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Estimation of the Parameters of Type-II Discrete Weibull Distribution Under Type-I Censoring

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Abstract

Maximum likelihood and proportion estimators of the parameters of the discrete Weibull type II distribution with type I censored data are discussed. A simulation study is performed to generate data from this distribution for suggested values of its parameters and to get the Maximum likelihood estimates of the parameters numerically. The method of proportions suggested by Khan et al. (1989) is also used to estimate the model's parameters. Numerical examples are used to perform a comparison study between the two method results according the values of the estimates and their corresponding mean squared errors.

Keywords: discrete Weibull distribution, Maximum likelihood estimates, method of proportions, type-I censoring

1. Introduction

Discrete versions of some continuous distributions were suggested by some researchers using different discretizing methods. These methods and distributions were very good reviewed by Chakraborty (2015) and extensively studied by Muiftah (2018). In this paper, the maximum likelihood (ML) and proportion estimators of the parameters of the discrete Weibull type II [DW(II)] distribution with type I censored data are discussed. The derivation of the first and second types of the discrete Weibull distribution is introduced firstly. Methods used to estimate the distribution parameters are then introduced. A simulation study is performed to generate data from the suggested distribution and to get the ML estimates of the parameters. The proportion estimates are also calculated. The results are compared. Tables and graphs are used to illustrate the distribution and the results in a good manner. Finally the dissection and conclusion are given.

2. Derivation of the Discrete Weibull Distribution

The first discrete version of the Weibull distribution DW(I) was introduced by Nakagawa and Osaki (1975) as follows:

If T is a random variable following the continuous Weibull distribution with a shape parameter $\alpha > 0$ and a scale parameter $\theta > 0$ [$T \sim W(\alpha, \theta)$], then the probability density function (pdf), the cumulative distribution function (cdf), the survival function (sf), and the failure rate (fr) of T are respectively:

$$f_T(t) = \alpha\, \theta\, t^{\alpha-1} e^{-\theta t^{\alpha}}, \quad t > 0$$

$$F_T(t) = P(T \le t) = 1 - e^{-\theta t^{\alpha}},$$

$$S_T(t) = e^{-\theta t^{\alpha}}, \tag{2.1}$$

and

$$h_T(t) = \alpha \theta t^{\alpha-1}, \tag{2.2}$$

then, by inserting the sf given by (1) in the discretization formula:

$$P(Y = y) = S_T(y) - S_T(y+1)$$

the pmf of the DW(I) distribution will be given by:

$$P(Y=y)=e^{-\theta y^{\alpha}}-e^{-\theta(y+1)^{\alpha}}, \quad y=0,1,\cdots$$

By substituting $e^{-\theta}=q$, the probability mass function (pmf), sf, and fr of the DW(I) are respectively:

$$P(Y=y)=P_y=q^{y^{\alpha}}-q^{(y+1)^{\alpha}}, \quad 0<q<1, \quad y=0,1,\cdots,$$

$$S_Y(y)=e^{-\theta y^{\alpha}}=q^{y^{\alpha}},$$

and

$$h_Y(q,\alpha)=\frac{P_y}{\sum_{i=y}^{\infty}P_i}=1-q^{(y+1)^{\alpha}-y^{\alpha}}.$$

The first two moments 0f the DW(I) distribution are given by:

$$E(Y)=\sum_{y=0}^{\infty}q^{y^{\alpha}}, \quad \text{and} \quad E(Y^2)=2\sum_{y=0}^{\infty}y\,q^{y^{\alpha}}+E(Y).$$

Khan et al. (1989) compared between the estimates of the parameters q and α obtained by method of moments and method of proportions. Kulasecara (1994) discussed the approximate ML estimation of the parameters under right censoring and compared his estimators with the proportion estimators introduced by Khan et al. (1989).

Stein and Dattero (1984) introduced their discrete analogue of the Weibull distribution [DW(II)] by inserting the fr given by (2) in the discretization formula:

$$\begin{aligned}P(Y=y)&=[1-h_T(0)]\,[1-h_T(1)]\cdots[1-h_T(y-1)]\{1-[1-h_T(y)]\}\\&=[1-h_T(0)]\,[1-h_T(1)]\cdots[1-h_T(y-1)]\,h_T(y)\end{aligned}$$

$$=\begin{cases}h_T(0), & y=0\\ [1-h_T(0)]\,[1-h_T(1)]\cdots[1-h_T(y-1)]\,h_T(y), & y=1,2,\cdots,d\\ 0, & otherwise\end{cases}$$

and defined it by the fr: $h_Y(y)=\begin{cases}C\,y^{\alpha-1}, & y=1,2,\cdots,d\\ 0, & otherwise\end{cases}$,

where, $0<C=\alpha\theta\le 1$, and the parameter d is determined in such a way that $0\le h_T(t)<1$,

where, with $\alpha>0$ and $0<C\le d$, $d=\begin{cases}int\{C^{-(\alpha-1)^{-1}}\}^{+}, & \alpha>1\\ +\infty, & \alpha\le 1\end{cases}$,

where, $int\{.\}^{+}$ represents the integer part of the quantity inside the braces.

Khan et al. (1989) presented the pmf of the DW(II) distribution to be:

$$\begin{aligned}P(Y=y)&=S_Y(y-1)h_Y(y)\\&=[1-h_T(1)]\,[1-h_T(2)]\cdots[1-h_T(y-1)]\,h_T(y)\end{aligned}$$

$$=C\,y^{\alpha-1}\prod_{i=1}^{y-1}(1-C\,i^{\alpha-1}), \qquad y=1,2,\cdots,d \tag{2.3}$$

Chakraborty (2015) gave the sf of this distribution in the form:

$$S_Y(y)=\prod_{i=1}^{y-1}(1-Ci^{\alpha-1}), \qquad y=1,2,\cdots,d$$

The first two moments of the DW(II) distribution are:

$$E(Y)=\sum_{y=1}^{d}Cy^{\alpha}\prod_{i=1}^{y-1}(1-Ci^{\alpha-1}), \quad \text{and} \quad E(Y^2)=\sum_{y=1}^{d}Cy^{\alpha+1}\prod_{i=1}^{y-1}(1-Ci^{\alpha-1}).$$

The DW(II) distribution reduces to the geometric distribution for C= 1- q and α = 1.

