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Statistical Properties of a New Bathtub Shaped Failure Rate Model With Applications in Survival and Failure Rate Data

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Abstract

In this study, we proposed a flexible lifetime model identified as the modified exponentiated Kumaraswamy (MEK) distribution. Some distributional and reliability properties were derived and discussed, including explicit expressions for the moments, quantile function, and order statistics. We discussed all the possible shapes of the density and the failure rate functions. We utilized the method of maximum likelihood to estimate the unknown parameters of the MEK distribution and executed a simulation study to assess the asymptotic behavior of the MLEs. Four suitable lifetime data sets we engaged and modeled, to disclose the usefulness and the dominance of the MEK distribution over its participant models.

Keywords: Kumaraswamy distribution, bathtub shaped hazard rate function, maximum likelihood estimation, order statistics, hydrology, reliability engineering, petroleum engineering

2000 Mathematics Subject Classification: 60E05, 62P30, 62P12

1. Introduction

In this world of science, the significance of probability distributions has an imperative role to elucidate the real-world random phenomenon. In this scenario, Kumaraswamy (1980) proposed a much better choice against the beta distribution, the Kumaraswamy distribution. It is defined over the interval bounded in (0, 1)). Several characteristics like uni-anti-modal, uni-modal, decreasing, increasing, or constant failure rate, which the Kumaraswamy distribution and the beta distribution shared alike. For details, readers are referred to as Jones (2009). He highlighted some significant and common features of Kumaraswamy distribution involved simple normalizing constant, uncomplicated explicit expressions for the density function, distribution function, order statistics, and quantile function. Beta and Kumaraswamy distributions, both are the special cases of the generalized beta distribution see McDonald (1984), Ali *et al.* (2017), and Mukhtar *et al.* (2019). To model in hydrology, atmosphere temperature, clinical trials, engineering, and geology, among other real word random phenomena, Kumaraswamy distribution considers a far better choice than beta distribution.

Let X be a random variable follow by the Kumaraswamy distribution. The associated cumulative distribution function (CDF) and corresponding probability density function (PDF) with two shape parameters $(\alpha, \beta > 0)$ with$0 < x < 1$, are given by, respectively

$$P(x;\alpha,\beta) = \int_0^x p(x)dx = \alpha\beta\int_0^x x^{\alpha-1}(1-x^\alpha)^{\beta-1}dx = 1- \ (1-x^\alpha)^\beta,$$

$$p(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}(1-x^\alpha)^{\beta-1}.$$

The capability of Kumaraswamy distribution was raised by Cordeiro and de Castro (2011) in introducing a new generalized class, called the Kumaraswamy-G (short Kum-G) family. The cumulative distribution function (CDF) and probability density function (PDF) of the Kum-G family, is defined by, respectively

$$F(x;\alpha,\beta,\xi) = 1- \ \left(1-G^\alpha(x;\xi)\right)^\beta,$$

$$f(x;\alpha,\beta,\xi)=\alpha\beta\,\mathrm{g}(x;\xi)G^{\beta-1}(x;\xi)\big(1-G^{\alpha}(x;\xi)\big)^{\beta-1}.$$

where $G(x;\xi)$ is CDF of arbitrary baseline model based on the parametric vector ξ with $\alpha,\beta>0$ are the two shape parameters, respectively. Let $\mathrm{g}(x;\xi)=dG(x;\xi)/dx$ is the probability density function of any baseline model.

To study further modifications and generalizations using the Kum-G family, see the exemplar work of Bourguignon *et al.* (2013). They developed the Kumaraswamy Pareto (KP) distribution and discussed their vital characteristics and explored their application to the hydrological data. Lemonte *et al.* (2013) developed two versions of the Kumaraswamy distribution named (i) exponentiated Kumaraswamy distribution, and (ii) Log Exponentiated Kumaraswamy distribution. They derived numerous mathematical and reliable characters and discussed the application with the assistance of Log Exponentiated Kumaraswamy distribution. Alizadeh *et al.* (2015) developed the Kumaraswamy version of the Marshall-Olkin (1997) family. Afify *et al.* (2016) initiated the Kumaraswamy version of Marshall-Olkin Fréchet distribution (Krishna *et al*. (2013)) and explored their application in the medical science and reliability engineering data. Ibrahim (2017) developed the Kumaraswamy version of the power function distribution and explored their application in medical science data. Bursa and Ozel (2017) discussed the exponentiated version of Kumaraswamy power function distribution and explored their application in the metrology data. Mahmoud *et al.* (2018) developed a five-parameter Kumaraswamy edition of the exponentiated Fréchet distribution. They explored twenty-seven models and explored their application in reliability engineering data. Nawaz *et al*. (2018) generalized Kappa distribution via Kumaraswamy G class with the intention that it would be a better alternative to the generalized Kappa distribution and exploring their application in the hydrology data. Silva *et al.* (2019) proposed the exponentiated Kum-G class and explored their application in the reliability engineering data. Cribari-Neto and Santos (2019), introduced an interesting work according to some specific nature of data included exactly zero, exactly one, or both the cases were involved known as the inflated Kumaraswamy distributions. This distribution was the mixture of Kumaraswamy and Bernoulli distributions.

This article is organized in the following sections. We define the linear expressions, shapes, quantile function, reliability, and other mathematical measures in Section 2. The estimation of the model parameters by the method of maximum likelihood and simulation results is performed in Section 3. Applications to real data sets are discussed in Section 4 to illustrate the importance and flexibility of the proposed model and finally, some conclusions are reported in Section 5.

1.1 New Model

The new model is based on the Type II Half Logistic G family of distributions attributed to Hassan *et al*. (2017) with associated CDF is given as follows:

$$F(x;\phi,\zeta)=1-\int_0^{-\log W(x;\zeta)}\frac{2\phi e^{-\phi t}}{(1+e^{-\phi t})^2}dt=\frac{2[W(x;\zeta)]^{\phi}}{1+[W(x;\zeta)]^{\phi}},\tag{1}$$

where $W(x;\zeta)$ is any arbitrary baseline model based on $\zeta\in\Omega$, and $\phi>0$ is a shape parameter with $x>0$.

For deep understanding, we suggest the reader see some notable efforts including Balakrishnan (1985), extended half logistic distribution by Altun *et al*. (2018), type II half logistic exponential by Elgarhy *et al*. (2019), Kumaraswamy inverse Lindley distribution by Hemeda *et al*. (2020), Al-Marzouki *et al*. (2021), and among others.

The new model is:

(i) flexible enough and bounded in (0, 1) interval,

(ii) exhibits a bathtub-shaped failure rate function,

(iii) offers more realistic and rationalized results specifically on the complex skewed symmetric and sophisticated random phenomena,

(iv) provides consistently a better fit over its competitors as shown in the application section using four real data sets,

(v) provides simple and uncomplicated CDF, PDF, and likelihood functions.

Formally, a random variable X is said to follow the modified exponentiated Kumaraswamy (MEK) distribution if the baseline model $W(x;\alpha,\beta,\gamma)$ by Lemonte *et al*. (2013) with associated CDF,

$$W(x;\alpha,\beta,\gamma)=\big(1-(1-x^{\alpha})^{\beta}\big)^{\gamma},\tag{2}$$

is placed in equation (1) with ϕ=1. The associated CDF with three shape parameters $\alpha,\beta,\gamma>0$ and the corresponding PDF is given by respectively

$$F(x;\alpha,\beta,\gamma)=\frac{2}{1+(1-(1-x^{\alpha})^{\beta})^{-\gamma}}, \tag{3}$$

$$f(x;\alpha,\beta,\gamma)=\frac{2\alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma-1}}{(1+(1-(1-x^{\alpha})^{\beta})^{-\gamma})^{2}}, \tag{4}$$

2. Distributional Properties

2.1 Linear Representation

Linear combination provides a much informal approach to discuss the CDF and PDF than the conventional integral computation when determining the mathematical properties. For this, we consider the following binomial expansion:

$$(1-z)^{\beta}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta}{i}z^{i}\,,|z|<1,$$

From Equation (3), linear expression of CDF is given by

$$F(x)=2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{-1}{i}\binom{-\gamma i}{j}\binom{\beta j}{k}x^{\alpha k}. \tag{5}$$

From Equation (4), linear expression of PDF is given by

$$f(x)=2\alpha\beta\gamma x^{\alpha-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j}\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}(1-x^{\alpha})^{\beta j+\beta-1}. \tag{6}$$

$$f(x)=2\alpha\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}\binom{\beta j+\beta-1}{k}x^{\alpha k+\alpha-1}. \tag{7}$$

Expression in Equation (6) will be quite helpful in the forthcoming computations of various mathematical properties of the MEK distribution.

