International Journal of Statistics and Probability; Vol. 8, No. 6; November 2019
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

Expected Winning Probabilities in Sequential Truels Under Uniform Distributions

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Received: October 20, 2019 Accepted: November 14, 2019 Online Published: November 18, 2019

doi:10.5539/ijsp.v8n6p82 URL: https://doi.org/10.5539/ijsp.v8n6p82

Abstract

In this paper we examine the expected probabilities of survival of each of the three participants in sequential truels assuming uniform distributions for their marksmanships. We start by discussing the two most common sequential truels, one in which everyone has to attempt to eliminate someone on each turn, and the other in which the weakest marksman has the option to abstain. We conjecture that the expected winning probabilities in sequential truels cannot be calculated analytically, and so we estimate these probabilities using multiple computational approaches. We also calculate the expected winning probabilities of each of the participants for one fixed marksmanship at a time. At the end we show that as the three marksmanships approach 0, the sequential truel approaches the simultaneous truel and this explains some aspects of the sequential truel for small marksmanships.

1. Introduction

Consider the following popular mathematics puzzle:

> Alice, Bob, and Carol arrange a three-way duel (which in our paper will be referred to as a truel). Alice is a poor shot, hitting her target only 1/3 of the time on average. Bob is better, hitting his target 2/3 of the time. Carol is a sure shot. They take turns shooting as follows:
>
> Alice goes first, followed by Bob, and then Carol, then back to Alice, and so on until only one of them is left. We assume that once someone hits their target, the target gets eliminated. What is Alice's best course of action?

The solution, surprisingly, is that Alice should shoot into the ground, that is, she should not even attempt to eliminate Bob or Carol. Many people will find this counterintuitive which makes it an interesting puzzle which appears in many books such as “A Connoisseur's Collection” (Winkler, 2004). This provides a motivation for many interesting questions, some of which have been addressed in previous works about truels.

A truel is essentially an extrapolation of the classical concept of a duel between two persons who take turns shooting at each other until one of them is eliminated, and then the two survivors then continue the process through a duel until only survivor is left. Although the idea of three persons fighting it out is probably a very old one, the term truel was possibly first used by Shubik (1975). Each person’s goal is to maximize their probability of being the lone survivor.

Different variations of truels have been examined by various people. For example, Random Truels, in which the next shooter is randomly selected (as opposed to everyone getting “equal” turns) are discussed by Amengual and Toral (2006). Simultaneous Truels, in which everyone fires at the same time, have been discussed by Kilgour (1971). Kilgour (1975) later discusses sequential truels in detail. A quite comprehensive overview of truels is provided by Kilgour and Brams (1997). According to this reference, the first mathematical problem/puzzle involving truels (without using the term) appeared in Encyclopedia of Puzzles and Pastimes (Kinnaird, 1946).

In the current study we shall be mainly discussing sequential truels, and in a later section we shall also consider some results about simultaneous truels that help us understand some aspect of the sequential truels. In a simultaneous truel, the three persons are shooting at the same time, while in a sequential truel they take turns at firing. The winner in each will be the last person standing. Note that we are guaranteed exactly one winner in the sequential truels that we shall be discussing in this paper, while in a simultaneous truel, it may happen that all the three persons are eliminated.

We begin by calculating the winning probabilities of each of the participants in the sequential truel under two main strategies as described below.

The first strategy we discuss is the “Pure Stronger-Opponent strategy” for which we require:

(a) Each person has a fixed probability of hitting their target. Thus, this probability is not a function of the number of attempts they have already made, and also not a function of the person they are targeting. Moreover, this fixed probability for each person is also known to their competitors.

(b) The persons fire sequentially in the reverse order of their marksmanship (that is, the weakest marksman shoots first and the strongest shoots last). This condition may be viewed as an attempt to remove some of the advantage that a superior marksman has and thus possibly make the game fairer. The process repeats until there is only one survivor. Obviously, once someone is eliminated, they are not a part of the sequence any more.

(c) Each person, in their turn, must try to eliminate one of their competitors. Thus, abstaining is not an option for anyone on any turn.

Since no one would like to be in a subsequent duel with the stronger of their two opponents, each will try to eliminate that opponent first. Thus, the resulting optimal strategies will be that the strongest marksman (call them A) shoots at the second strongest marksman (call them B), B shoots at A, and the weakest marksman (call them C) also shoots at A; hence, we will use the abbreviation BAA to represent the “Pure Stronger-Opponent strategy.”