3. Estimation of the Parameters

3.1 The Maximum Likelihood Estimation

Let $Y_1, Y_2, ..., Y_n$ be *n* identically independently distributed (i.i.d.) observations from the DW(II) distribution given by (2.3) each is right censored by a fixed termination time γ. Suppose that r observations from this sample are less than *γ* and (*n-r*) observations are greater than *γ*, then the likelihood function of these observations will be:

$$L=\frac{n!}{(n-r)!}\prod_{j=1}^{r}\{[P(Y=y_j)][S_Y(y)]^{(n-r)}\}$$

$$=\frac{n!}{(n-r)!}\prod_{j=1}^{r}\{[\alpha\theta y_j^{\alpha-1}\prod_{i=1}^{y_j-1}(1-\alpha\theta i^{\alpha-1})][\prod_{i=1}^{y_j-1}(1-\alpha\theta i^{\alpha-1})]^{(n-r)}\}$$

The log-likelihood function is then given by:

$$\log L=\log[\frac{n!}{(n-r)!}]+\log\prod_{j=1}^{r}\{[\alpha\theta y_j^{\alpha-1}\prod_{i=1}^{y_j-1}(1-\alpha\theta i^{\alpha-1})][\prod_{i=1}^{y_j-1}(1-\alpha\theta i^{\alpha-1})]^{(n-r)}\}$$

$$=\log[\frac{n!}{(n-r)!}]+r\log\alpha+r\log\theta+(\alpha-1)\sum_{j=1}^{r}\log y_j+(n-r+1)\sum_{j=1}^{r}\sum_{i=1}^{y_j-1}\log(1-\alpha\theta i^{\alpha-1})$$

The first partial derivatives of $\log L$ w.r.t α and θ are respectively:

$$\frac{\partial \log L}{\partial\alpha}=\frac{r}{\alpha}+\sum_{j=1}^{r}\log y_j-(n-r+1)\sum_{j=1}^{r}\sum_{i=1}^{y_j-1}\left(\frac{\theta i^{\alpha-1}(\alpha\log i+1)}{1-\alpha\theta i^{\alpha-1}}\right),$$

and

$$\frac{\partial \log L}{\partial\theta}=\frac{r}{\theta}-(n-r+1)\sum_{j=1}^{r}\sum_{i=1}^{y_j-1}\left(\frac{\alpha\, i^{\alpha-1}}{1-\alpha\theta i^{\alpha-1}}\right).$$

which will be numerically maximized to get the ML estimates of *θ* and *α*.

3.2 Estimation Using the Method of Proportions

In the DW(II) model $P(Y=1)=P_1=C$ may be estimated through the proportion of ones in the sample, $\frac{f_1}{n}$, where, f_1 is the number of ones in the sample, and n is the sample size, then an estimate of C is given by: $\hat{C}=\hat{P}_1=\frac{f_1}{n}$.

Similarly, $P(Y=2)=P_2=C2^{\alpha-1}(1-C)$ may be estimated through the proportion of 2's in the sample, $\frac{f_2}{n}$, where, f_2 is the number of 2's in the sample, then:

$$\hat{P}_2=\hat{C}2^{\hat{\alpha}-1}(1-\hat{C}) \Rightarrow 2^{\hat{\alpha}-1}=\frac{\hat{P}_2}{\hat{C}(1-\hat{C})} \Rightarrow (\hat{\alpha}-1)\log 2=\log\left[\frac{\hat{P}_2}{\hat{C}(1-\hat{C})}\right]$$

$$\Rightarrow \hat{\alpha}\log 2=\log\left[\frac{\hat{P}_2}{\hat{C}(1-\hat{C})}\right]+\log 2$$

Substituting for $\hat{C}$ and $\hat{P}_2$, then an estimator of α is given by:

$$\hat{\alpha}_p=(\log 2)^{-1}\left[\log\left(\frac{\frac{f_2}{n}}{\frac{f_1}{n}(1-\frac{f_1}{n})}\right)+\log 2\right] \tag{3.1}$$

then, the estimator of θ is given by: $$\hat{\theta}_p=\frac{\hat{C}}{\hat{\alpha}_p}=\frac{(\frac{f_1}{n})}{\hat{\alpha}_p}. \tag{3.2}$$

The cdfs $\hat{\bar{F}}_Y(1)=\hat{P}(Y>1)=1-\hat{C}=1-\frac{f_1}{n}$ and $\hat{\bar{F}}_Y(2)=\hat{P}(Y>2)=1-\frac{f_1}{n}-\frac{f_2}{n}$ are empirical cdfs and are unbiased and consistent estimators of the actual cdfs $\bar{F}_Y(1)$ and $\bar{F}_Y(2)$ respectively. [Khan et al. (1989)], then, replacing $\frac{f_1}{n}$ by $[1-\hat{\bar{F}}_Y(1)]$ and $\frac{f_2}{n}$ by $[\hat{\bar{F}}_Y(2)-\bar{F}_Y(1)]$, equations (3.1) and (3.2) may be rewritten as:

$$\hat{\alpha}_p=(\log 2)^{-1}\left[\log\left(\frac{\hat{\bar{F}}_Y(2)-\hat{\bar{F}}_Y(1)}{\hat{\bar{F}}_Y(1)[1-\hat{\bar{F}}_Y(1)]}\right)+\log 2\right],\quad \hat{\theta}_p=\frac{1-\hat{\bar{F}}_Y(1)}{\hat{\alpha}_p}$$

Since the empirical probabilities are used to estimate $\bar{F}_Y(1)=P(Y>1)$ and $\bar{F}_Y(2)=P(Y>2)$, then, $(1-\frac{f_1}{n})$ and $(1-\frac{f_1}{n}-\frac{f_2}{n})$ are unbiased and consistent estimators for $\bar{F}_Y(1)$ and $\bar{F}_Y(2)$ respectively, consequently, $\hat{\alpha}_p$ and $\hat{\theta}_p$ are also unbiased and consistent estimators of α and θ respectively.

4. The Simulation Study (Numerical Examples)

Different values for α and β were suggested such that $0<\alpha\beta<1$ and d is finite (the infinite case is not considered in our study), i.e. ($\alpha>1, \beta<1$), and a Mathcad program is used to simulate data from the suggested distribution and to obtain the ML estimates using 1000 replications. A sample of *150* samples is randomly chosen from the simulated data of each DW(II) distribution used in the ML estimation, a Microsoft Excel program is used to calculate the number of 1's and

2's in each chosen sample and to obtain the proportion estimates $\breve{\alpha}_i$ and $\breve{\beta}_{p\,i}$ ($i = 1, ..., 150$) from each sample, then the mean of the sample estimates is used to be the proportion estimate of the distribution, i.e. $\breve{\alpha}_p = \frac{\sum_{i=1}^{150} \breve{\alpha}_{p\,i}}{150}$ and $\breve{\beta} = \frac{\sum_{i=1}^{150} \breve{\beta}_i}{150}$.

In the case of $n = 15$, some selected samples are found not to contain 1's or 2's, hence no proportion estimates are available, so these samples were excluded and replaced by other randomly selected samples.

The ML and the proportion estimates for the parameters α and β from DW(II) (α, β) are obtained for different γ which are found to be equivalent to the values of r (r = *80%n, 90%n, 95%n, 98%n, and 100%n*) for each n = *15, 25, 50, 100*, and *150*.

4.1 For (α = 1.8, β = 0.1, d= 9)

The pmf of DW(II) (*1.8, 0.1*) is given by Table 4.1a and the bar chart given by Figure 4.1.