2.2 Shapes

Different plots of density and failure rate functions of the MEK distribution are displayed in Figures 1 and 2, for various choices of the parameters. Possible shapes of the density function including increasing, decreasing, symmetric, and upside-down bathtub shapes and, Figure 2 illustrates the increasing, decreasing, U - shaped, and upside-down bathtub-shaped failure rate function.

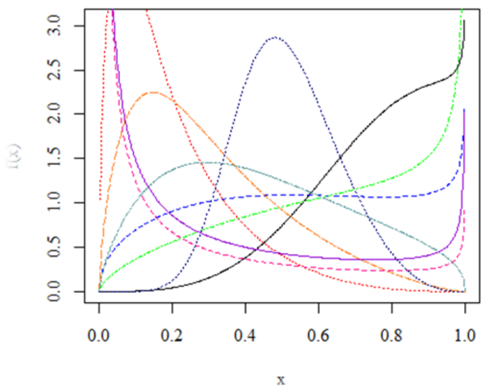

Figure 1. Plot of the density function for Parameters Black($\alpha = 4.7, \beta = 0.9, \gamma = 0.9$),Blue($\alpha = 0.8, \beta = 0.8, \gamma = 1.8$),Red($\alpha = 0.9, \beta = 3.7, \gamma = 1.7$), Green($\alpha = 1.0, \beta = 0.6, \gamma = 1.6$), chocolate1($\alpha = 1.1, \beta = 2.5, \gamma = 1.5$), Cadet blue($\alpha = 1.2, \beta = 1.4, \gamma = 1.4$), Darkviolet ($\alpha = 1.3, \beta = 0.5, \gamma = 0.3$), Deeppink($\alpha = 1.4, \beta = 0.6, \gamma = 0.2$), Navy($\alpha = 1.5, \beta = 3.7, \gamma = 5.1$)

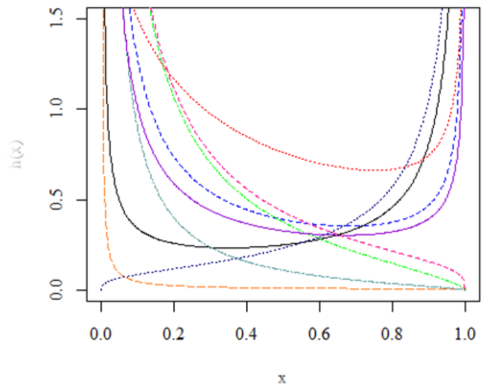

Figure 2. Plot of the failure rate function for Parameters Black($\alpha = 0.1, \beta = 0.1, \gamma = 1.5$), Blue($\alpha = 1.1, \beta = 0.3, \gamma = 0.3$),Red($\alpha = 2.1, \beta = 0.5, \gamma = 0.4$),Green($\alpha = 0.1, \beta = 1.7, \gamma = 0.5$), chocolate1($\alpha = 0.01, \beta = 0.9, \gamma = 0.7$), Cadet blue($\alpha = 0.3, \beta = 1.7, \gamma = 0.8$), Darkviolet ($\alpha = 0.2, \beta = 0.5, \gamma = 0.9$), Deeppink($\alpha = 0.5, \beta = 1.3, \gamma = 1.1$), Navy($\alpha = 1.1, \beta = 0.1, \gamma = 1.2$)

2.3 Quantiles

Hyndman and Fan (1996) introduced the concept of quantile function. The p^{th} quantile function of X~MEK$(x; \alpha, \beta, \gamma)$ with $\alpha, \beta, \gamma > 0$, is obtained by inverting the CDF mention in Equation (3). Quantile function is defined by $p = F(x_p) = P(X \le x_p),\ p \in (0,1)$.

Quantile function of X is given by

$$x_p = \left(1 - \left(1 - \left(\frac{p}{2-p}\right)^{1/\gamma}\right)^{1/\beta}\right)^{1/\alpha}. \tag{8}$$

One may obtain 1st quartile, median and 3rd quartile of X by setting p = 0.25, 0.5, and 0.75 in Equation (8) respectively. Henceforth, to generate random numbers, we assume that CDF (5) follows uniform distribution u= U (0, 1).

2.4 Skewness, Kurtosis, and Mean Deviation

The Skewness and kurtosis of MEK distribution can be calculated by the following two useful measures

$$B = \frac{Q_{0.75} + Q_{0.25} - 2Q_{0.50}}{Q_{0.75} - Q_{0.25}}, \qquad \text{and} \qquad M = \frac{Q_{0.375} - Q_{0.125} - Q_{0.625} + Q_{0.875}}{Q_{0.75} - Q_{0.25}},$$

by Bowley (1920) and Moors (1988) respectively. These descriptive measures, based on quartiles and octiles, provide more robust estimates than the traditional skewness and kurtosis measures. Moreover, these measures are almost less reactive to outliers and work more effectively for the distributions, deficient in moments. The following Table-1, presents some results of the first four moments about the origin, variance, skewness, and kurtosis of MEK distribution for some choices of parameters place in S-I($\alpha = 1.07, \beta = 5, \gamma = 1.1$), S-II($\alpha = 1.1, \beta = 5, \gamma = 1.07$), S-III($\alpha = 1.09, \beta = 5, \gamma = 1.1$), S-IV($\alpha = 1.1, \beta = 5, \gamma = 1.09$), S-V($\alpha = 1.1, \beta = 5, \gamma = 1.1$), S-VI($\alpha = 1.1, \beta = 1.1, \gamma = 5$), S-VII($\alpha = 1.1, \beta = 1.2, \gamma = 5$), S-VIII($\alpha = 1.1, \beta = 1.3, \gamma = 5$), S-IX($\alpha = 1.01, \beta = 5, \gamma = 1.3$), and S-X($\alpha = 1.02, \beta = 5, \gamma = 1.4$). The behavior of variance, skewness, and kurtosis has decreasing trend as per the results indicate in Table-1.

Table 1. Some results of moments, variance, skewness, and kurtosis

$\boldsymbol{\mu'_s}$	**S-I**	**S-II**	**S-III**	**S-IV**	**S-V**
μ'_1	2.2094	2.1831	2.1857	2.1772	2.1743
μ'_2	4.4551	4.3292	4.3346	4.2921	4.2741
μ'_3	9.8507	9.4252	9.4155	9.2805	9.2114
μ'_4	23.176	21.849	21.709	21.294	21.033
Variance	0.8224	0.7061	0.6777	0.6404	0.6082
Skewness	1.0973	1.0949	1.0885	1.0893	1.0867
Kurtosis	1.1677	1.1658	1.1554	1.1558	1.1514
$\boldsymbol{\mu'_s}$	**S-VI**	**S-VII**	**S-VIII**	**S-IX**	**S-X**
μ'_1	2.1743	2.1487	2.1282	2.2315	2.1974
μ'_2	4.2741	4.1162	3.9881	4.5144	4.3161
μ'_3	9.2113	8.6180	8.1575	9.9463	9.2237
μ'_4	21.033	18.903	17.364	22.952	20.505
Variance	0.6081	0.3139	0.0613	0.6987	0.3811
Skewness	1.0867	1.0649	1.0491	1.0752	1.0581
Kurtosis	1.1514	1.1157	1.0917	1.1262	1.1007

2.5 Reliability Characteristics

One of the imperative roles of probability distribution in reliability engineering is to analyze and predicts the life of a component. Numerous reliability measures for the MEK distribution are discussed here. One may explain the reliability function as the probability of a component that survives till the time *x* and analytically it is written as$R(x) = 1 - F(x)$.