The second strategy will be called the “Simplest Hybrid Stronger-Opponent strategy.” For this strategy, (a) and (b) remain as above while (c) is replaced by:

(c^*) Each of A and B, in their turn, must try to eliminate one of their competitors, while C may try to eliminate a competitor or may abstain. Note that if C does try to eliminate someone, it will be A, as previously explained. As in BAA, A and B will try to eliminate each other in order to get into a duel with C. Thus, the abbreviation BAØ will represent the “Simplest Hybrid Stronger-Opponent strategy.”

2. BAA Strategies

Let the probabilities of *A*, *B*, and *C* hitting their targets be denoted by *a*, *b*, and *c*, respectively. Then, since we are assuming *A* to be the strongest marksman, and *C* to be the weakest, we have $a \geq b \geq c$. We assume that once a person hits a target, the target person is eliminated. The goal of each truelist is to adopt a strategy that maximizes their chance of eventually being the only survivor. As each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing their own strategy as long as the others’ strategies do not change, the optimal strategy of each player implies a Nash Equilibrium. Kilgour (1977) has examined these equilibrium points under a variety of conditions.

3. Calculation of Survival Probabilities

The main results of this section have been derived by Toral and Amengual (2006) using Markov chains, but here we use infinite series to obtain them.

Let

$$\bar{a} = 1 - a,$$

$$\bar{b} = 1 - b,$$

$$\bar{c} = 1 - c.$$

Recalling that *C* shoots first, *B* second, and *A* third, we first derive the probabilities of all possible first eliminations.

P (A eliminates B) = P (A eliminates B on 1st turn) + P (A eliminates B on 2nd turn) + …

$$= \bar{c}\bar{b}a + \bar{c}\bar{b}\bar{a}\bar{c}\bar{b}a + \bar{c}\bar{b}\bar{a}\bar{c}\bar{b}\bar{a}\bar{c}\bar{b}a + \cdots = \frac{a\bar{b}\bar{c}}{1 - \bar{a}\bar{b}\bar{c}}$$

Similarly, we get:

$$P\ (B \text{ eliminates } A) = \frac{b\bar{c}}{1 - \bar{a}\bar{b}\bar{c}}$$

and

$$P\ (C \text{ eliminates } A) = \frac{b\bar{c}}{1 - \bar{a}\bar{b}\bar{c}}\ .$$

Next we derive the probabilities of each person winning a duel once someone has been eliminated. Defining an *XY*-duel as the duel between *X* and *Y*, in which *X* shoots first, we have

$$P\ (C \text{ wins the } CA \text{ duel}) = c + \bar{c}\bar{a}c + \bar{c}\bar{a}\bar{c}\bar{a}c + \cdots = \frac{c}{1-\bar{a}\bar{c}}.$$

Similarly, we see that

$$P\ (A \text{ wins the } CA \text{ duel}) = \frac{a\bar{c}}{1-\bar{a}\bar{c}}\ ,$$

$$P\ (C \text{ wins the } BC \text{ duel}) = \frac{\bar{b}c}{1-\bar{b}\bar{c}}\ ,$$

$$P\ (B \text{ wins the } BC \text{ duel}) = \frac{b}{1-\bar{b}\bar{c}}\ ,$$

$$P\ (C \text{ wins the } CB \text{ duel}) = \frac{c}{1-\bar{b}\bar{c}}$$

and

$$P\ (B \text{ wins the } CB \text{ duel}) = \frac{b\bar{c}}{1-\bar{b}\bar{c}}\ .$$

Note that since only *A* is firing at *B*, *A* and *C* can get in a duel only if *A* eliminates *B*; therefore, in the subsequent duel between *A* and *C*, *C* will always get the first turn. Thus, an AC-duel will never occur.