Table 4.1a. The pmf of DW(II) (*1.8, 0.1*) distribution

y	1	2	3	4	5
P(y)	0.18	0.256987	0.244055	0.174042	0.094529
y	6	7	8	9	Σ
P(y)	0.038029	0.010551	0.001717	9.42×10^{-5}	1.000004

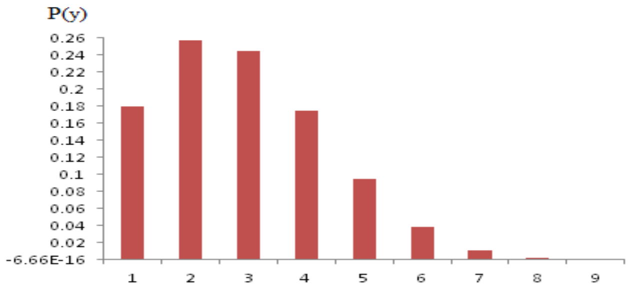

Figure 4.1 The pmf of DW(II) (*1.8, 0.1*)

The first two moments and the variance of DW(II) (1.8, 0.1) are respectively given by:

$$E(Y) = 2.912,\ E(Y^2) = 10.556,\ \text{and } Var(Y) = 2.076.$$

The ML and the proportion estimates for the parameter α from DW(II) (*1.8, 0.1*) for different γ (γ = *4, 5, 6, 7*, and *9*) which are found to be respectively equivalent to the values of r (r = *80%n, 90%n, 95%n, 98%n*, and *100%n*) for each sample size and the corresponding MSEs are given in table (4.1b) which constitutes of six columns, the first column contains the sample size n (n=*15, 25, 50, 100*, and *150*), the second column contains r (the number of observations less than γ) for each n, the third column contains the ML estimate of α, the fourth column contains the MSE of the ML estimate of α, the fifth column contains the proportion estimate of α, and the sixth column contains the MSE of the proportion estimate of α. The estimates of the parameter β from DW(II) (*1.8, 0.1*) for different γ and the corresponding MSEs are given in Table 4.1c which constitutes of the same contents of Table 4.1b considering β instead of α**.**

The estimates of the parameter α from DW(II) (*1.8, 0.1*) for different n and the corresponding MSEs are given in Table 4.1d which constitutes of seven columns, the first column contains the percentage *%* from n (*80%, 90%, 95%, 98%*, and *100%*), the second column contains the termination time γ (γ = *4, 5, 6, 7*, and *9*), the third column contains r (the number of observations less than γ), the fourth column contains the ML estimate of α, the fifth column contains the MSE of the ML estimate of α, the sixth column contains the proportion estimate of α, and the seventh column contains the MSE of the proportion estimate of α. The estimates of the parameter β from DW(II) (*1.8, 0.1*) for different n and the corresponding MSEs are given in Table 4.1e which constitutes of the same contents of Table 4.1d after replacing α by β**.**

4.2 For (α = 1.9, β = 0.05 , d= 14)

The pmf of DW(II) (*1.9, 0.05*) is given by Table 4.2a and the bar chart given by Figure 4.2.

Table 4.2a. The pmf of DW(II) (*1.9, 0.05*) distribution

y	1	2	3	4	5
P(y)	0.095	0.16045	0.19012	0.18341	0.15004
y	6	7	8	9	10
P(y)	0.1053	0.06333	0.03232	0.01375	0.00474
y	11	12	13	14	Σ
P(y)	0.00127	0.00024	2.904×10^{-5}	1.379×10^{-6}	1.00000

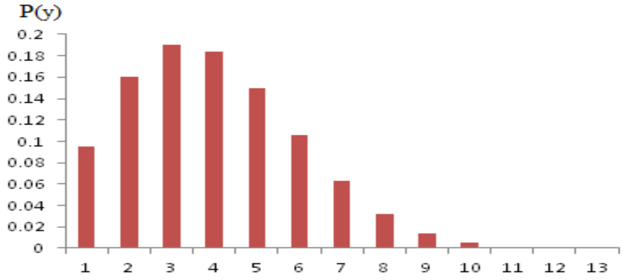

Figure 4.2 The pmf of DW(II) (*1.9, 0.05*)

The first two moments and the variance of DW(II) (1.9, 0.05) are respectively given by:

$$E(Y)=3.992,\ E(Y^2)=19.878, \text{ and } Var(Y)=3.942.$$

The ML and the proportion estimates for the parameter *α* and *β* from DW(II) *(1.9, 0.05)* for different *γ* (*γ* = *6, 7, 8, 9,* and *14*) which are found to be respectively equivalent to the values of *r* (r = *80%n, 90%n, 95%n, 98%n*, and *100%n*) for each *n* and the corresponding MSEs are given respectively in tables (4.2b) and (4.2c). Tables (4.2d) and (4.2e) represent respectively the estimates of *α* and *β* from DW(II) (*1.9, 0.05*) for different *n* and their corresponding MSEs.

4.3 For (α = 1.7, β = 0.1 , d= 13)

The pmf DW(II) (*1.7, 0.1*) is given by Table 4.3a and the bar chart given by Figure 4.3.

Table 4.3a The pmf of DW(II) (*1.7, 0.1*)

y	1	2	3	4	5
P(y)	0.17	0.22922	0.22037	0.17067	0.11001
y	6	7	8	9	10
P(y)	0.05943	0.026755	0.009877	0.002909	0.000653
y	11	12	13	Σ	
P(y)	0.000103	$10^{-6}\times9.79$	3.31×10^{-7}	1.00000	

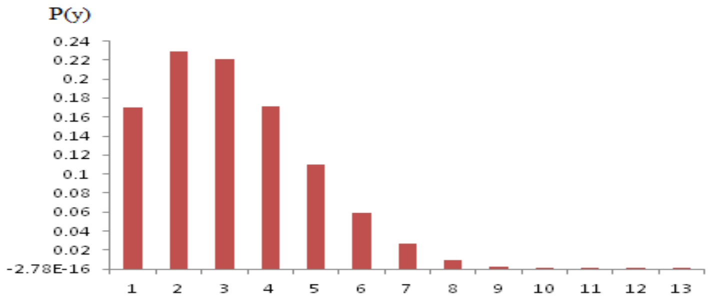

Figure 4.3 The pmf of DW(II) (1.7, 0.1)

The first two moments and the variance of DW(II) (*1.7, 0.1*) are respectively given by:

$$E(Y)=3.179,\ E(Y^2)=12.949, \text{ and } Var(Y)=2.843.$$

The estimates of the parameters α and β from DW(II) (*1.7, 0.1*) for different γ (γ = *4, 5, 6, 7,* and *13*) which are found to be respectively equivalent to the values of *r* (*r = 80%n, 90%n, 95%n, 98%n,* and *100%n*) and the corresponding MSEs are given respectively in Table 4.3b and Table 4.3c. The estimates of α and β from DW(II) (*1.7, 0.1*) for different *n* and the corresponding MSEs are given respectively in Table 4.3d and Table 4.3e.

4.4 For (α = 1.4, β = 0.2 , d= 25)

The PMF DW(II) (*1.4, 0.2*) is given by Table 4.4a and the bar chart given by Figure 4.4.

The first two moments and the variance of DW(II)(1.4, 0.2) are respectively given by:

$$E(Y)= 2.665,\ E(Y^2)=9.676,\ \text{and}\ Var(Y) =2.574.$$

The estimates of the parameters α and β from DW(II) (*1.4, 0.2*) for different γ (γ = *4, 5, 6, 7,* and *25*) which are found to be respectively equivalent to the values of *r* (*r = 80%n, 90%n, 95%n, 98%n,* and *100%n*) and the corresponding MSEs are given respectively in Table 4.3b and Table 4.4c. The estimates of α and β from DW(II) (*1.7, 0.1*) for different *n* and the corresponding MSEs are given respectively in Table 4.4d and Table 4.4e.