Reliability function of *X* is given by

$$R(x) = 1 - \frac{2}{1 + (1 - (1 - x^{\alpha})^{\beta})^{-\gamma}}. \tag{9}$$

In reliability theory, a significant contribution of a function, most of the time considers as a failure rate function or hazard rate function, and sometimes it is called the force of mortality. Time depended this function is used to measure the failure rate of a component in a particular period *x* and mathematically it is written as $h(x) = f(x)/R(x)$.

Hazard rate function of *X* is given by

$$h(x) = \frac{2\alpha\beta\gamma x^{\alpha-1}(1 - x^{\alpha})^{\beta-1}\left(1 - (1 - x^{\alpha})^{\beta}\right)^{-\gamma-1}}{\left((1 + (1 - (1 - x^{\alpha})^{\beta})^{-\gamma})\right)(1 + (1 - (1 - x^{\alpha})^{\beta})^{-\gamma} - 2)}, x > 0. \tag{10}$$

The conditional survivor function is the probability that a component whose life says *x*, survives in an additional interval at *z*. It can be written as $R(^{z}/_{x}) = P(X > z + x/X > t) = \frac{R(X > z+x)}{P(X > x)} = \frac{R(x+z)}{R(x)}$.

Conditional survivor function of *X* is given by

$$R(^{z}/_{x}) = \frac{\left(\left(1 - (1 - (x+z)^{\alpha})^{\beta}\right)^{-\gamma} - 1\right)\left(1 + \left(1 - (1 - x^{\alpha})^{\beta}\right)^{-\gamma}\right)}{(1 + (1 - (1 - (x+z)^{\alpha})^{\beta})^{-\gamma})(1 + (1 - (1 - x^{\alpha})^{\beta})^{-\gamma} - 1)}.$$

Most of the time, it is assumed that the mechanical components/parts of some systems follow the bathtub-shaped failure rate phenomena. For this, several well-established and useful reliability measures are available in the literature to discuss the significance of EM distribution. the cumulative hazard rate function is expressed by $h_c(x) = -\log\left(R(x)\right)$.

Cumulative hazard rate function of *X* is given by

$$h_c(x) = -\,log\left(1-\frac{2}{1+(1-(1-x^{\alpha})^{\beta})^{-\gamma}}\right)$$

The reverse hazard rate function is expressed by $h_r(x) = f(x)/R(x)$.

Reverse hazard rate function of *X* is given by

$$\frac{2\alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma-1}\left(1+\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma}\right)}{(1+(1-(1-x^{\alpha})^{\beta})^{-\gamma})^{2}((1-(1-x^{\alpha})^{\beta})^{-\gamma}-1)}.$$

Mills ratio is expressed by $M(x) = R(x)/f(x)$.

Mills ratio of *X* is given by

$$\frac{\left(1+\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma}\right)^{2}\left(\left(1-(1-x^{\alpha})^{\beta}\right)^{-\gamma}-1\right)}{2\alpha\beta\gamma x^{\alpha-1}(1-x^{\alpha})^{\beta-1}(1-(1-x^{\alpha})^{\beta})^{-\gamma-1}(1+(1-(1-x^{\alpha})^{\beta})^{-\gamma})}.$$

Odd function is expressed by $O(x) = F(x)/R(x)$.

Odd function of *X* is given by

$$O(x) = \frac{2}{((1-(1-x^{\alpha})^{\beta})^{-\gamma}-1)}$$

We may develop the linear expressions for reliability characteristics, mention in section 1.2. The reliability and hazard rate functions of *X* are given by

$$R^{*}(x) = 1-2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{-1}{i}\binom{-\gamma i}{j}\binom{\beta j}{k}x^{\alpha k},$$

and

$$h^{*}(x) = \frac{2\alpha\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}\binom{\beta j+\beta-1}{k}x^{\alpha k+\alpha-1}}{1-2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{-1}{i}\binom{-\gamma i}{j}\binom{\beta j}{k}x^{\alpha k}}$$

2.6 Limiting Behavior

Here we study the limiting behavior of distribution function, density function, reliability function, and failure rate function of the MEK distribution present in Equations (3), (4), (9), and (10) at $x\rightarrow 0$ and $x\rightarrow 1$.

Proposition-1

Limiting behavior of distribution function, density function, reliability function, and failure rate function of the MEK distribution at $x\rightarrow 0$ *is followed by*

$$F(x)\sim 0,$$
$$f(x)\sim 0,$$
$$R(x)\sim 1,$$
$$h(x)\sim 0.$$

Proposition-2

Limiting behavior of distribution function, density function, reliability function, and failure rate function of the MEK distribution at $x\rightarrow 1$ *is followed by*

$$F(x)\sim 1,$$
$$f(x)\sim 0,$$
$$R(x)\sim 0,$$
$$h(x)\sim\text{Indeterminate}.$$

The above limiting behaviors of distribution, density, reliability, and failure rate functions illustrate that there is no effect of parameters on the tail of the MEK distribution.

2.7 Moments and Its Associated Measures

Moments have a remarkable role in the discussion of distribution theory, to study the significant characteristics of a probability distribution.

Theorem 1: If X ~MEK $(x;\alpha,\beta,\gamma)$, for $\alpha,\beta,\gamma>0$, then the *r*-th ordinary moment (say $\mu_r^{/}$) of X is given by

$$\mu_r^{/}=2\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}B\left(\frac{r}{\alpha}+1,\beta(j+1)\right)$$

Proof: $\mu_r^{/}$ can be written by following Equation (6), as

$$\mu_r^{/}=2\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}\int_0^1 x^r\left(\alpha x^{\alpha-1}(1-x^{\alpha})^{\beta j+\beta-1}\right)dx,$$

by simple computation on the prior expression leads to the final form of the *r*-th ordinary moment and it is given by

$$\mu_r^{/}=2\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}B\left(\frac{r}{\alpha}+1,\beta(j+1)\right), \tag{11}$$

where $B(x;\alpha,\beta)$=$\int_0^x t^{\alpha}(1-t)^{\beta-1}dt$ and $\alpha,\beta,\gamma>0$ are the beta function and shape parameters, control the tail behavior of *X*, respectively.

The derived expression in Equation (11) provides a supportive and useful role in the development of numerous statistics. For instance: to deduce the mean of *X,* place *r* =1 in Equation (11) and it is given by

$$\mu_1^{/}=2\beta\gamma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}B\left(\frac{1}{\alpha}+1,\beta(j+1)\right).$$

The higher-order ordinary moments of *X* approximating to 2nd, 3rd, and 4th, can be formulated by setting *r* = 2, 3, and 4 in Equation (11) respectively. Further to discuss the variability in *X,* the Fisher index F.I = $(Var(X)/E(X))$ plays a supportive role. One may perhaps further determine the well-established statistics for instance: skewness $(\beta_1=\mu_3^2/\mu_2^3)$, kurtosis $(\beta_2=\mu_4/\mu_2^2)$, and mode = $\left(\sqrt{\beta_1}(\beta_2+3)\mathrm{SD}/\left(2(5\beta_2-6\beta_1-9)\right)\right)$ of *X* by integrating Equation (11).

Moment generating function $M_X(t)$ can be presented by

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r^{/}.$$

Moment generating function of *X* is followed by equation (9)

$$M_X(t)=2\beta\gamma\sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}B\left(\frac{r}{\alpha}+1,\beta(j+1)\right).$$

A well-established recurrence relationship between the ordinary moments $(\mu_r^{/})$ and central moments (μ_s) to derive the cumulants is $\mu_s=\sum_{k=0}^{s}\binom{s}{k}(-1)^k\left(\mu_1^{/}\right)^s\mu_{s-k}^{/}$. Hence, the first four cumulants are: $K_1=\mu_1^{/}$, $K_2=\mu_2^{/}-\mu_1^{/2}$, $K_3=\mu_3^{/}-3\mu_2^{/}\mu_1^{/}+2\mu_1^{/3}$, and $K_4=\mu_4^{/}-4\mu_3^{/}\mu_1^{/}-3\mu_2^{/2}+12\mu_2^{/}\mu_1^{/2}-6\mu_1^{/4}$.