Now we utilize the above results to derive each person's probability of being the lone survivor.

$$PA := P\ (A \text{ wins}) = P\ (A \text{ eliminates } B)\ P(\ A \text{ wins the } CA \text{ duel}) = \frac{a^2\bar{b}\bar{c}^2}{(1-\bar{a}\bar{b}\bar{c})(1-\bar{a}\bar{c})}$$

P_B:= P (B wins) = P (B eliminates A)P(B wins the CB duel) + P (C eliminates A)P(B wins the BC duel)

$$= \frac{bc + b^2\bar{c}^2}{(1-\bar{a}\bar{b}\bar{c})(1-\bar{b}\bar{c})}$$

$$PC := P\ (C \text{ wins}) = 1 - \frac{a^2\bar{b}\bar{c}^2}{(1-\bar{a}\bar{b}\bar{c})(1-\bar{a}\bar{c})} - \frac{bc + b^2\bar{c}^2}{(1-\bar{a}\bar{b}\bar{c})(1-\bar{b}\bar{c})}$$

4. BAA Strategy Results

We attempt to answer the following question: given a random point (a,b,c) in the set

$$S = \{(x,y,z): 0 < z < y < x < 1\},$$

what are the expected values: $E(P_A)$, $E(P_B)$ and $E(P_C)$?

In order to calculate these, we need to make some assumption about the underlying distributions of *a*, *b*, and *c*. A natural choice would be that they follow the following uniform distributions:

$$a \sim U(0,1), \quad b \sim U(0,a), \quad c \sim U(0,b).$$

This choice would result in the following:

$$E(P_A) = \frac{1}{6}\int_0^1\int_0^a\int_0^b P_A\, dc\, db\, da$$

$$E(P_B) = \frac{1}{6}\int_0^1\int_0^a\int_0^b P_B\, dc\, db\, da$$

$$E(P_C) = \frac{1}{6}\int_0^1\int_0^a\int_0^b P_C\, dc\, db\, da$$

We attempted to evaluate each of these triple integrals using both Maple 10 and Mathematica. Each was able to provide exact results for the two inner integrals, but were unable to evaluate the triple integrals. Thus we conjecture that there is no way to evaluate these expected values exactly. We therefore resorted to two computational techniques: numerical integration and Monte-Carlo simulations. We performed numerical integration using the midpoint rule. For the Monte-Carlo method, we generated 11 billion random points in S; for each point we then calculated P_A, P_B, and P_C. From these we were able to estimate the expected values and variances of these probabilities. The results are summarized in Table 1 below:

Table 1. Numerical Expected Values for BAA

Expected Values	Numeric Integration Estimate	Monte-Carlo Estimate (95% Confidence Interval)
$E(P_A)$	0.260889	0.260890 ±0.000004
$E(P_B)$	0.479578	0.479578 ±0.000003
$E(P_C)$	0.259523	0.259532 ±0.000002

Notice that for each expected value, the estimates using the two methods differ by less than 0.00001.

It would have been interesting to view graphs of each of P_A, P_B, and P_C as a function of points in S. However, this would require four-dimensional plots. So instead, we provide plots of these probabilities versus each of a, b and c in Figures 1, 2, and 3, respectively. In Figure 1 every point in each of the three graphs corresponds to all points in S for a fixed value of a. Similarly, in Figures 2 and 3 every point in each of the three graphs corresponds to all points in S for fixed values of b and c, respectively.