Table 4.4a. The pmf of DW(II) (*1.4, 0.2*)

y	1	2	3	4	5	
P(y)	0.28	0.26601	0.19727	0.12515	0.070129	
y	6	7	8	9	10	
P(y)	0.03523	0.01599	0.00658	0.00246	0.00084	
y	11	12	13	14	15	
P(y)	0.00026	7.18×10^{-5}	1.81×10^{-5}	4.07×10^{-6}	8.18×10^{-7}	
y	16	17	18	19	20	
P(y)	1.45×10^{-7}	2.25×10^{-8}	3.00×10^{-9}	3.37×10^{-10}	3.13×10^{-11}	
y	21	22	23	24	25	Σ
P(y)	2.3×10^{-12}	1.25×10^{-13}	4.58×10^{-15}	8.67×10^{-17}	1.54×10^{-19}	1.00000

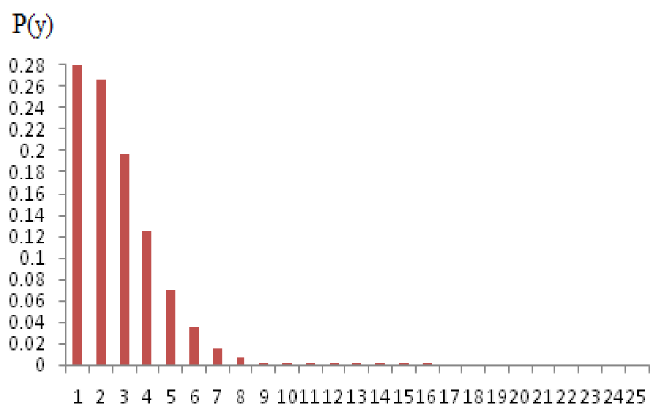

Figure 4.4 The pmf of DW(II) (*1.4, 0.2*)

5. Discussion and Conclusion

It is observed that the simulated data in all cases don't contain all integer values in the theoretical range of the random variable, for example in the case of DW(II) (*1.4, 0.2*) (where, theoretically Y= 1, 2, ..., 25), the random variable Y takes only the values between 1 and 13. It takes the value 12 in twelve, seven, three, one, and one sample(s) for *n = 150, 100, 50, 25,*and *15* respectively, takes the value 13 only in three, two, two, one, and zero sample(s) for *n = 150, 100, 50, 25,* and *15* respectively, and doesn't take any of the values 14, 15, ..., or 25 for any sample size.

From Tables 4.1b, 4.1c, 4.1d, 4.1e, 4.2b, 4.2c, 4.2d, 4.2e, 4.3b, 4.3c, 4.3d, 4.3e, 4.4b, 4.4c, 4.4d, and 4.1e, it is observed that the ML estimates for both parameters α and β are closer to the true parameter value than the proportion estimates, and that the MSE of the ML estimates is smaller in most cases than that of the proportion estimates. We may refer this to the number of data used in each of the estimation procedures, as the whole data set are used in the ML estimation, whereas, only a part of the data (1's and 2's only) is used in the proportion estimation, i.e. the ML estimators are more sufficient than the proportion estimators which consider only the number of 1's and 2's in calculating the estimates.

It seems that the estimate of the scale parameter β is less affected by censoring than the estimate of the shape parameter α in most cases, especially when heavy censoring takes place.

It also seems that the ML estimates are more affected by the proportion of censored data than the proportion estimates.

Table 4.1b. Estimated α from DW(II) (*1.8, 0.1*) for different γ

(γ = 9) ≡ (r = 100%n)					
n	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	15	**1.658**	0.2410	**1.345**	0.8835
25	25	**1.602**	0.1740	**1.296**	0.8428
50	50	**1.545**	0.1690	**1.239**	0.7954
100	100	**1.517**	0.1780	**1.240**	0.7073
150	150	**1.513**	0.1770	**1.237**	0.6909
(γ = 7) ≡ (r= 98%n)					
n	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	15	**1.658**	0.2410	**1.345**	0.8835
25	25	**1.602**	0.1740	**1.296**	0.8428
50	49	**1.796**	0.0680	**1.253**	0.7632
100	98	**1.853**	0.0470	**1.253**	0.6775
150	147	**1.893**	0.0490	**1.250**	0.6612
(γ = 6) ≡ (r= 95%n)					
n	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	15	**1.658**	0.2410	**1.345**	0.8835
25	24	**1.940**	0.2610	**1.325**	0.7746
50	48	**1.959**	0.1410	**1.267**	0.7309
100	95	**2.086**	0.2320	**1.274**	0.6310
150	143	**2.058**	0.2080	**1.268**	0.6209
(γ = 5) ≡ (r= 90%n)					
n	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	14	**2.336**	0.9965	**1.394**	0.7797
25	23	**2.216**	0.9330	**1.358**	0.7022
50	45	**2.300**	0.6790	**1.314**	0.6285
100	90	**2.145**	0.6210	**1.313**	0.5498
150	135	**1.007**	1.2680	**1.372**	0.4222
(γ = 4) ≡ (r= 80%n)					
n	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	12	**2.990**	0.8967	**1.525**	0.5510
25	20	**2.836**	0.8725	**1.485**	0.4668
50	40	**2.943**	0.6350	**1.415**	0.4414
100	80	**1.234**	1.1490	**1.410**	0.3728
150	120	**1.000**	1.2800	**1.406**	0.3595

Note. $\hat{\alpha}$ = ML estimate for α , $\breve{\alpha}$ = Proportion estimate for α.

Table 4.1c. Estimated β from DW(II) (*1.8, 0.1*) for different γ

(γ = 9) ≡ (r = 100%n)					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	15	**0.192**	0.0300	**0.489**	0.9582
25	25	**0.189**	0.0240	**0.382**	0.4836
50	50	**0.191**	0.0190	**0.292**	0.0986
100	100	**0.190**	0.0180	**0.265**	0.0628
150	150	**0.189**	0.0170	**0.260**	0.0574
(γ = 7) ≡ (r= 98%n)					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	15	**0.192**	0.0300	**0.489**	0.9582
25	25	**0.189**	0.0240	**0.382**	0.4836
50	49	**0.094**	0.0150	**0.293**	0.0984
100	98	**0.063**	0.0030	**0.267**	0.0640
150	147	**0.047**	0.0060	**0.262**	0.0586
(γ = 6) ≡ (r= 95%n)					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	15	**0.192**	0.0300	**0.489**	0.9582
25	24	**0.095**	0.0040	**0.360**	0.3033
50	48	**0.062**	0.0040	**0.294**	0.0983
100	95	**0.030**	0.0090	**0.270**	0.0657
150	143	**0.024**	0.0120	**0.265**	0.0604
(γ = 5) ≡ (r= 90%n)					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	14	**0.025**	0.0059	**0.409**	0.4491
25	23	**0.064**	0.0047	**0.347**	0.2305
50	45	**0.030**	0.0100	**0.297**	0.0979
100	90	**0.027**	0.0110	**0.274**	0.0684
150	135	**0.078**	0.0010	**0.237**	0.0418
(γ = 4) ≡ (r= 80%n)					
n	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	12	**0.135**	0.0064	**0.344**	0.2050
25	20	**0.159**	0.0072	**0.323**	0.1457
50	40	**0.018**	0.8240	**0.301**	0.0970
100	80	**0.068**	0.0030	**0.284**	0. 0743
150	120	**0.074**	0.0010	**0.281**	0.0704

Note. $\hat{\beta}$ = ML estimate *for* β , $\breve{\beta}$ = Proportion estimate for β.