The *s*-th central moment (μ_s) of *X* is given by

$$\mu_s = \sum_{k=0}^{s} \binom{s}{k} (-1)^k \left(\frac{\left(2\beta\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j \binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}}{B\left(\frac{1}{\alpha}+1, \beta(j+1)\right)} \right)^s}{2\beta\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j \binom{-2}{i}\binom{-\gamma i-\gamma-1}{j}}{B\left(\frac{s-k}{\alpha}+1, \beta(j+1)\right)}} \right).$$

2.8 Incomplete Moments

Incomplete moments are classified into lower incomplete moments and upper incomplete moments. Lower incomplete moments are defined as $M_r(v) = E_{X\le v}(x^r) = \int_0^l x^r f(x)dx$.

Lower incomplete moments of X is given by

$$M_r(v) = 2\beta\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (-1)^j \binom{-2}{i}\binom{-\gamma i-\gamma-1}{j} B_l\left(\frac{r}{\alpha}+1, \beta(j+1)\right).$$

Upper incomplete moments are defined as $M_s^*(u) = E_{X>u}(x^r) = \int_l^1 x^r f(x)dx$ or more convenient, it can be written as $M_s^*(u) = \int_0^1 x^r f(x)dx - \int_0^l x^r f(x)dx$.

Upper incomplete moments of X is given by

$$M_s^*(u) = 2\beta\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (-1)^j \binom{-2}{i}\binom{-\gamma i-\gamma-1}{j} \begin{pmatrix} B\left(\frac{r}{\alpha}+1, \beta(j+1)\right) - \\ B_l\left(\frac{r}{\alpha}+1, \beta(j+1)\right) \end{pmatrix}.$$

Let be the residual life (RL) function $m_n(w) = E[(X-w)^n / X \le w] = \frac{1}{S(w)}\int_w^1 (x-w)^n f(x)dx$ has the n-th moment

$m_n(w) = \frac{1}{1-F(w)} \sum_{r=0}^{n} \binom{n}{r} (-w)^{n-r} \left(\int_0^1 x^r f(x)dx - \int_0^w x^r f(x)dx\right)$.

Residual life function X is given by

$$m_n(w) = \frac{2\beta\gamma \sum_{r=0}^{n}\binom{n}{r}(-w)^{n-r}}{1-F(w)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j\binom{-2}{i}}{\binom{-\gamma i-\gamma-1}{j}} \begin{pmatrix} B\left(\frac{r}{\alpha}+1, \beta(j+1)\right) - \\ B_w\left(\frac{r}{\alpha}+1, \beta(j+1)\right) \end{pmatrix}.$$

The life expectancy or mean residual life (MRL) function, $m_1(w)$, *of X, follows from the above equation with n = 1.*

Let be the reverse residual life (RRL) function $R_n(w) = E[(w-X)^n / X \le w] = \frac{1}{F(w)}\int_0^1 (w-x)^n f(x)dx$ has the n-th moment. $R_n(w) = \frac{1}{F(w)} \sum_{r=0}^{n} \binom{n}{r} (-1)^r w^{n-r} \int_0^1 x^r f(x)dx$.

Reverse residual life (RRL) function of X is given by

$$R_n(w) = \frac{2\beta\gamma}{F(w)} \left(\sum_{r=0}^{n} \binom{n}{r} (-1)^r w^{n-r} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j\binom{-2}{i}}{\binom{-\gamma i-\gamma-1}{j}} \right) B_w\left(\frac{r}{\alpha}+1, \beta(j+1)\right).$$

The mean waiting time or mean inactivity time of X, follows from the above Equation with n = 1.

Kayid and Izadkhah (2014) defined, strong mean inactivity time (SMIT). It can be written as $M(t) = t^2 - \frac{1}{f(t)}\int_0^t x^2 f(x)dx$ for g , $t > 0$.

Strong mean inactivity time of X is given by

$$M(t) = t^2 - \frac{\left(\left(1+\left(1-(1-x^\alpha)^\beta\right)^{-\gamma}\right)^2\right)\left(2\beta\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \begin{matrix} (-1)^j \binom{-2}{i}\binom{-\gamma i-\gamma-1}{j} \\ B_t\left(\frac{2}{\alpha}+1, \beta(j+1)\right)\end{matrix}\right)}{2\alpha\beta\gamma x^{\alpha-1}(1-x^\alpha)^{\beta-1}(1-(1-x^\alpha)^\beta)^{-\gamma-1}}.$$

Mean past lifetime (MPL) for the conditional random variable $(x - X/X \leq x)$ is given by $k(x) = E(x - X/X \leq x)$. It can be written as $k(x) = x - \frac{\int_0^t xf(x)dx}{F(x)}$.

Mean past life time of X is given by

$$k(x) = x - \frac{1}{2}\left(1+\left(1-(1-x^\alpha)^\beta\right)^{-\gamma}\right)\left(2\beta\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \begin{matrix} (-1)^j \binom{-2}{i}\binom{-\gamma i-\gamma-1}{j} \\ B_t\left(\frac{1}{\alpha}+1, \beta(j+1)\right)\end{matrix}\right).$$

2.9 Order Statistics

In reliability analysis and life testing of a component in quality control, order statistics (OS) and moments have noteworthy consideration. Let $X_1, X_2, \dots, X_n$ be a random sample of size *n* follows to the MEK distribution and $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ be the corresponding order statistics. The random variables$X_{(i)}$, $X_{(1)}$, and $X_{(n)}$ be the *i*th, minimum, and maximum order statistics of X.

The PDF of $X_{(i)}$is given by

$$f_{(i)}(x) = \frac{1}{B(i, n-i+1)!}\left(F(x)\right)^{i-1}\left(1-F(x)\right)^{n-i} f(x), \qquad i = 1,2,3, \dots, n.$$

By incorporating Equations (3) and (4), the PDF of $X_{(i)}$takes the form

$$f_{(i:n)}(x) = \frac{\alpha\beta\gamma}{B(i, n-i+1)!}\left(\begin{matrix} \left(\frac{2}{1+(1-(1-x^\alpha)^\beta)^{-\gamma}}\right)^{i-1} \\ \left(1-\frac{2}{1+(1-(1-x^\alpha)^\beta)^{-\gamma}}\right)^{n-i} \\ \left(\frac{2\alpha\beta\gamma x^{\alpha-1}(1-x^\alpha)^{\beta-1}\left(1-(1-x^\alpha)^\beta\right)^{-\gamma-1}}{(1+(1-(1-x^\alpha)^\beta)^{-\gamma})^2}\right)\end{matrix}\right).$$

The last equation is quite helpful in computing the *w*-th moment order statistics of the MEK distribution. Further, the minimum and maximum order statistics of X follow directly from the above equation with *i*=1 and *i*= *n*, respectively.

The *w*-th moment order statistics, $E(X_{OS}^w)$, of *X* is

$$E(X_{OS}^w) = \frac{2\alpha\beta\gamma}{B(i, n-i+1)!}, \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{j+l}(2)^\alpha \binom{n-i}{j}\binom{\alpha}{k}\binom{\beta}{l} B\left(\frac{r}{\alpha}+1, \beta l+1\right). \tag{12}$$

2.10 Entropy

When a system is quantified by disorderedness, randomness, diversity, or uncertainty, in general, it is known as entropy. Rényi (1961) entropy of *X* is described by

$$H_\zeta(X) = \frac{1}{1-\zeta} log \int_0^1 f^\zeta(x)dx, \qquad \zeta > 0 \; and \; \zeta \neq 1. \tag{13}$$

First, we simplify $f(x)$ in terms of $f^\zeta(x)$, we get

$$f^\zeta(x) = (2\alpha\beta\gamma)^\zeta x^{\zeta(\alpha-1)}(1-x^\alpha)^{\zeta(\beta-1)}\left(1-(1-x^\alpha)^\beta\right)^{-\zeta(\gamma+1)}\left(1+\left(1-(1-x^\alpha)^\beta\right)^{-\gamma}\right)^{2\zeta}.$$

$$f^{\zeta}(x) = (2\alpha\beta\gamma)^{\zeta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j}\binom{-2\zeta}{i}\binom{-\lambda}{j}x^{\zeta(\alpha-1)}(1-x^{\alpha})^{\beta j+\zeta(\beta-1)},$$

by shifting the above equation in Equation (13), we get

$$H_{\zeta}(X) = \frac{1}{1-\zeta}log\left((2\alpha\beta\gamma)^{\zeta}\int_{0}^{1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{j}\binom{-2\zeta}{i}\binom{-\lambda}{j}x^{\zeta(\alpha-1)}(1-x^{\alpha})^{\beta j+\zeta(\beta-1)}dx\right),$$

by solving simple mathematics on the prior equation we will be provided the reduced form of the Rényi entropy for X and it is given by

$$H_{\zeta}(X) = \frac{1}{1-\zeta}log\big((2\alpha\beta\gamma)^{\zeta}\big)\frac{1}{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tau_{i,j}\,B\left(\frac{\zeta(\alpha-1)+1}{\alpha},\beta j+\zeta(\beta-1)+1\right), \tag{14}$$

where $\lambda = \zeta(\gamma+1)+i\gamma, \tau_{i,j} = (-1)^{j}\binom{-2\zeta}{i}\binom{-\lambda}{j}$.