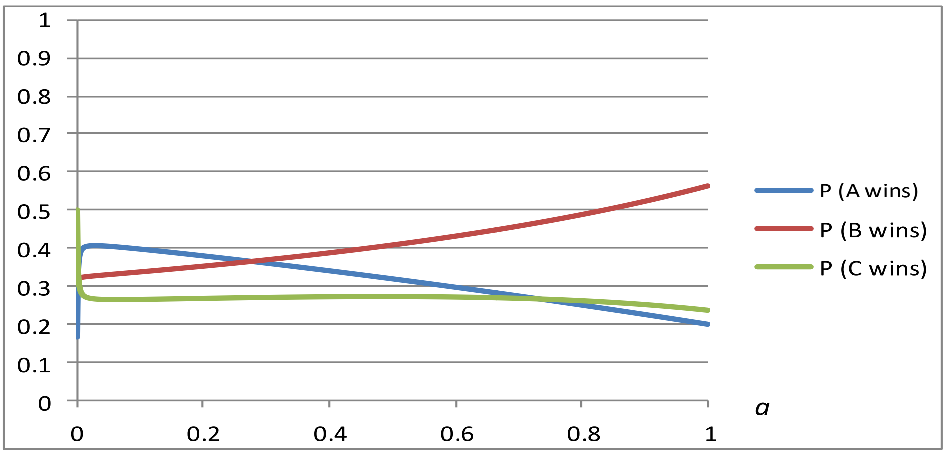

Figure 1. Winning Probabilities Under BAA Versus a

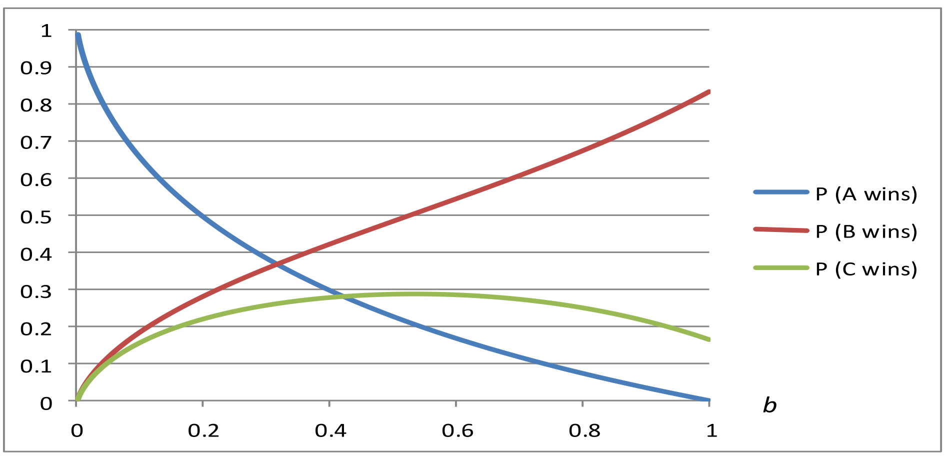

Figure 2. Winning Probabilities Under BAA Versus b

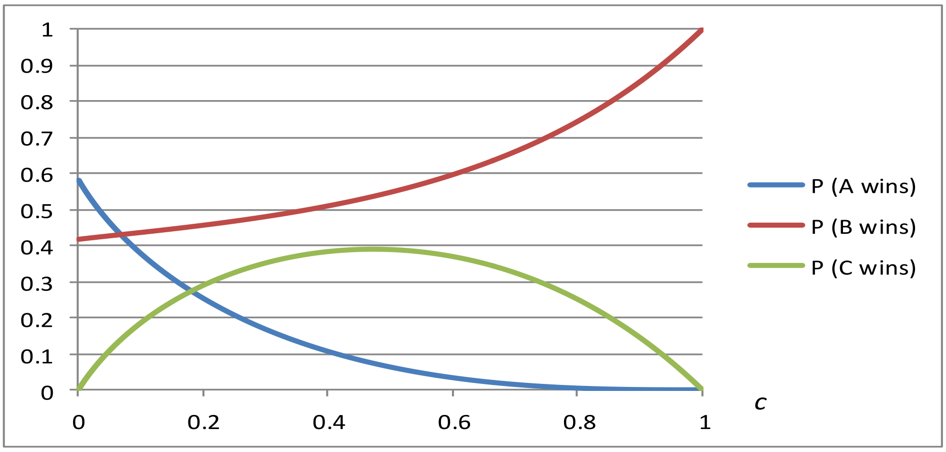

Figure 3. Winning Probabilities Under BAA Versus c

In Figure 1, we see what appears to be an initial rise in P_A, the peak occurring at about $a = 0.0285$, and then a decrease thereafter. It is not clear from the graph (or even from our data values) what $\lim_{a\to 0} P_A$ is or whether it even exists. We investigate this further in section 7. Also note that, perhaps somewhat counterintuitively, that as A's marksmanship increases, their probability of winning decreases, with B being the primary beneficiary.

We also observe that P_B decreases when $a < 0.0025$ (the range is too small to be captured on the graph, but the data bear this out)) and then increases afterwards as seen in the graph. As for P_C, it decreases for $a < 0.0635$, increases for $0.0635 < a < 0.2673$, and then decreases again thereafter. Also, the graph of P_C shows an initial instability similar to that shown by P_A. This, too, will be examined further in section 7.

Figure 2 shows that P_A is decreasing and P_B is increasing throughout, while P_C is increasing for $b < 0.533$ and decreasing thereafter.

