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Characterizations of the Exponentiated Marshall-Olkin Discrete Uniform Distribution

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Abstract

In this short paper, we consider certain characterizations of the Exponentiated Marshall-Olkin Discrete Uniform (EMOD-U) distribution introduced by Gharib et al. (2017) to complete in some manner, the authors' work.

Keywords: discrete distribution, characterizations, conditional expectation, hazard function, reverse hazard function

1. Introduction

The cumulative distribution function (cdf), $F(x)$, probability mass function (pmf), $f(x)$, hazard function, $h_F(x)$, and reverse hazard rate function, $r_F(x)$, are give by

$$F(x;\alpha,\theta,\gamma)=\left(\frac{x+1}{\alpha\theta+(1-\theta)(x+1)}\right)^{\gamma}, \qquad x=0,1,...\alpha, \tag{1}$$

$$f(x;\alpha,\theta,\gamma)=\left(\frac{x+1}{\alpha\theta+(1-\theta)(x+1)}\right)^{\gamma}-\left(\frac{x}{\alpha\theta+(1-\theta)x}\right)^{\gamma}, \qquad x=0,1,...,\alpha, \tag{2}$$

$$h_F(x)=\frac{1-\left(\frac{x}{\alpha\theta+(1-\theta)x}\right)^{\gamma}}{1-\left(\frac{x+1}{\alpha\theta+(1-\theta)(x+1)}\right)^{\gamma}}-1, \qquad x=0,1,...,\alpha, \tag{3}$$

$$r_F(x)=1-\left(\frac{x(\alpha\theta+(1-\theta)(x+1))}{(x+1)(\alpha\theta+(1-\theta)x)}\right)^{\gamma}, \qquad x=0,1,...,\alpha, \tag{4}$$

Respectively, where α, a positive integer and $\beta>0,\gamma>0$ are parameters.

1.1 Remark

In defining the cdf of EMODU, Gharib et al. did not make the distinction we have made here for $x=0$.

2. Characterization Results

We present our characterizations via three subsections 2.1, 2.2 and 2.3. In the following subsections, we let $I=\{0,1,...,\alpha\}$.

2.1 Characterizations of EMODU in Terms of the Conditional Expectation of Certain Function of the Random Variable

2.1.1 Proposition

Let $X:\Omega\rightarrow I$ be a random variable. The pmf of X is (2) if and only if

$$
\begin{aligned}
& E\left\{\left[\left(\frac{X+1}{\alpha\theta+(1-\theta)(X+1)}\right)^{\gamma}+\left(\frac{X}{\alpha\theta+(1-\theta)X}\right)^{\gamma}\right] \mid X \leq k\right\} \\
= & \left(\frac{k+1}{\alpha\theta+(1-\theta)(k+1)}\right)^{\gamma}, \quad k \in I. \qquad (5)
\end{aligned}
$$

Proof: If X has pmf (2), then the left-hand side of (5) , using the telescoping series property, will be

$$
\begin{aligned}
& (F(k))^{-1} \sum_{x=0}^{k}\left\{\left[\left(\frac{x+1}{\alpha\theta+(1-\theta)(x+1)}\right)^{2\gamma}-\left(\frac{x}{\alpha\theta+(1-\theta)x}\right)^{2\gamma}\right]\right\} \\
= & \left(\frac{k+1}{\alpha\theta+(1-\theta)(k+1)}\right)^{-\gamma}\left\{\left(\frac{k+1}{\alpha\theta+(1-\theta)(k+1)}\right)^{2\gamma}\right\} \\
= & \left(\frac{k+1}{\alpha\theta+(1-\theta)(k+1)}\right)^{\gamma}.
\end{aligned}
$$

Conversely, if (5) holds, then

$$
\begin{aligned}
& \sum_{x=0}^{k}\left\{\left[\left(\frac{x+1}{\alpha\theta+(1-\theta)(x+1)}\right)^{\gamma}+\left(\frac{x}{\alpha\theta+(1-\theta)x}\right)^{\gamma}\right] f(x)\right\} \\
= & F(k)\left(\frac{k+1}{\alpha\theta+(1-\theta)(k+1)}\right)^{\gamma} \\
= & [F(k+1)-f(k+1)]\left(\frac{k+1}{\alpha\theta+(1-\theta)(k+1)}\right)^{\gamma}, \qquad (6)
\end{aligned}
$$

where we have used $F(k)=F(k+1)-f(k+1)$.

From (5), we also have

$$
\begin{aligned}
& \sum_{x=0}^{k+1}\left\{\left[\left(\frac{x+1}{\alpha\theta+(1-\theta)(x+1)}\right)^{\gamma}+\left(\frac{x}{\alpha\theta+(1-\theta)x}\right)^{\gamma}\right] f(x)\right\} \\
= & F(k+1)\left(\frac{k+2}{\alpha\theta+(1-\theta)(k+2)}\right)^{\gamma}. \qquad (7)
\end{aligned}
$$

Subtracting (6) from (7), we come up with

$$
\begin{aligned}
& F(k+1)\left[\left(\frac{k+2}{\alpha\theta+(1-\theta)(k+2)}\right)^{\gamma}-\left(\frac{k+1}{\alpha\theta+(1-\theta)(k+1)}\right)^{\gamma}\right] \\
= & \left[\left(\frac{k+2}{\alpha\theta+(1-\theta)(k+2)}\right)^{\gamma}\right] f(k+1),
\end{aligned}
$$

and hence

$$
r_F(k+1)=\frac{f(k+1)}{F(k+1)}=1-\left(\frac{(k+1)(\alpha\theta+(1-\theta)(k+2))}{(k+2)(\alpha\theta+(1-\theta)(k+1))}\right)^{\gamma},
$$

which, considering (4), suggests that X has pmf (2).

