data set is made up of failure time in hours of Kevlar 49/epoxy strands with pressure at 90% and was already studied by Andrews and Herzberg (2012). The data consists of 101 observations and the numbers are: 0.01, 0.01, 0.02, 0.02,0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09,0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20,0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42,0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68,0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85,0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10,1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40,1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58,1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17,2.33, 3.03, 3.03, 3.34, 4.20,4.69, 7.89.

The descriptive statistics of the two data sets are given in Table 3 and the graph of Total Time on Test plot is given in figure 5 and Boxplot in figure 6. From this table, it can be observed that the two data sets are over-dispersed, leptokurtic, and positively skewed. Also, from figure 5, it can be observed that data 1 exhibits decreasing failure rate and data 2 exhibit a non-monotone failure rate

Table 3. Exploratory data Analysis of Failure data

Discriptive statistics	*Data* 1	*Data* 2
Sample size	50	101
Mean	7.82	1.03
Lower quartile	1.39	0.24
Upper quartile	10.04	1.45
Median	5.32	0.80
Variance	84.75	1.25
Kurtosis	7.23	14.41
Skewness	2.38	3.05

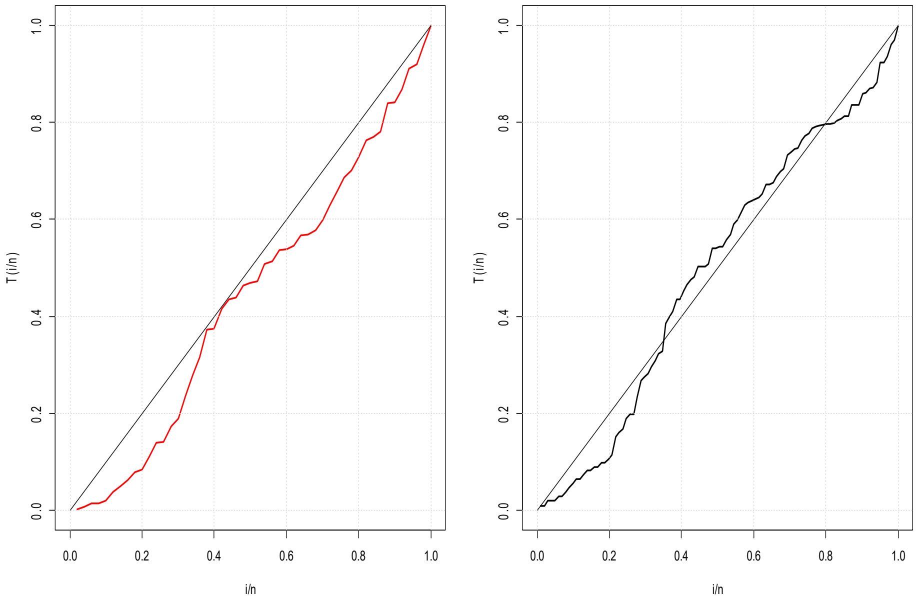

Diagram I Diagram 2

Figure 5. TTT Plot for the two failure data

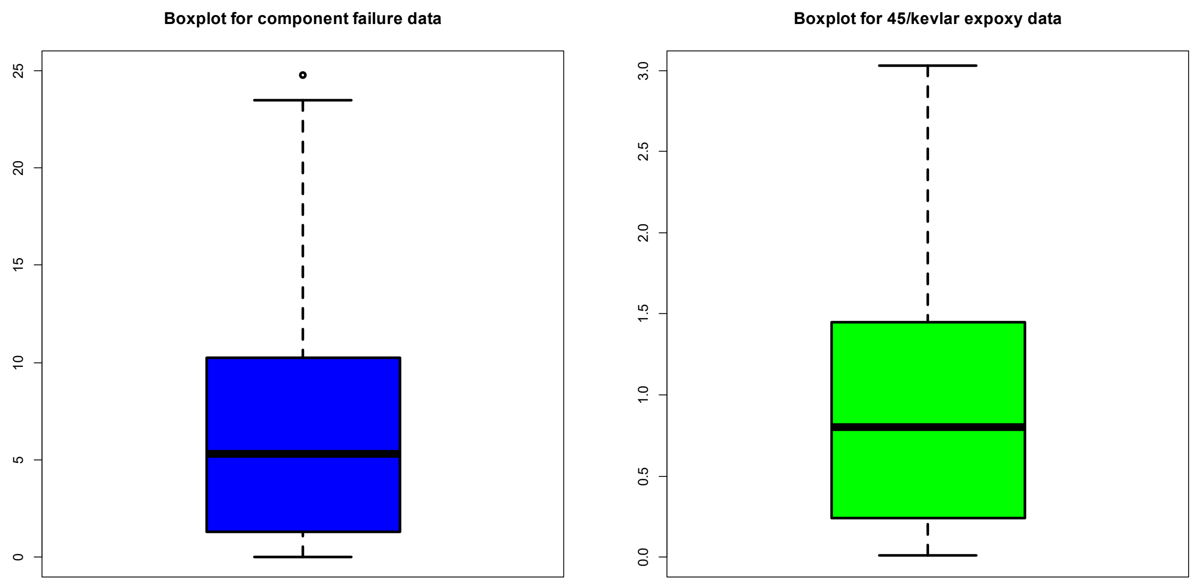

Diagram I Diagram 2

Figure 6. Box plot for the two failure data

The ML estimates along with their standard error (SE) and the confidence interval in a curly bracket of the model parameters are provided in Tables 4 and 5. In the same tables, the analytical measures including; minus 2*log-likelihood(-2log L), Akaike Information Criterion (AIC), Bayesian information criterion (BIC), and Kolmogorov Smirnov (KS) test statistic are obtained for the model considered. The fit of the proposed $APEIWP$ distribution is compared with three other competitive models namely the conventional Inverse Weibull distribution, Alpha Power

Poisson inverse exponential distribution, Alpha Power Extended Inverse Weibull distribution, and Alpha Power Extended Inverse Exponential distribution, with the following PDFs:

$$f(x) = \frac{\eta\lambda x^{-2} log\alpha e^{-\eta x^{-1}} \alpha^{e^{-\eta x^{-1}}} exp\left[-\lambda\left(\frac{\alpha^{e^{-\eta x^{-1}}}-1}{\alpha-1}\right)\right]}{(\alpha-1)(e^{-\lambda}-1)}, x > 0; \alpha, \eta, \lambda > 0$$

$$f(x) = \frac{\eta\omega x^{-\omega-1} log\alpha e^{-\eta x^{-\omega}} \alpha^{e^{-\eta x^{-\omega}}}}{(\alpha-1)}, \qquad x > 0;\ \alpha, \eta, \omega > 0$$

$$f(x) = \frac{\eta x^{-2} log\alpha e^{-\eta x^{-1}} \alpha^{e^{-\eta x^{-1}}}}{(\alpha-1)}, \qquad x > 0;\ \alpha, \eta > 0$$

Table 4. Analytical results of the APEIWP model and other competing models for Kevlar 45/epoxy data

$Model$	α	λ	η	ω	$-2l$	AIC	BIC	K
$APEIWP$	0.07 (0.08) {−0.09,0.22}	8.36 (2.93) {2.62,14.10}	2.94 (0.39) {2.18,3.70}	0.23 (0.03) {0.17,0.29}	232.06	240.07	250.53	0.11
$APPIE$	6.91 (3.618) {−0.19,14.01}	−3.39 (0.763) {−4.87, −1.91}	0.048 (0.011) {0.03,0.07}	− (−) {−}	247.40	253.41	261.25	0.18
$APEIW$	0.02 (0.01) {0.00,0.04}	− (−) {−}	1.64 (0.30) {1.05,2.23}	0.32 (0.04) {0.24,0.40}	247.32	253.31	261.16	0.19
$APEIE$	8.21 (2.34) {3.62.12.77}	− (−) {−}	0.10 (0.01) {0.08,0.12}	− (−) {−}	279.40	283.40	288.63	0.32
IW	− (−) {−}	− (−) {−}	0.42 (0.06) {0.30,0.54}	0.62 (0.04) {0.54,0.70}	264.88	268.88	274.11	0.19

Table 5. Analytical results of the APEIWP model and other competing models for failure time of components

$Model$	α	λ	η	ω	$-2l$	AIC	BIC	K
$APEIP$	7.81 (3.86) {0.24,15.38}	−3.82 (1.43) {−6.62, −1.02}	0.20 (0.08) {0.04,0.36}	0.69 (0.07) {0.55,0.83}	323.00	331.01	338.65	0.17
$APPIE$	10.76 (5.73) {−0.47,21.99}	−4.99 (1.09) {−7.13, −2.85}	0.13 (0.03) {0.07,0.19}	− (−) {−}	343.42	349.41	355.15	0.30
$APEIW$	8.40 (5.35) {−2.09,18.89}	− (−) {−}	0.59 (0.13) {0.34,0.85}	0.61 (0.05) {0.51,0.71}	330.54	336.55	342.28	0.21
$APEIE$	19.27 (6.98) {5.59,32.95}	− (−) {−}	0.27 (0.04) {0.19,0.35}	− (−) {−}	380.64	384.70	388.52	0.46
$1W$	− (−) {−}	− (−) {−}	1.117 (0.17) {0.79,1.45}	0.48 (0.05) {0.38,0.58}	337.28	341.28	345.11	0.20

Based on Tables 4 and 5, it is evident that $APEIWP$ model provides the best fit among other competing models, and can therefore be taken as the best model based on the data considered. Also, Figures 7 and 8 provide more information on the applicability of the $APEIWP$ distribution in modeling lifetime data.