Table 4.1d. Estimated α from DW(II) (*1.8, 0.1*) for different *n*

(n= 150)						
%	γ	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	4	120	**1.000**	1.2800	**1.406**	0.3595
90%	5	135	**1.007**	1.2680	**1.372**	0.4222
95%	6	143	**2.058**	0.2080	**1.268**	0.6209
98%	7	147	**1.893**	0.0490	**1.250**	0.6612
100%	9	150	**1.513**	0.1770	**1.237**	0.6909
(n= 100)						
%	γ	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	4	80	**1.234**	1.1490	**1.410**	0.3728
90%	5	90	**2.145**	0.6210	**1.313**	0.5489
95%	6	95	**2.086**	0.2320	**1.274**	0.6310
98%	7	98	**1.853**	0.0470	**1.253**	0.6775
100%	9	100	**1.517**	0.1780	**1.240**	0.7073

(n= 50)						
%	γ	r	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	4	40	**2.943**	0.6350	**1.415**	0.4414
90%	5	45	**2.300**	0.6790	**1.314**	0.6285
95%	6	48	**1.959**	0.1410	**1.267**	0.7309
98%	7	49	**1.796**	0.0680	**1.253**	0.7632
100%	9	50	**1.545**	0.1690	**1.239**	0.7954
(n= 25)						
%	γ	r	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	4	20	**2.836**	0.8725	**1.485**	0.4668
90%	5	23	**2.216**	0.9330	**1.358**	0.7022
95%	6	24	**1.940**	0.2610	**1.325**	0.7746
100%	9	25	**1.602**	0.1740	**1.296**	0.8428
(n= 15)						
%	γ	r	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	4	12	**2.990**	0.8967	**1.525**	0.5510
90%	5	14	**2.336**	0.9965	**1.394**	0.7797
100%	9	15	**1.658**	0.2410	**1.345**	0.8835

Note. $\hat{\alpha}$ = ML estimate for α , $\breve{\alpha}$ = Proportion estimate for α.

Table 4.1e. Estimated β from DW(II) (*1.8, 0.1*) for different *n*

(n= 150)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	4	120	**0.074**	0.0013	0.281	0.0704
90%	5	135	**0.078**	0.0010	0.237	0.0418
95%	6	143	**0.024**	0.0120	0.265	0.0604
98%	7	147	**0.047**	0.0058	0.262	0.0586
100%	9	150	**0.189**	0.0170	0.260	0.0574
(n= 100)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	4	80	**0.068**	0.0025	**0.284**	0.0743
90%	5	90	**0.027**	0.0110	**0.274**	0.0684
95%	6	95	**0.030**	0.0935	**0.270**	0.0657
98%	7	98	**0.063**	0.0031	**0.267**	0.0640
100%	9	100	**0.190**	0.0180	**0.265**	0.0628
(n= 50)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	4	40	**0.018**	0.8240	**0.301**	0.0970
90%	5	45	**0.030**	0.0100	**0.297**	0.0979
95%	6	48	**0.062**	0.0040	**0.294**	0.0983
98%	7	49	**0.094**	0.0150	**0.293**	0.0984
100%	9	50	**0.885**	0.0190	**0.292**	0.0986
(n= 25)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	4	20	**0.159**	0.0072	**0.323**	0.1457
90%	5	23	**0.064**	0.0047	**0.347**	0.2305
95%	6	24	**0.095**	0.0035	**0.360**	0.3033
100%	9	25	**0.189**	0.0240	**0.382**	0.4836
(n= 15)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	4	12	**0.135**	0.0064	**0.344**	0.2050
90%	5	14	**0.025**	0.0059	**0.409**	0.4491
100%	9	15	**0.192**	0.0300	**0.489**	0.9582

Note. $\hat{\beta}$ = ML estimate *for* β , $\breve{\beta}$ = Proportion estimate for β.

Table 4.2b. Estimated α from DW(II) (*1.9, 0.05*) for different γ

$(\gamma = 14) \equiv (r = 100\%n)$					
n	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	15	**1.771**	0.2030	**1.329**	1.7901
25	25	**1.744**	0.1440	**1.555**	0.9735
50	50	**1.690**	0.1310	**1.496**	0.6553
100	100	**1.662**	0.1320	**1.394**	0.6465
150	150	**1.658**	0.1300	**1.410**	0.5708
$(\gamma = 9) \equiv (r= 98\%n)$					
n	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	15	**1.771**	0.2030	**1.329**	1.7901
25	25	**1.744**	0.1440	**1.555**	0.9735
50	49	**1.902**	0.0620	**1.502**	0.6439
100	98	**1.926**	0.0330	**1.401**	0.6326
150	147	**1.976**	0.0390	**1.416**	0.5586
$(\gamma = 8) \equiv (r= 95\%n)$					
n	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	15	**1.771**	0.2030	**1.329**	1.7901
25	24	**2.014**	0.1790	**1.567**	0.9507
50	48	**2.010**	0.0970	**1.508**	0.6324
100	95	**2.005**	0.1050	**1.410**	0.6130
150	143	**2.100**	0.1410	**1.425**	0.5411
$(\gamma = 7) \equiv (r= 90\%n)$					
n	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	14	**2.121**	0.5050	**1.354**	1.7127
25	23	**2.195**	0.4130	**1.581**	0.9258
50	45	**2.267**	0.4360	**1.529**	0.5945
100	90	**2.279**	0.4070	**1.428**	0.5766
150	135	**2.014**	0.4506	**1.444**	0.5045
$(\gamma = 6) \equiv (r= 80\%n)$					
n	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	12	**2.379**	1.9989	**1.420**	1.5227
25	20	**2.677**	1.9410	**1.631**	0.8415
50	40	**2.632**	1.3700	**1.571**	0.5232
100	80	**1.044**	1.5420	**1.471**	0.4942
150	120	**1.000**	1.6200	**1.486**	0.4273

Note. $\hat{\alpha}$ = ML estimate for α, $\breve{\alpha}$ = Proportion estimate for α.