The quadratic entropy is a special case of Rényi entropy, called quadratic Rényi entropy (QRE). It has a wide range of applications in economics, signal processing, and physics. It is obtained by substituting ζ by 2 in Equation (14).

A generalization of the Boltzmann-Gibbs entropy is the η – entropy. Although in physics, it is referred to as the Tsallis entropy. Tsallis (1988) entropy / η – entropy is described by

η – entropy is described by

$$H_{\eta}(X) = \frac{1}{\eta-1}\left(1-\int_{0}^{1}f^{\eta-1}(x)dx\right), \qquad \eta > 0\ and\ \eta \neq 1.$$

η – entropy of X is given by

$$H_{\eta}(X) = \frac{1}{\eta-1}\left(\begin{array}{c}1-((2\alpha\beta\gamma)^{\eta-1})\frac{1}{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tau_{i,j}\\ B\left(\frac{(\eta-1)(\alpha-1)+1}{\alpha},\beta j+(\eta-1)(\beta-1)+1\right)\end{array}\right). \tag{15}$$

Mathai and Haubold (2013) generalized the classical Shannon entropy known as $\phi-$ entropy. It is presented by

$$H_{\phi}(X) = \frac{1}{\phi-1}\left(\int_{0}^{1}f^{2-\phi}(x)dx-1\right), \qquad \phi > 0\ and\ \phi \neq 1.$$

$\phi-$ entropy of X is given by

$$H_{\phi}(X) = \frac{1}{\phi-1}\left(\left(\begin{array}{c}\big((2\alpha\beta\gamma)^{\phi}\big)\frac{1}{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tau_{i,j}\\ B\left(\frac{(2-\phi)(\alpha-1)+1}{\alpha},\beta j+(2-\phi)(\beta-1)+1\right)\end{array}\right)-1\right). \tag{16}$$

Another generalized version of the Shannon entropy is the $\bar{\varphi}$ - entropy. It is presented by

$$H_{\bar{\varphi}}(X) = \frac{1}{\bar{\varphi}-1}\left(1-\int_{0}^{1}f^{\bar{\varphi}}(x)dx\right), \qquad \bar{\varphi} \neq 1.$$

$\bar{\varphi}$ - entropy of X is given by

$$H_{\bar{\varphi}}(X)=\frac{1}{\bar{\varphi}-1}\left(1-\left(((2\alpha\beta\gamma)^{\bar{\varphi}})\frac{1}{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tau_{i,j}\,B\left(\frac{\bar{\varphi}(\alpha-1)+1}{\alpha},\beta j+\bar{\varphi}(\beta-1)+1\right)\right)\right). \quad (17)$$

Havrda and Charvat (1967) introduced $\omega -$ entropy measure. It is presented by

$$H_{\omega}(X)=\frac{1}{2^{1-\omega}-1}\left(\int_0^1 f^{\omega}(x)dx-1\right), \qquad \omega>0 \; and \; \omega\neq 1.$$

$\omega -$ entropy of X is given by

$$H_{\omega}(X)=\frac{1}{2^{1-\omega}-1}\left(\left(((2\alpha\beta\gamma)^{\omega})\frac{1}{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\tau_{i,j}\,B\left(\frac{\omega(\alpha-1)+1}{\alpha},\beta j+\omega(\beta-1)+1\right)\right)-1\right). \quad (18)$$

where $\lambda=\zeta(\gamma+1)+i\gamma,\tau_{i,j}=(-1)^j\binom{-2\zeta}{i}\binom{-\lambda}{j}$.

3. Estimation

In this section, we utilize the method of maximum likelihood estimation which provides the maximum information about the unknown model parameters.

By Equation (4), the likelihood function, $L(\vartheta)=\prod_{i=1}^{n}f(x_i;\alpha,\beta,\gamma)$,of the MEK distribution is:

$$L\,(\vartheta)=(2\alpha\beta\gamma)^n\prod_{i=1}^{n}\frac{x_i^{\alpha-1}(1-x_i{}^{\alpha})^{\beta-1}\left(1-(1-x_i{}^{\alpha})^{\beta}\right)^{-\gamma-1}}{(1+(1-(1-x_i{}^{\alpha})^{\beta})^{-\gamma})^2}.$$

The log-likelihood function,$l(\vartheta)$, reduces to

$$l(\vartheta)=\left(\begin{array}{c} n(log2+log\alpha+log\beta+log\gamma)+(\alpha-1)\sum_{i=1}^{n}log(x_i)+(\beta-1)\sum_{i=1}^{n}log(1-x_i{}^{\alpha})- \\ (\gamma+1)\sum_{i=1}^{n}log\left(1-(1-x_i{}^{\alpha})^{\beta}\right)-2\sum_{i=1}^{n}log\left(1+\left(1-(1-x_i{}^{\alpha})^{\beta}\right)^{-\gamma}\right)\end{array}\right).$$

The maximum likelihood estimates (MLEs) of the MEK model parameters can be obtained by maximizing the last equation for α,β, and γ, or by solving the following nonlinear Equations,

$$\frac{\partial l}{\partial\alpha}=\left(\begin{array}{c}\frac{n}{\alpha}+\sum_{i=1}^{n}log(x_i)+\sum_{i=1}^{n}log(x_i)-(\beta-1)\sum_{i=1}^{n}\frac{x_i{}^{\alpha}logx_i}{1-x_i{}^{\alpha}}- \\ (\gamma-1)\sum_{i=1}^{n}\frac{\beta x_i{}^{\alpha}logx_i(1-x_i{}^{\alpha})^{\beta-1}}{1-(1-x_i{}^{\alpha})^{\beta}} \\ +2\sum_{i=1}^{n}\frac{\beta\gamma x_i{}^{\alpha}logx_i(1-x_i{}^{\alpha})^{\beta-1}\left(1-(1-x_i{}^{\alpha})^{\beta}\right)^{-\gamma-1}}{1+(1-(1-x_i{}^{\alpha})^{\beta})^{-\gamma}}\end{array}\right),$$

$$\frac{\partial l}{\partial\beta}=\left(\begin{array}{c}\frac{n}{\beta}+\sum_{i=1}^{n}log(1-x_i{}^{\alpha})-(1+\gamma)\sum_{i=1}^{n}\frac{(1-x_i{}^{\alpha})^{\beta}log(1-x_i{}^{\alpha})}{1-(1-x_i{}^{\alpha})^{\beta}} \\ +2\sum_{i=1}^{n}\frac{\gamma log(1-x_i{}^{\alpha})(1-x_i{}^{\alpha})^{\beta}\left(1-(1-x_i{}^{\alpha})^{\beta}\right)^{-\gamma-1}}{1+(1-(1-x_i{}^{\alpha})^{\beta})^{-\gamma}}\end{array}\right),$$

$$\frac{\partial l}{\partial \gamma}=\left(\begin{array}{c}\frac{n}{\gamma}-\sum_{i=1}^{n} log\left(1-(1-{x_i}^{\alpha})^{\beta}\right)+\\ 2\sum_{i=1}^{n}\frac{\left(\left(1-(1-{x_i}^{\alpha})^{\beta}\right)^{-\gamma}\right)log\left(1-(1-{x_i}^{\alpha})^{\beta}\right)}{1+(1-(1-{x_i}^{\alpha})^{\beta})^{-\gamma}}\end{array}\right).$$

The last three non-linear Equations do not provide the analytical solution for MLEs and the optimum value of α, β, and γ. The Newton-Raphson is considered an appropriate algorithm which plays a supportive role in such kind of MLEs. For numerical solutions, the R statistical software (package name, *Adequacy-Model*) is preferred to estimate the MEK distribution parameters.