Figure 3 shows a very similar trend in that P_A is decreasing and P_B is increasing throughout, while P_C is increasing for $c < 0.472$ and decreasing thereafter.

5. BAØ Strategy

Now we consider the case where C has the option of abstaining. Let $g = a^2\bar{b}^2\bar{c} - ab(1-bc) - b^2c$. It was shown by Kilgour (1975) that abstaining is the optimal strategy for C if $g < 0$ while C is better off firing at A if $g > 0$. There exist values of a, b, and c for which $g < 0, (ex. a = 1, b = 0.9, c = 0.1 \implies g = -0.891)$ and values for which $g > 0, (ex. a = 1, b = 0.2, c = 0.1 \implies g = 0.376)$. Therefore, if $g > 0$, the strategies are BAA, as given in section 4. If, however, $g < 0$, Amengual and Toral (2006) used Markov chains to derive the following:

Table 2. P_A, P_B, and P_C for BAØ with C Abstaining

Winner	A	B	C
Probability	$\frac{a^2\bar{b}\bar{c}}{(1-\bar{a}\bar{c})(1-\bar{a}\bar{b})}$	$\frac{b^2\bar{c}}{(1-\bar{b}\bar{c})(1-\bar{a}\bar{b})}$	$1-\frac{a^2\bar{b}\bar{c}}{(1-\bar{a}\bar{c})(1-\bar{a}\bar{b})}-\frac{b^2\bar{c}}{(1-\bar{b}\bar{c})(1-\bar{a}\bar{b})}$

As in section 3, we can also derive these results by taking the infinite series approach using the following table and results from section 3.

Table 3. Winning Sequences for each assuming C abstains

Winner	A	B	C
Sequences	(A eliminates B)(A wins CA duel)	(B eliminates A)(B wins CB duel)	(B eliminates A)(C wins CB duel) or (A eliminates B)(C wins CA duel)

6. BAØ Strategy Results

As in section 4, we were interested in the expected values $E(P_A)$, $E(P_B)$, and $E(P_C)$. Again we had to approximate these values. We repeated our work of section 4 except that we used 13 billion rather than 11 billion points. We obtained the following results:

Table 4. Numerical Expected Values for BAØ

Expected Values	Numeric Integration Estimate	Monte-Carlo Estimate (95% Confidence Interval)
$E(P_A)$	0.309389	0.309428 ±0.000003
$E(P_B)$	0.354378	0.354407 ±0.000003
$E(P_C)$	0.336233	0.336165 ±0.000004

Notice that even for this strategy, the expected values obtained from the two methods differ by less than 0.0001.

The graphs analogous to those in section 4 are given below:

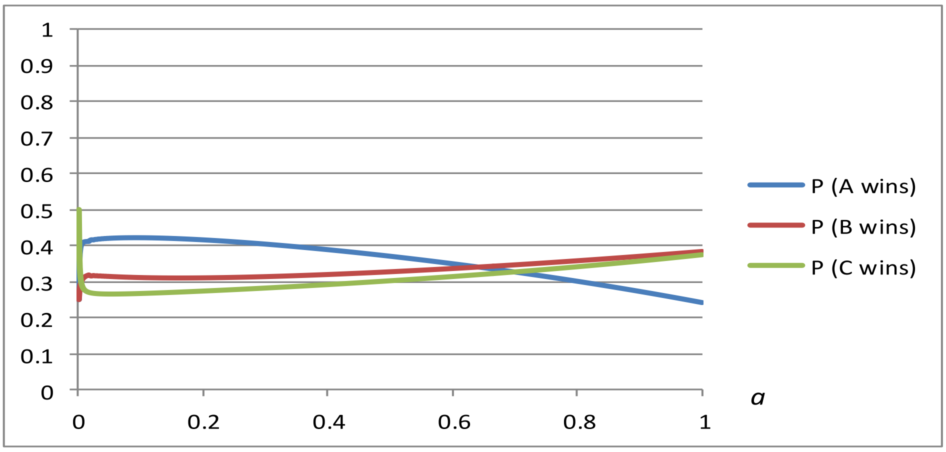

Figure 4. Winning Probabilities Under BAØ Versus *a*

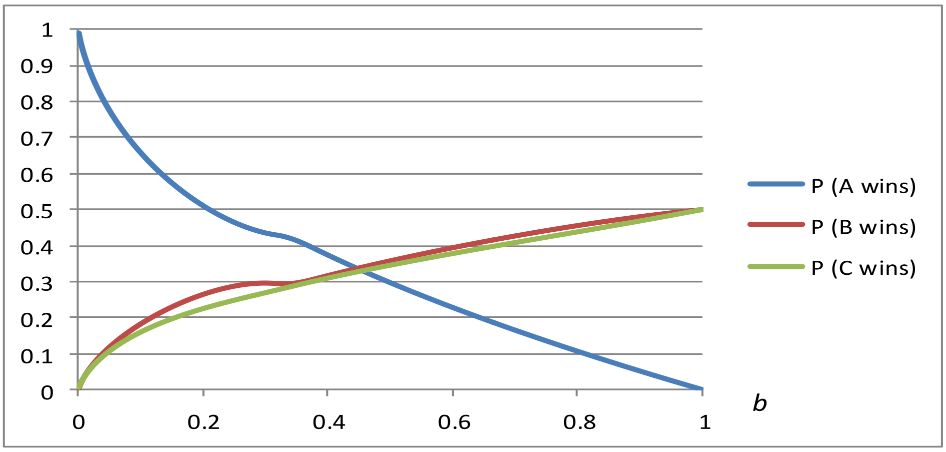

Figure 5. Winning Probabilities Under BAØ Versus b

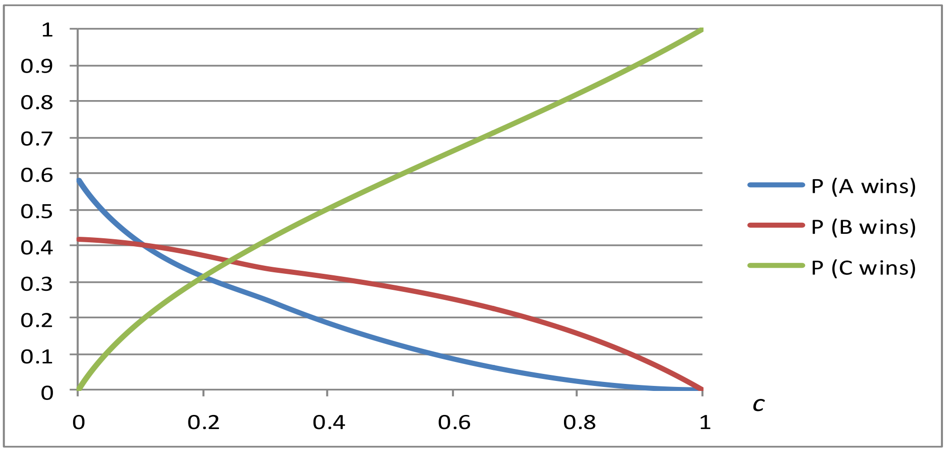

Figure 6. Winning Probabilities Under BAØ Versus c

In Figure 4, all the three winning probabilities have the kind of initial instability that was shown by P_A and P_C in Figure 1. Section 7, once again, addresses this issue further.

Note that P_A in Figure 4 is increasing for $a < 0.0915$ (as indicated by our data points) and decreases afterwards. Also, P_B increases initially, peaking at $a = 0.0175$, then decreases until $a = 0.1575$, and then decreases thereafter. Finally, P_C decreases initially, achieving a minimum at $a = 0.0525$, and then increases thereafter.

In Figures 5 and 6, in contrast to all the previous figures, all the winning probabilities appear to be montone, either increasing throughout, or decreasing throughout. However, our actual data values reveal that P_B in Figure 5 actually decreases slightly for $0.3005 < b < 0.3285$, but is otherwise monotone increasing as shown by the graph.