Table 4.2c. Estimated β from DW(II) (*1.9, 0.05*) for different γ

$(\gamma = 14) \equiv (r = 100\%n)$					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	15	**0.099**	0.0098	**0.131**	0.1756
25	25	**0.094**	0.0068	**0.171**	0.0602
50	50	**0.093**	0.0044	**0.137**	0.0222
100	100	**0.092**	0.0043	**0.137**	0.0200
150	150	**0.092**	0.0040	**0.128**	0.0139
$(\gamma = 9) \equiv (r= 98\%n)$					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	15	**0.099**	0.0098	**0.131**	0.1756
25	25	**0.094**	0.0068	**0.171**	0.0602
50	49	**0.043**	0.0046	**0.139**	0.0229
100	98	**0.028**	0.0010	**0.139**	0.0207
150	147	**0.020**	0.0018	**0.130**	0.0146

(γ = 8) ≡ (r= 95%n)					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	15	**0.099**	0.0098	**0.131**	0.1756
25	24	**0.044**	0.0011	**0.173**	0.0604
50	48	**0.028**	0.0011	**0.141**	0.0237
100	95	**0.015**	0.0024	**0.142**	0.0219
150	143	**0.010**	0.0032	**0.132**	0.0154
(γ = 7) ≡ (r= 90%n)					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	14	**0.025**	0.0019	**0.115**	0.1736
25	23	**0.029**	0.0014	**0.171**	0.0602
50	45	**0.013**	0.0027	**0.147**	0.0262
100	90	**0.007**	0.0035	**0.148**	0.0241
150	135	**0.042**	0.0002	**0.138**	0.0175
(γ = 6) ≡ (r= 80%n)					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	12	**0.125**	0.0188	**0.066**	0.2379
25	20	**0.015**	0.0027	**0.184**	0.0632
50	40	**0.007**	0.0037	**0.158**	0.0310
100	80	**0.040**	0.0024	**0.159**	0.0290
150	120	**0.041**	0.0002	**0.150**	0.0222

Note. $\hat{\beta}$ = ML estimate *for* β , $\breve{\beta}$ = Proportion estimate for *β.*

Table 4.2d. Estimated *α* from DW(II) (*1.9, 0.05*) for different *n*

(n= 150)						
%	*γ*	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	6	120	**1.000**	1.6200	**1.486**	0.4273
90%	7	135	**2.014**	0.4506	**1.444**	0.5045
95%	8	143	**2.100**	0.1410	**1.425**	0.5411
98%	9	147	**1.976**	0.0390	**1.416**	0.5586
100%	14	150	**1.658**	0.1300	**1.410**	0.5708
(n= 100)						
%	*γ*	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	6	80	**1.044**	1.5420	**1.471**	0.4942
90%	7	90	**2.279**	0.4070	**1.428**	0.5766
95%	8	95	**2.005**	0.1050	**1.410**	0.6130
98%	9	98	**1.926**	0.0330	**1.401**	0.6326
100%	14	100	**1.662**	0.1320	**1.394**	0.6465
(n= 50)						
%	*γ*	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	6	40	**2.632**	1.3700	**1.571**	0.5232
90%	7	45	**2.267**	0.4360	**1.529**	0.5945
95%	8	48	**2.010**	0.0970	**1.508**	0.6324
98%	9	49	**1.902**	0.0620	**1.502**	0.6439
100%	14	50	**1.690**	0.1310	**1.496**	0.6553
(n= 25)						
%	*γ*	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	6	20	**2.677**	1.9410	**1.631**	0.8415
90%	7	23	**2.195**	0.4130	**1.581**	0.9258
95%	8	24	**2.014**	0.1790	**1.567**	0.9507
100%	14	25	**1.744**	0.1440	**1.555**	0.9735
(n= 15)						
%	*γ*	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	6	12	**2.379**	1.9989	**1.420**	1.5227
90%	7	14	**2.121**	0.5050	**1.354**	1.7127
100%	14	15	**1.771**	0.2030	**1.329**	1.7901

Note. $\hat{\alpha}$ = ML estimate for α , $\breve{\alpha}$ = Proportion estimate for *α*.

Table 4.2e. Estimated β from DW(II) (*1.9, 0.05*) for different *n*

(n= 150)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	6	120	**0.041**	0.0002	**0.150**	0.0222
90%	7	135	**0.042**	0.0002	**0.138**	0.0175
95%	8	143	**0.010**	0.0032	**0.132**	0.0154
98%	9	147	**0.020**	0.0018	**0.130**	0.0146
100%	14	150	**0.092**	0.0040	**0.128**	0.0139
(n= 100)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	6	80	**0.040**	0.0024	**0.137**	0.0200
90%	7	90	**0.007**	0.0035	**0.139**	0.0207
95%	8	95	**0.015**	0.0024	**0.142**	0.0219
98%	9	98	**0.028**	0.0010	**0.148**	0.0241
100%	14	100	**0.093**	0.0043	**0.159**	0.0290
(n= 50)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	6	40	**0.007**	0.0037	**0.158**	0.0310
90%	7	45	**0.013**	0.0027	**0.147**	0.0262
95%	8	48	**0.028**	0.0011	**0.141**	0.0237
98%	9	49	**0.043**	0.0046	**0.139**	0.0229
100%	14	50	**0.092**	0.0044	**0.137**	0.0222
(n= 25)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	6	20	**0.015**	0.0027	**0.184**	0.0632
90%	7	23	**0.029**	0.0014	**0.175**	0.0608
95%	8	24	**0.044**	0.0011	**0.173**	0.0604
100%	14	25	**0.094**	0.0068	**0.171**	0.0602
(n= 15)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	6	12	**0.125**	0.0188	**0.066**	0.2379
90%	7	13	**0.025**	0.0019	**0.115**	0.1736
100%	14	15	**0.099**	0.0098	**0.131**	0.1756

Note. $\hat{\beta}$ = ML estimate *for* β , $\breve{\beta}$ = Proportion estimate for β

Table 4.3b Estimated α from DW(II) (*1.7, 0.1*) for different γ

(γ = 13) ≡ (r = 100%n)					
n	r	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	15	**1.779**	0.2230	**1.514**	0.7090
25	25	**1.738**	0.1000	**1.499**	0.3754
50	50	**1.716**	0.0500	**1.381**	0.3273
100	100	**1.449**	0.1430	**1.240**	0.5169
150	150	**1.445**	0.1420	**1.233**	0.4910
(γ = 7) ≡ (r= 98%n)					
n	r	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	15	**1.779**	0.2230	**1.514**	0.7090
25	25	**1.738**	0.1000	**1.499**	0.3754
50	49	**1.877**	0.132	**1.392**	0.3119
100	98	**1.719**	0.035	**1.251**	0.4956
150	147	**1.735**	0.027	**1.244**	0.4699

$(\gamma = 6) \equiv (r= 95\%n)$					
n	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	15	**1.779**	0.2230	**1.514**	0.7090
25	24	**1.915**	0.2440	**1.520**	0.3529
50	48	**1.939**	0.1920	**1.404**	0.2960
100	95	**1.925**	0.1610	**1.270**	0.4613
150	143	**1.689**	0.1870	**1.260**	0.4405
$(\gamma = 5) \equiv (r= 90\%n)$					
n	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	14	**2.060**	1.0530	**1.554**	0.7148
25	23	**2.047**	0.4770	**1.544**	0.3300
50	45	**2.149**	0.5570	**1.442**	0.2482
100	90	**2.218**	0.6526	**1.303**	0.4033
150	135	**1.000**	0.9800	**1.296**	0.3787
$(\gamma = 4) \equiv (r= 80\%n)$					
n	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	12	**2.768**	1.6336	**1.660**	0.5659
25	20	**2.564**	1.2430	**1.634**	0.2658
50	40	**2.370**	1.5970	**1.523**	0.1670
100	80	**3.167**	4.5853	**1.386**	0.2781
150	120	**1.000**	0.9800	**1.379**	0.2552

Note. $\hat{\alpha}$ = ML estimate for α , $\breve{\alpha}$ = Proportion estimate for α.