3.1 Simulation Study

In this section, to observe the performance of MLE's, the following algorithm is adopted.

Step-1: A random sample $x_1, x_2, x_3, \ldots, x_n$ of sizes n = 25, 50, and 100 are generated from Equation (5).

Step-2: Each sample is replicated 1000 times.

Step-3: The required results are obtained based on the different combinations of the parameters place in S-XI, S-XII, and S-XIII.

Step-4: Gradual decrease in S.Es and pretty close ML estimates to the true parameters for the increases of sample size help out to declare that the method of maximum likelihood estimation works quite well for MEK distribution.

4. Application

In this section, we report the flexibility and potentiality of the MEK distribution by modeling in various disciplines of applied sciences. For this, we consider four suitable lifetime data sets. The ***first dataset*** presents the 20 observations of flood including 0.265, 0.269, 0.297, 0.315, 0.3235, 0.338, 0.379, 0.379, 0.392, 0.402, 0.412, 0.416, 0.418, 0.423, 0.449, 0.484, 0.494, 0.613, 0.654, 0.74, discussed by Dumonceaux and Antle (1973). The ***second dataset*** discussed by Caramanis *et al*. (1983) and Mazumdar and Gaver (1984). They estimated the unit capacity factors by comparing two different algorithms called SC16 and P3. The observations are 0.853, 0.759, 0.866, 0.809, 0.717, 0.544, 0.492, 0.403, 0.344, 0.213, 0.116, 0.116, 0.092, 0.070, 0.059, 0.048, 0.036, 0.029, 0.021, 0.014, 0.011, 0.008, 0.006. The ***third dataset*** refers to 20 mechanical parts failure times. This data set was analyzed by Murthy *et al*. (2004) and the observations are 0.067, 0.068, 0.076, 0.081, 0.084, 0.085, 0.085, 0.086, 0.089, 0.098, 0.098, 0.114, 0.114, 0.115, 0.121, 0.125, 0.131, 0.149, 0.160, 0.485 and finally the ***forth dataset*** refers to the measurement on 48 samples of petroleum rock obtained from petroleum reservoirs. This data was discussed by Cordeiro and Brito (2012) and the observations are: 0.0903296, 0.2036540, 0.2043140, 0.2808870, 0.1976530, 0.3286410, 0.1486220, 0.1623940, 0.2627270, 0.1794550, 0.3266350, 0.2300810, 0.1833120, 0.1509440, 0.2000710, 0.1918020, 0.1541920, 0.4641250, 0.1170630, 0.1481410, 0.1448100, 0.1330830, 0.2760160, 0.4204770, 0.1224170, 0.2285950, 0.1138520, 0.2252140, 0.1769690, 0.2007440, 0.1670450, 0.2316230, 0.2910290, 0.3412730, 0.4387120, 0.2626510, 0.1896510, 0.1725670, 0.2400770, 0.3116460, 0.1635860, 0.1824530, 0.1641270, 0.1534810, 0.1618650, 0.2760160, 0.2538320, 0.2004470.

Some descriptive statistics are presented in Table 3. The MEK distribution is compared with its competing models (mention in Table-4), based on some criteria called, -Log-likelihood (-LL), Bayesian information criterion (BIC), Cramer-Von Mises (W^*), Anderson-Darling (A^*), and Kolmogorov Smirnov (K-S) test statistics. Tables 5-8, confirm the parameter estimates and their standard errors (in parenthesis) and the goodness-of-fit criteria, respectively. The MEK distribution is a better fit among all competitors, based on the results in Tables 5-8. Further, fitted density and distribution functions, Kaplan-Meier survival, and Probability- Probability (PP) plots are presented in Figures 3-6, respectively, provide close fits to the four datasets.

Table 2. Average MLEs and Standard Errors (in parenthesis)

	S-XI ($\alpha = 0.1, \beta = 0.5, \gamma = 0.5$) Parameter estimate (Standard Error)			S-XII ($\alpha = 0.2, \beta = 0.5, \gamma = 0.7$) Parameter estimate (Standard Error)		
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$
25	0.1846 (0.2493)	0.4643 (0.2518)	0.4647 (0.5686)	0.2146 (0.5654)	0.6297 (0.3091)	0.8348 (1.9998)
50	0.0964 (0.1442)	0.5622 (0.2283)	0.5111 (0.7047)	0.1405 (0.2377)	0.5124 (0.1945)	0.8625 (1.2555)
100	0.0971 (0.0964)	0.5005 (0.1516)	0.4974 (0.4481)	0.1949 (0.2040)	0.5225 (0.1407)	0.6845 (0.6362)

	S-XIII ($\alpha = 1.1, \beta = 1.7, \gamma = 0.2$) Parameter estimate (Standard Error)		
n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$
25	1.4605 (0.0020)	1.4963 (0.8981)	0.1374 (0.0236)
50	1.2108 (0.0935)	1.5576 (0.1588)	0.1690 (0.0243)
100	1.1019 (0.0249)	1.6998 (0.0424)	0.2000 (0.0175)

Table 3. Descriptive Information

Data set	Minimum	Median	Mean	Maximum	Skewness	Kurtosis
Flood data	0.011	0.041	0.045	0.125	1.1672	4.324
Unit capacity data	0.006	0.116	0.288	0.866	0.718	1.974
Failure times data	0.067	0.098	0.121	0.485	3.585	15.203
Petroleum rock data	0.090	0.198	0.218	0.464	1.169	4.109

Table 4. Competitive Models

Abbr.	Model	Parameters/ variable Range	Reference
L-I	$G(x) = x^{\alpha}$	$\alpha > 0, 0 < x < 1$	Lehmann (1953)
L-II	$G(x) = 1 - (1 - x)^{\alpha}$	$\alpha > 0, 0 < x < 1$	Lehmann (1953)
TL	$G(x) = (2x - x^2)^{\alpha}$	$\alpha > 0, 0 < x < 1$	Topp and Leone (1955)
Kum	$G(x) = 1 - (1 - x^{\alpha})^{\beta}$	$\alpha, \beta > 0, 0 < x < 1$	Kumaraswamy (1980)
GPF	$G(x) = 1 - \left(\frac{g - x}{g - m}\right)^{\alpha}$	$\alpha > 0, m \leq x \leq g$	Saran and Pandey (2004)
EK	$G(x) = \left(1 - (1 - x^{\alpha})^{\beta}\right)^{\gamma}$	$\alpha, \beta, \gamma > 0, 0 < x < 1$	Lemonte *et al*. (2013)
WPF	$G(x) = 1 - e^{-\alpha\left(\frac{x^{\beta}}{g^{\beta} - x^{\beta}}\right)^{\gamma}}$	$\alpha, \beta, \gamma > 0, 0 < x \leq g$	Tahir *et al*. (2014)
KPF	$G(x) = 1 - \left(1 - \left(\frac{x}{g}\right)^{\alpha\beta}\right)^{\gamma}$	$\alpha, \beta, \gamma > 0, 0 < x \leq g$	Ibrahim (2017)
MT-II	$G(x) = e^{x^{\alpha} ln2} - 1$	$\alpha > 0, 0 < x < 1$	Muhammad (2017)

Topp-Leone (TL), Kumaraswamy (Kum), Lehmann -I and Lehmann-II (L-I, L-II), generalized power function (GPF), exponentiated Kumaraswamy (EK), Weibull power function (WPF), Kumaraswamy power function (KPF), and Mustapha Type-II (MT-II).