2.2 Characterization of EMODU Based on Hazard Function

2.2.1 Proposition

If $X : \Omega \to I$ is a random variable, then the pmf of X is (2) if and only if its hazard rate function satisfies the difference equation

$$h_F(k+1) - h_F(k) = \frac{1-\left(\frac{k+1}{\alpha\theta+(1-\theta)(k+1)}\right)^{\gamma}}{1-\left(\frac{k+2}{\alpha\theta+(1-\theta)(k+2)}\right)^{\gamma}} - \frac{1-\left(\frac{k}{\alpha\theta+(1-\theta)k}\right)^{\gamma}}{1-\left(\frac{k+1}{\alpha\theta+(1-\theta)(k+1)}\right)^{\gamma}}, \tag{8}$$

$k \in I$, with the boundary condition $h_F(0) = \frac{(\alpha\theta+1-\theta)^{\gamma}}{(\alpha\theta+1-\theta)^{\gamma}-1} - 1$.

Proof: If X has pmf (2), then obviously (8) holds. Now, if (8) holds, then for every $x \in I$, we have

$$\begin{aligned}&\sum_{k=0}^{x-1}\{h_F(k+1)-h_F(k)\}\\ =\ &\sum_{k=0}^{x-1}\left\{\frac{1-\left(\frac{k+1}{\alpha\theta+(1-\theta)(k+1)}\right)^{\gamma}}{1-\left(\frac{k+2}{\alpha\theta+(1-\theta)(k+2)}\right)^{\gamma}} - \frac{1-\left(\frac{k}{\alpha\theta+(1-\theta)k}\right)^{\gamma}}{1-\left(\frac{k+1}{\alpha\theta+(1-\theta)(k+1)}\right)^{\gamma}}\right\},\end{aligned}$$

or

$$h_F(x) - h_F(0) = \frac{1-\left(\frac{x}{\alpha\theta+(1-\theta)x}\right)^{\gamma}}{1-\left(\frac{x+1}{\alpha\theta+(1-\theta)(x+1)}\right)^{\gamma}} - \frac{(\alpha\theta+1-\theta)^{\gamma}}{(\alpha\theta+1-\theta)^{\gamma}-1}.$$

Considering the fact that $h_F(0) = \frac{(\alpha\theta+1-\theta)^{\gamma}}{(\alpha\theta+1-\theta)^{\gamma}-1} - 1$, from the last equation we have

$$h_F(x) = \frac{1-\left(\frac{x}{\alpha\theta+(1-\theta)x}\right)^{\gamma}}{1-\left(\frac{x+1}{\alpha\theta+(1-\theta)(x+1)}\right)^{\gamma}} - 1,$$

which, considering (3), suggests that X has pmf (2).

2.3 Characterization of EMODU Based on Reverse Hazard Function

2.3.1 Proposition

If $X : \Omega \to I$ is a random variable, then the pmf of X is (2) if and only if its reverse hazard rate function satisfies the difference equation

$$\begin{aligned}r_F(k+1) - r_F(k) &= \left(\frac{k(\alpha\theta+(1-\theta)(k+1))}{(k+1)(\alpha\theta+(1-\theta)k)}\right)^{\gamma}\\ &-\left(\frac{(k+1)(\alpha\theta+(1-\theta)(k+2))}{(k+2)(\alpha\theta+(1-\theta)(k+1))}\right)^{\gamma},\end{aligned} \tag{9}$$

$k \in I$, with the boundary condition $r_F(0) = 1$.

Proof: If X has pmf (2), then obviously (9) holds. Now, if (9) holds, then for every $x \in I$, we have

$$\begin{aligned}&\sum_{k=0}^{x-1}\{r_F(k+1)-r_F(k)\}\\ =\ &\sum_{k=0}^{x-1}\left\{\left(\frac{k(\alpha\theta+(1-\theta)(k+1))}{(k+1)(\alpha\theta+(1-\theta)k)}\right)^{\gamma} - \left(\frac{(k+1)(\alpha\theta+(1-\theta)(k+2))}{(k+2)(\alpha\theta+(1-\theta)(k+1))}\right)\right\},\end{aligned}$$

or

$$r_F(x) - r_F(0) = -\left(\frac{x(\alpha\theta + (1-\theta)(x+1))}{(x+1)(\alpha\theta + (1-\theta)x)}\right)^{\gamma}.$$

In view of the condition $r_F(0) = 1$, from the above equation we have

$$r_F(x) = 1 - \left(\frac{x(\alpha\theta + (1-\theta)(x+1))}{(x+1)(\alpha\theta + (1-\theta)x)}\right)^{\gamma},$$

which, in view of (4), implies that X has pmf (2).

2.4 Remark

Sandhya and Prasanth (2014) considered the following discrete distribution, called Marshall-Olkin Discrete Uniform (MODU), with pmf given by

$$f(x;\alpha,\theta) = \frac{\alpha\theta}{(\alpha\theta + (1-\theta)x)(\alpha\theta + (1-\theta)(x+1))}, \quad x = 0.1, ..., \alpha.$$

The EMODU is an extension of MODU for $\gamma = 1$.

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