Table 4.3c. Estimated β from DW(II) (*1.7, 0.1*) for different γ

$(\gamma = 13) \equiv (r = 100\%n)$					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	15	**0.155**	0.0170	**0.290**	0.1979
25	25	**0.147**	0.0110	**0.204**	0.0332
50	50	**0.141**	0.0065	**0.214**	0.0342
100	100	**0.183**	0.0160	**0.242**	0.0488
150	150	**0.182**	0.0150	**0.236**	0.0414
$(\gamma = 7) \equiv (r= 98\%n)$					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	15	**0.155**	0.0170	**0.290**	0.1979
25	25	**0.147**	0.0110	**0.204**	0.0332
50	49	**0.076**	0.0023	**0.216**	0.0351
100	98	**0.389**	0.0030	**0.244**	0.0498
150	147	**0.046**	0.0060	**0.238**	0.0425
$(\gamma = 6) \equiv (r= 95\%n)$					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	15	**0.155**	0.0170	**0.290**	0.1979
25	24	**0.083**	0.0033	**0.207**	0.0346
50	48	**0.054**	0.0044	**0.218**	0.0360
100	95	**0.029**	0.0100	**0.248**	0.0518
150	143	**0.038**	0.0085	**0.241**	0.0442
$(\gamma = 5) \equiv (r= 90\%n)$					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	14	**0.091**	0.0056	**0.280**	0.1515
25	23	**0.062**	0.0048	**0.211**	0.0363
50	45	**0.029**	0.0100	**0.224**	0.0388
100	90	**0.018**	0.0140	**0.253**	0.0546
150	135	**0.080**	0.0007	**0.247**	0.0476

(γ = 4) ≡ (r= 80%n)					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	12	**0.110**	0.0089	**0.268**	0.1109
25	20	**0.088**	0.0069	**0.223**	0.0415
50	40	**0.027**	0.0110	**0.235**	0.0440
100	80	**0.010**	0.0162	**0.264**	0.0608
150	120	**0.077**	0.0010	**0.260**	0.0550

Note. $\hat{\beta}$ = ML estimate *for* β , $\breve{\beta}$ = Proportion estimate for β.

Table 4.3d. Estimated α from DW(II) (*1.7, 0.1*) for different *n*

(n= 150)						
%	γ	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	4	120	**1.000**	0.9800	**1.379**	0.2552
90%	5	135	**1.000**	0.9800	**1.296**	0.3787
95%	6	143	**1.689**	0.1870	**1.260**	0.4405
98%	7	147	**1.735**	0.0270	**1.244**	0.4699
100%	13	150	**1.445**	0.1420	**1.233**	0.4910
(n= 100)						
%	γ	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	4	80	**3.167**	4.5853	**1.386**	0.2781
90%	5	90	**2.218**	0.6526	**1.303**	0.4033
95%	6	95	**1.925**	0.1610	**1.270**	0.4613
98%	7	98	**1.719**	0.0350	**1.251**	0.4956
100%	13	100	**1.449**	0.1430	**1.240**	0.5169
(n= 50)						
%	γ	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	4	40	**2.370**	1.5970	**1.523**	0.1670
90%	5	45	**2.149**	0.5570	**1.442**	0.2482
95%	6	48	**1.939**	0.1920	**1.404**	0.2960
98%	7	49	**1.877**	0.1320	**1.392**	0.3119
100%	13	50	**1.716**	0.0500	**1.381**	0.3273
(n= 25)						
%	γ	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	4	20	**2.564**	1.2430	**1.634**	0.2658
90%	5	23	**2.047**	0.4770	**1.544**	0.3300
95%	6	24	**1.915**	0.2440	**1.520**	0.3529
100%	13	25	**1.738**	0.1000	**1.499**	0.3754
(n= 15)						
%	γ	r	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	4	12	**2.768**	1.6336	**1.660**	0.5659
90%	5	14	**2.060**	1.0530	**1.554**	0.7148
100%	13	15	**1.779**	0.2230	**1.514**	0.7090

Note. $\hat{\alpha}$ = ML estimate for α , $\breve{\alpha}$ = Proportion estimate for α.

Table 4.3e. Estimated β from DW(II) (*1.7, 0.1*) for different *n*

(n= 150)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	4	120	**0.077**	0.0010	**0.260**	0.0550
90%	5	135	**0.080**	0.0007	**0.247**	0.0476
95%	6	143	**0.038**	0.0085	**0.241**	0.0442
98%	7	147	**0.046**	0.0060	**0.238**	0.0425
100%	13	150	**0.182**	0.0150	**0.236**	0.0414
(n= 100)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	4	80	**0.0102**	0.0162	**0.264**	0.0608
90%	5	90	**0.068**	0.0027	**0.253**	0.0546
95%	6	95	**0.029**	0.0100	**0.248**	0.0518
98%	7	98	**0.389**	0.0034	**0.244**	0.0498
100%	13	100	**0.183**	0.0160	**0.242**	0.0488
(n= 50)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	4	40	**0.0270**	0.0110	**0.235**	0.0440
90%	5	45	**0.029**	0.0100	**0.224**	0.0388
95%	6	48	**0.054**	0.0044	**0.218**	0.0360
98%	7	49	**0.076**	0.0023	**0.216**	0.0351
100%	13	50	**0.141**	0.0065	**0.214**	0.0342
(n= 25)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	4	20	**0.088**	0.0069	**0.223**	0.0415
90%	5	23	**0.062**	0.0048	**0.211**	0.0363
95%	6	24	**0.083**	0.0033	**0.207**	0.0346
100%	13	25	**0.147**	0.0110	**0.204**	0.0332
(n= 15)						
%	$\boldsymbol{\gamma}$	$\boldsymbol{r}$	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	4	12	**0.110**	0.0089	**0.268**	0.1109
90%	5	14	**0.091**	0.0056	**0.280**	0.1515
100%	13	15	**0.155**	0.0170	**0.290**	0.1979

Note. $\hat{\beta}$ = ML estimate *for* β , $\breve{\beta}$ = Proportion estimate for β.

Table 4.4b. Estimated α from DW(II) (*1.4, 0.2*) for different γ

(γ = 25) ≡ (r = 100%n)					
n	r	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	15	**1.332**	0.1140	**1.280**	0.2569
25	25	**1.265**	0.0940	**1.327**	0.1567
50	50	**1.204**	0.1030	**1.148**	0.2107
100	100	**1.179**	0.1110	**1.054**	0.2821
150	150	**1.171**	0.1150	**1.001**	0.3608
(γ = 8) ≡ (r= 98%n)					
n	r	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	15	**1.332**	0.1140	**1.280**	0.2569
25	25	**1.265**	0.0940	**1.327**	0.1567
50	49	**1.363**	0.0580	**1.168**	0.1904
100	98	**1.384**	0.1330	**1.074**	0.2546
150	147	**1.328**	0.0640	**1.022**	0.3283
(γ = 6) ≡ (r= 95%n)					
n	r	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	15	**1.332**	0.1140	**1.280**	0.2569

25	24	1.403	0.1033	**1.365**	0.1431
50	48	**1.487**	0.0940	**1.209**	0.1565
100	95	**1.471**	0.2700	**1.106**	0.2137
150	143	**1.408**	0.0510	**1.050**	0.1862
(γ = 5) ≡ (r= 90%n)					
n	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	14	**1.495**	0.1233	**1.433**	0.2013
25	23	**1.473**	0.2458	**1.408**	0.1353
50	45	**1.468**	0.3190	**1.256**	0.1179
100	90	**1.501**	0.2672	**1.166**	0.1483
150	135	**1.633**	0.2080	**1.114**	0.2028
(γ = 4) ≡ (r= 80%n)					
n	*r*	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
15	12	**1.447**	0.1688	**1.541**	0.2273
25	20	**1.435**	0.2858	**1.576**	0.1821
50	40	**1.433**	0.7600	**1.403**	0.0682
100	80	**1.573**	0.6621	**1.321**	0.0477
150	120	**1.590**	0.3200	**1.270**	0.0696

Note. $\hat{\alpha}$ = ML estimate for α , $\breve{\alpha}$ = Proportion estimate for α.