Table 5. Parameter Estimates and Standard Errors (parenthesis) for Flood data

Model	Parameters (Standard Errors)			Information Criterion				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	-LL	BIC	W*	A*	K-S
MEK	**0.766 (0.474)**	**4.537 (1.501)**	**25.346 (31.929)**	**-15.903**	**-22.820**	**0.059**	**0.369**	**0.142**
EK	0.684 (0.393)	5.002 (1.496)	35.178 (46.797)	-15.514	-22.041	0.074	0.454	0.161
K	3.363 (0.603)	11.792 (5.361)	-	-12.866	-19.741	0.166	0.972	0.211
TL	2.244 (0.502)	-	-	-7.367	-11.739	0.119	0.712	0.335
L-I	1.114 (0.249)	-	-	-0.112	2.771	0.122	0.731	0.394
L-II	1.727 (0.386)	-	-	-2.512	-2.027	0.128	0.764	0.413
MT-II	0.852 (0.211)	-	-	1.247	5.489	0.131	0.782	0.388
GPF	1.579 (0.353)	-	-	-16.277	-29.559	0.131	0.728	0.224
WPF	30.814 (16.071)	11.045 (20.466)	0.319 (0.590)	-13.264	-17.540	0.146	0.868	0.198
KPF	1.386 (173.04)	1.693 (211.35)	1.865 (0.572)	-9.884	-10.780	0.303	1.717	0.263

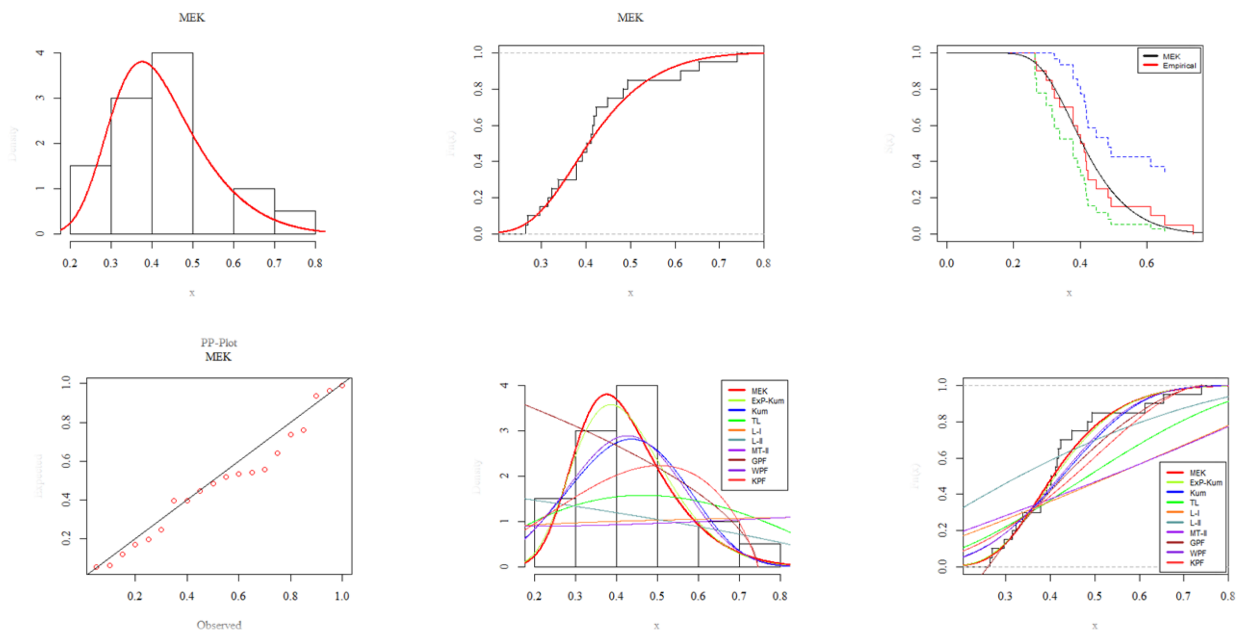

Figure 3. The Empirical Fitted PDF, CDF, Kaplan-Meier Survival, and PP-Plots of the MEK distribution for flood data

Table 6. Parameter Estimates and Standard Errors (parenthesis) for Unit Capacity Factors data

Model	Parameters (Standard Errors)			Information Criterion				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	-LL	BIC	W*	A*	K-S
MEK	**1.411 (9.021)**	**0.957 (0.436)**	**0.435 (2.767)**	**-10.151**	**-10.897**	**0.090**	**0.579**	**0.151**
EK	0.065 (0.117)	1.185 (0.235)	9.781 (20.034)	-9.849	-10.292	0.103	0.648	0.169
K	0.504 (0.129)	1.186 (0.326)	-	-9.671	-13.071	0.108	0.682	0.179
TL	0.594 (0.124)	-	-	-8.115	-13.095	0.119	0.746	0.169
L-I	0.454 (0.095)	-	-	-9.485	-15.833	0.107	0.675	0.189
L-II	1.989 (0.415)	-	-	-4.383	-5.630	0.112	0.703	0.347
MT-II	0.371 (0.086)	-	-	-8.921	-14.708	0.117	0.732	0.199
GPF	1.185 (0.247)	-	-	-3.516	-3.897	0.114	0.683	0.411
WPF	2.285 (1.167)	1.105 (0.679)	0.551 (0.244)	-9.234	-9.061	0.095	0.616	0.155
KPF	1.389 (72.029)	0.287 (14.865)	0.737 (0.187)	-11.752	-14.099	0.128	0.767	0.211

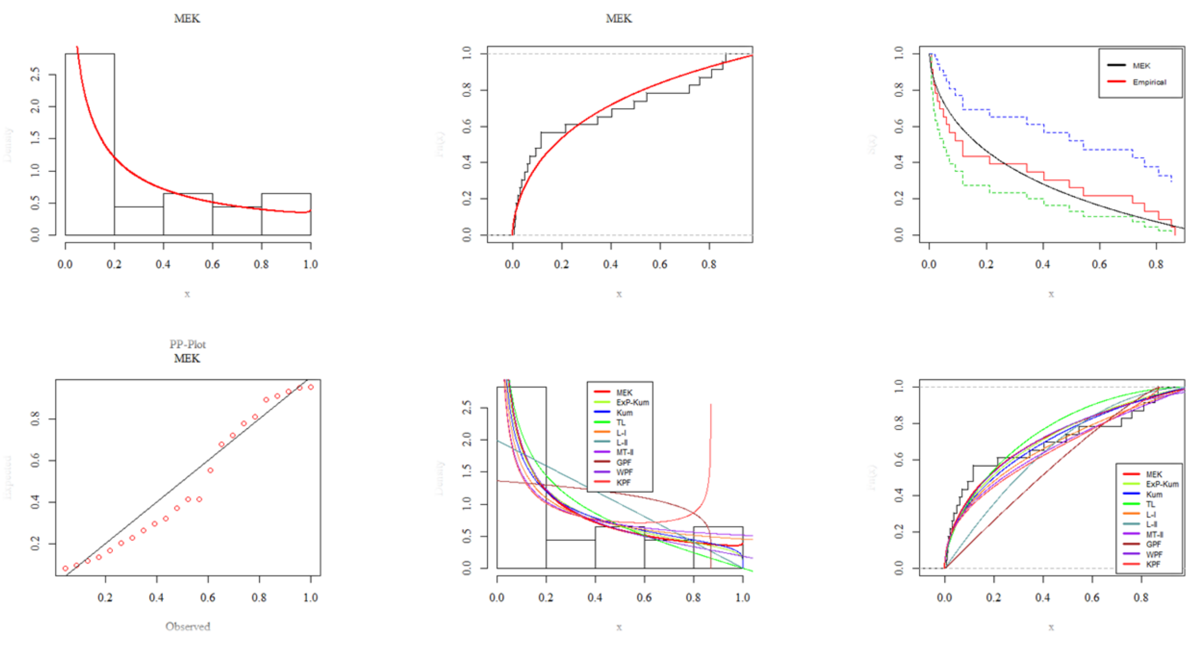

Figure 4. The Empirical Fitted PDF, CDF, Kaplan-Meier Survival, PP-Plots of the MEK distribution for unit capacity factors data

Table 7. Parameter Estimates and Standard Errors (parenthesis) for Failure Times data