7. Graph Behavior as $a \to 0$

We first show that as $a \to 0$, the sequential and simultaneous truels become indistinguishable. To see this, let X be the random variable which gives the number of successful hits in a single round of simultaneous truel. We then need to prove that $\lim_{a\to 0} P(X = 1|X \geq 1) = 1$. This will follow if, given $\epsilon > 0$, $\exists\, \delta > 0 \ni |P(X = 1|X \geq 1) - 1| < \epsilon$ whenever $a < \delta$

(Note that $a < \delta \Rightarrow b, c < \delta$). Now

$$P(X = 1|X \geq 1) = \frac{a\bar{b}\,\bar{c} + \bar{a}\,b\,\bar{c} + \bar{a}\,\bar{b}\,c}{1 - \bar{a}\,\bar{b}\,\bar{c}}$$

And

$$\left|\frac{a\bar{b}\,\bar{c} + \bar{a}\,b\,\bar{c} + \bar{a}\,\bar{b}\,c}{1 - \bar{a}\,\bar{b}\,\bar{c}} - 1\right| = \frac{ab + c(a + b - 2ab)}{a + b - ab + (1-a)(1-b)}$$

$$< \frac{ab + b(a + b - 2ab)}{a + b - ab}$$

$$= \frac{b(2a + b - 2ab)}{a + b(1-a)}$$

$$< \frac{a(2a + a)}{a}$$

$$= 3a$$

$$< \epsilon$$

whenever

$$a < \frac{\epsilon}{3}.$$

Therefore it suffices to let

$$\delta = \frac{\epsilon}{3}.$$

Thus in the limit, a sequential truel can be treated as a simultaneous truel. One advantage of this for our purposes is that the formulas for $P_A, P_B,$ and P_C are simpler in case of simultaneous truel. Indeed, for simultaneous truel, letting x be the marksmanship ability of X, where $x = a, b, c$ for $X = A, B, C$ respectively, we have:

$$\text{P (X is the first successful shot in the truel)} = \frac{\text{x}}{\text{(a+b+c)}}$$

$$P\ (X \text{ is the first successful shot in a duel with } Y) = \frac{x}{(x+y)}$$

P_A = P (A eliminates B) P(A wins the CA duel)
This gives:

$$P_A = \frac{a^2}{(a+b+c)(a+c)}$$

Similarly, we see that

$$P_B = P\ (B \text{ eliminates } A) P(B \text{ wins the } CB \text{ duel}) + P\ (C \text{ eliminates } A) P(B \text{ wins the } BC \text{ duel}$$

$$= \frac{b^2}{(a+b+c)(b+c)} + \frac{cb}{(a+b+c)(b+c)}$$

$$= \frac{b}{a+b+c}$$

Finally,

$$P_C = 1 - P_A - P_B$$

$$= \frac{c}{a+b+c} + \frac{ac}{(a+b+c)(a+c)}$$

We were then able to obtain an exact form for one of the triple integrals expected values (namely, the one with P_B), something that was not possible in case of sequential expressions. The remaining two triple integrals were still not possible to evaluate exactly but we did obtain numerical values (using numerical integration in both Mathematica and Maple) which clarified the initial behaviors of the graphs that were not clear in Figures 1 and 4. That is, we found that

$$\lim_{a\to 0} P_A \approx 0.416$$

$$\lim_{a\to 0} P_B = \frac{\ln(27)-2}{4} \approx 0.324$$

$$\lim_{a\to 0} P_C \approx 0.260$$

Also note that for small values of marksmanships, the simplest-hybrid stronger-opponent strategy reduces to the stronger opponent strategy because C will not benefit by abstaining since they will have a better survival chance in a duel against B than in a duel against A, and not abstaining in this case increases the probability of A being the first one to be eliminated.

8. Discussion and Conclusions

To summarize, our focus on this paper was on the average winning probabilities of each of the three participants. We first calculated these in the case when abstention was not an option for anyone, and then in the case where the weakest marksman had the option to abstain. In either case, the triple integrals that appeared could not be calculated in the exact form and so we resorted to numerical methods. Next we fixed one marksmanship at a time and calculated the average winning probabilities of each, and the results were displayed in Figures 1 through 6. The behavior of P_A and P_C was unclear in the graphs for a near 0. We cleared this up by first showing that as $a \to 0$ the sequential truel approaches the simultaneous truel. That allowed us to use numerical triple integration to calculate these values.

One idea for future research can be seen by noting that we obtained the Simplest Hybrid Stronger-Opponent Strategy from the Stronger Opponent Strategy by allowing C two choices, namely, firing and abstaining. A further hybridization could be obtained by allowing A and B those same options as well. New situations arise when everyone has the option to abstain. For instance, the truel may go on indefinitely, whereas when only C has the option to abstain, the probability that the truel will end approaches 1.

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