Table 4.4c. Estimated β from DW(II) (*1.4, 0.2*) for different γ

(γ = 25) ≡ (r = 100%n)					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	15	**0.307**	0.0410	**0.360**	0.0964
25	25	**0.321**	0.0420	**0.311**	0.0417
50	50	**0.332**	0.0430	**0.363**	0.0670
100	100	**0.338**	0.0420	**0.397**	0.0880
150	150	**0.339**	0.0420	**0.424**	0.1164
(γ = 8) ≡ (r= 98%n)					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	15	**0.307**	0.0410	**0.360**	0.0964
25	25	**0.321**	0.0420	**0.311**	0.0417
50	49	**0.189**	0.0048	**0.363**	0.0665
100	98	**0.133**	0.0110	**0.396**	0.0700
150	147	**0.119**	0.0150	**0.423**	0.1141
(γ = 6) ≡ (r= 95%n)					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	15	**0.307**	0.0410	**0.360**	0.0964
25	24	**0.231**	0.0233	**0.312**	0.0407
50	48	**0.130**	0.0130	**0.363**	0.0654
100	95	**0.162**	0.0040	**0.395**	0.0854
150	143	**0.172**	0.0020	**0.421**	0.1109
(γ = 5) ≡ (r= 90%n)					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	14	**0.258**	0.0477	**0.345**	0.0707
25	23	**0.365**	0.0384	**0.312**	0.0395
50	45	**0.120**	0.0160	**0.362**	0.0389
100	90	**0.144**	0.0047	**0.393**	0.0826
150	135	**0.168**	0.0020	**0.417**	0.1047
(γ = 4) ≡ (r= 80%n)					
n	*r*	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
15	12	**0.211**	0.0457	**0.336**	0.0592
25	20	**0.294**	0.0445	**0.312**	0.0524
50	40	**0.136**	0.0110	**0.358**	0.0584
100	80	**0.162**	0.0030	**0.386**	0.0748
150	120	**0.178**	0.0022	**0.406**	0.0916

Note. $\hat{\beta}$ = ML estimate *for* β , $\breve{\beta}$ = Proportion estimate for β.

Table 4.4d. Estimated α from DW(II) (*1.4, 0.2*) for different n

(n= 150)						
%	γ	r	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	4	120	**1.590**	0.3200	**1.270**	0.0696
90%	5	135	**1.633**	0.2080	**1.114**	0.2028
95%	6	143	**1.408**	0.0510	**1.050**	0.1862
98%	8	147	**1.328**	0.0640	**1.022**	0.3283
100%	25	150	**1.171**	0.1150	**1.001**	0.3608
(n= 100)						
%	γ	r	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	4	80	**1.573**	0.6621	**1.321**	0.0477
90%	5	90	**1.501**	0.2672	**1.166**	0.1483
95%	6	95	**1.471**	0.2700	**1.106**	0.2137
98%	8	98	**1.384**	0.1330	**1.074**	0.2546
100%	25	100	**1.179**	0.1110	**1.054**	0.2821
(n= 50)						
%	γ	r	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	4	40	**1.433**	0.7600	**1.403**	0.0682
90%	5	45	**1.468**	0.3190	**1.256**	0.1179
95%	6	48	**1.487**	0.0940	**1.209**	0.1565
98%	8	49	**1.363**	0.0580	**1.168**	0.1904
100%	25	50	**1.204**	0.1030	**1.148**	0.2107
(n= 25)						
%	γ	r	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	4	20	**1.435**	0.2858	**1.576**	0.1821
90%	5	23	**1.473**	0.2458	**1.408**	0.1353
95%	6	24	1.403	0.1033	**1.365**	0.1431
100%	25	25	**1.265**	0.0940	**1.327**	0.1567
(n= 15)						
%	γ	r	$\hat{\alpha}$	MSE($\hat{\alpha}$)	$\breve{\alpha}$	MSE($\breve{\alpha}$)
80%	4	12	**1.447**	0.1688	**1.541**	0.2273
90%	5	14	**1.495**	0.1233	**1.433**	0.2013
100%	25	15	**1.332**	0.1140	**1.280**	0.2569

Note. $\hat{\alpha}$ = ML estimate for α , $\breve{\alpha}$ = Proportion estimate for α.

Table 4.4e. Estimated β from DW(II) (*1.4, 0.2*) for different n

(n= 150)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	4	120	**0.178**	0.0022	**0.406**	0.0916
90%	5	135	**0.168**	0.0020	**0.417**	0.1047
95%	6	143	**0.172**	0.0016	**0.421**	0.1109
98%	8	147	**0.119**	0.0150	**0.423**	0.1141
100%	25	150	**0.339**	0.0420	**0.424**	0.1164
(n= 100)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	4	80	**0.162**	0.0030	**0.386**	0.0748
90%	5	90	**0.168**	0.0021	**0.393**	0.0826
95%	6	95	**0.162**	0.0038	**0.395**	0.0854
98%	8	98	**0.133**	0.0110	**0.396**	0.0700
100%	25	100	**0.338**	0.0420	**0.397**	0.0880

(n= 50)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	4	40	**0.136**	0.0110	**0.358**	0.0584
90%	5	45	**0.120**	0.0160	**0.362**	0.0389
95%	6	48	**0.130**	0.0130	**0.363**	0.0654
98%	8	49	**0.189**	0.0048	**0.363**	0.0665
100%	25	50	**0.332**	0.0430	**0.363**	0.0670
(n= 25)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	4	20	**0.294**	0.0445	**0.312**	0.0524
90%	5	23	**0.365**	0.0384	**0.312**	0.0395
95%	6	24	**0.231**	0.0233	**0.312**	0.0407
100%	25	25	**0.321**	0.0420	**0.311**	0.0417
(n= 15)						
%	γ	r	$\hat{\beta}$	MSE($\hat{\beta}$)	$\breve{\beta}$	MSE($\breve{\beta}$)
80%	4	12	**0.211**	0.0457	**0.336**	0.0592
90%	5	14	**0.258**	0.0477	**0.345**	0.0707
100%	25	15	**0.307**	0.0410	**0.360**	0.0964

Note. $\hat{\beta}$ = ML estimate *for* β , $\breve{\beta}$ = Proportion estimate for β.

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