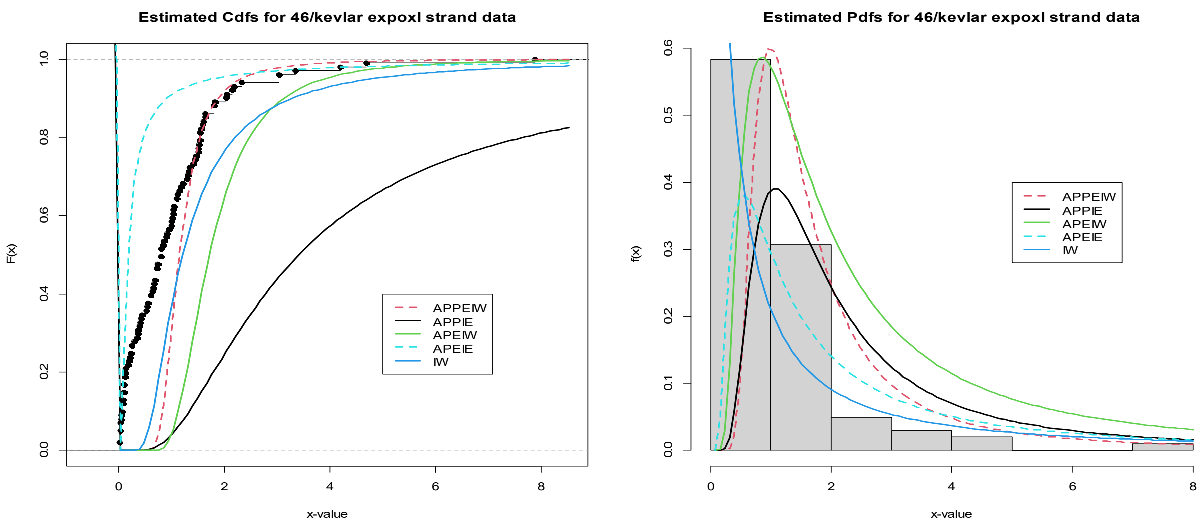

Figure 7. Estimated CDF and PDF function and other competing models for 46/Kevlar epoxy strand data

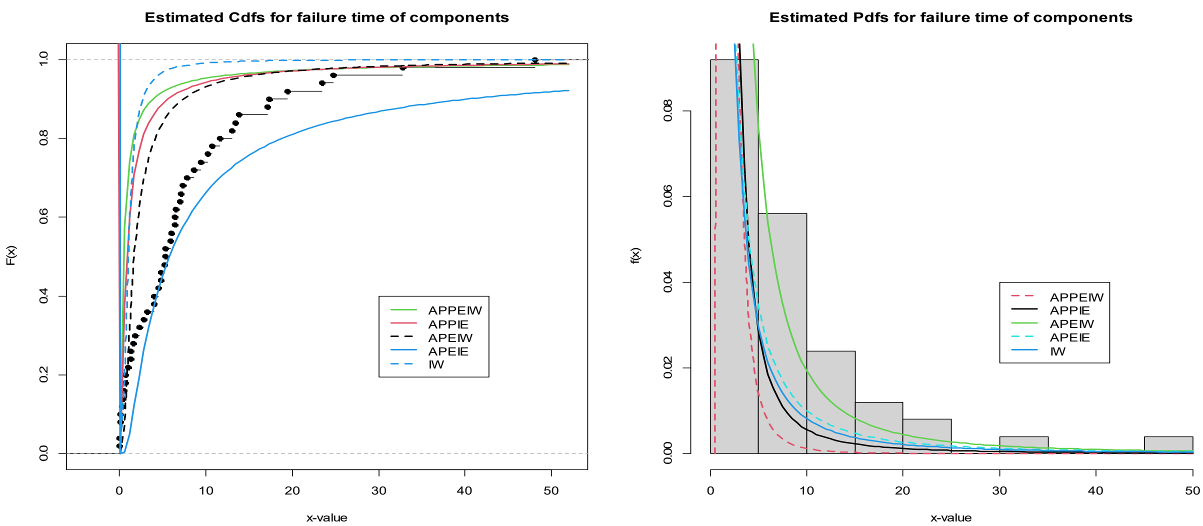

Figure 8. Estimated CDF and PDF function and other competing models for components failure data

7. Conclusion

In this work, we study the Alpha power extended inverse Weibull Poisson distribution. Some structural properties of the $APEIWP$ distribution are derived such as ordinary and incomplete moments, Renyi entropy, order statistics, mean residual life, mean inactivity time, Bonferroni and Lorenz curves, and stress strength reliability. Estimation of the population parameters is carried out by using the maximum likelihood estimation method. Simulation study and two life data sets are used to illustrate the applicability of $APEIWP$ distribution in modeling lifetime data. We recommend that further studies should be carried out by using different estimations techniques such as the Weighted Least Square method, Minimum spacing method, and Bayesian method, etc., and compare the performance of the estimation techniques.

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