Model	Parameters (Standard Errors)			Information Criterion				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	-LL	BIC	W*	A*	K-S
MEK	**0.568 (0.199)**	**10.893 (3.789)**	**40.053 (42.362)**	**-34.540**	**-60.094**	**0.143**	**1.053**	**0.166**
EK	0.517 (0.165)	11.897 (4.019)	63.739 (63.007)	-33.551	-58.115	0.172	1.232	0.168
K	1.587 (0.244)	21.868 (10.210)	-	-25.648	-60.094	0.143	1.053	0.166
TL	0.625 (0.139)	-	-	-13.742	-24.490	0.339	2.156	0.484
L-I	0.448 (0.100)	-	-	-8.558	-14.121	0.321	2.063	0.510
L-II	7.341 (1.641)	-	-	-22.593	-42.191	0.369	2.314	0.398
MT-II	0.340 (0.084)	-	-	-7.097	-11.197	0.339	2.153	0.500
GPF	3.135 (0.701)	-	-	-26.208	-50.417	0.416	2.501	0.426
WPF	25.321 (10.981)	8.698 (30.616)	0.189 (0.664)	-26.422	-43.857	0.397	2.452	0.264
KPF	1.053 (87.439)	0.959 (79.636)	2.224 (0.682)	-19.137	-29.286	0.762	4.159	0.370

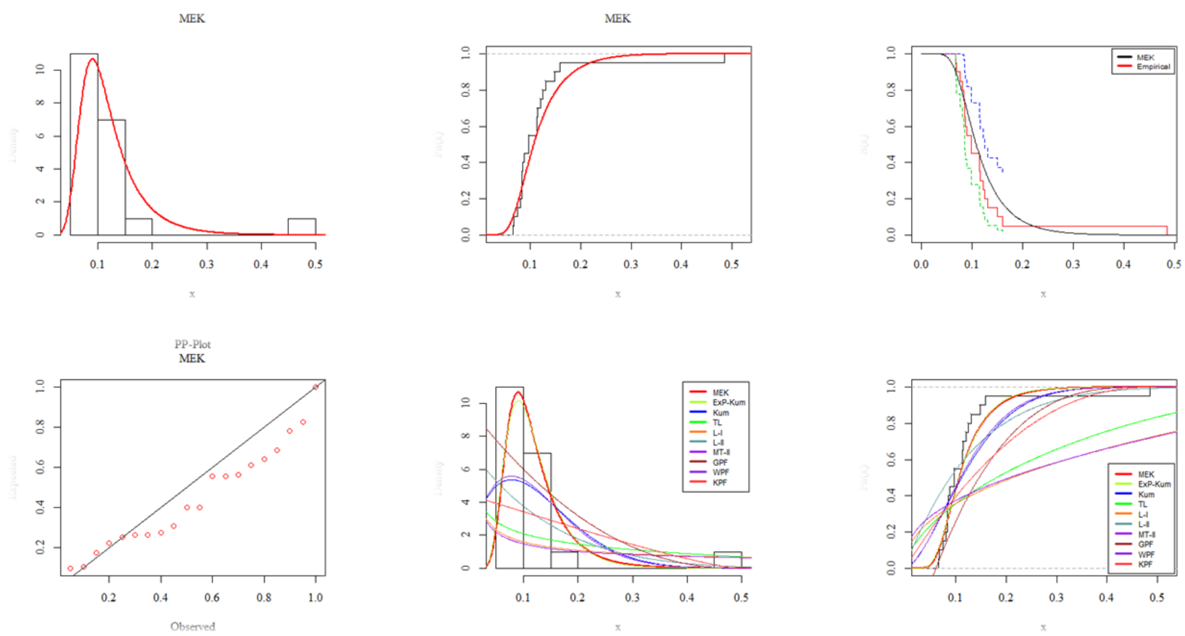

Figure 5. The Empirical Fitted PDF, CDF, Kaplan-Meier Survival, PP-Plots of the MEK distribution for failure times data

Table 8. Parameter Estimates and Standard Errors (parenthesis) for Petroleum Rock data

Model	Parameters (Standard Errors)			Information Criterion				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	-LL	BIC	W*	A*	K-S
MEK	**0.756 (0.450)**	**8.525 (3.693)**	**21.870 (31.023)**	**-58.371**	**-105.12**	**0.038**	**0.232**	**0.089**
EK	0.727 (0.271)	9.439 (2.784)	24.699 (24.178)	-57.859	-104.10	0.058	0.346	0.108
K	2.719 (0.294)	44.667 (17.587)	-	-52.491	-97.241	0.208	1.280	0.153
TL	0.989 (0.143)	-	-	-21.166	-38.461	0.119	0.721	0.368
L-I	0.630 (0.091)	-	-	-6.011	-8.152	0.114	0.690	0.429
L-II	3.965 (0.572)	-	-	-30.221	-56.569	0.128	0.778	0.359
MT-II	0.479 (0.077)	-	-	-25.54	-1.238	1.225	0.743	0.424
GPF	1.788 (0.258)	-	-	-52.703	-101.534	0.232	1.442	0.156
WPF	42.995 (15.791)	8.774 (28.625)	0.313 (1.021)	-52.741	-93.869	0.200	1.225	0.149
KPF	1.441 (90.546)	1.405 (88.274)	2.632 (0.555)	-46.042	-80.471	0.417	2.545	0.186

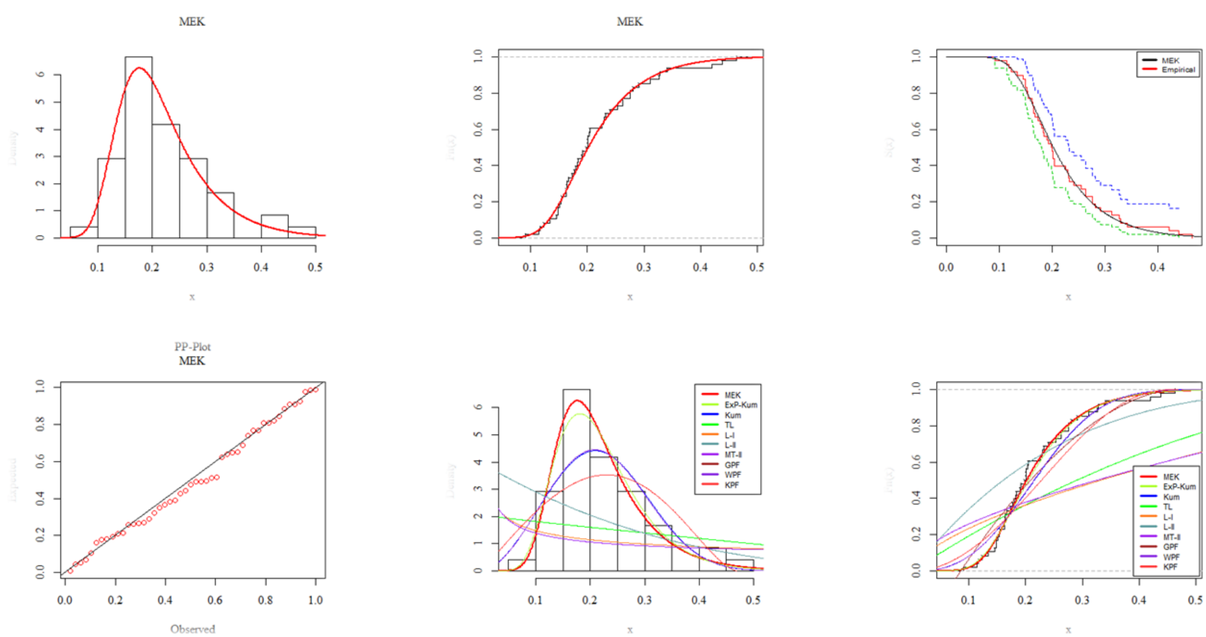

Figure 6. The Empirical Fitted PDF, CDF, Kaplan-Meier Survival, PP-Plots of the MEK distribution for petroleum rock data

5. Conclusion

In this article, we developed a flexible lifetime model that demonstrated the increasing, decreasing, and upside-down bathtub-shaped density and failure rate functions. The proposed model is referred to as the modified exponentiated Kumaraswamy (MEK) distribution. Numerous mathematical and reliability measures were derived and discussed. For estimation of the model parameters, we followed the method of maximum likelihood and executed a simulation study to observe the asymptotic behavior of MLEs. The MEK distribution explored its dominance by modeling in four-lifetime datasets and we hope it will be considered as a choice against the baseline model.

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