Random Measure Algebras Under O-dot Product and Morse-Transue Integral Convolution

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Abstract

In this article, we consider two operations of random measures: O-dot product and the convolution product by Morse-Transue integral. With these two operations, we construct algebras of random measures. Also we investigate further on the explicit forms of the products of Wiener processes by O-dot operation and by Morse-Transue integral convolution.

Keywords: random measure, convolution, probability, algebra

1. Introduction

Convolution is a very interesting subject in classical analysis. The convolution product is of interest to us since the operation is often closed in the same class. This allows us to build an algebra structure. One of the classical examples is L^2 space. Let $(X, \mathcal{M}, \mu)$ be a measurable space and let $L^2(\mu)$ be be the set of all measurable function $f : X \to \mathbb{C}$ such that $\int |f|^2 d\mu < \infty$. The convolution of $f, g \in L^2$ is defined by $f * g(x) = \int f(x - y)g(y)dy$. $f * g$ is also in L^2, therefore L^2 is an algebra. L^2 is also a complete normed space, therefore it is a Banach algebra. A Banach algebra is well-studied area of functional analysis. By constructing algebra structure of L^2 functions, the study of functional analysis has expanded since we can apply thousands of algebra theorems. Then we can ask if similar things can be done in probability, in other words, 'Can we build an algebra of stochastic processes?' If this can be done, it will enrich the study of statistics and probability.

To build an algebra structure, we need two operations. For instance, we have the addition and multiplication in real numbers $\mathbb{R}$. In L^2, the first operation is addition. If we add $f, g \in L^2$ then $f + g \in L^2$ also. However, if we choose the second operation as multiplication, $f \cdot g$ is not necessarily in L^2. Therefore, we need another operation, and that is the convolution.

Defining the convolution is the key to build an algebra. In the past, some researchers have defined the convolution of stochastic processes with restricted conditions or something related to stochastic processes. J.E. Huneycutt (1972) has introduced a convolution of vector measures, and D. Dehay (1991) has shown the product of two L^p-harmonizable series is harmonizable. C. Graham and B. Schreiber (1984) defined a convolution of bilinear forms and built an algebra of bilinear forms. M.M. Rao (2012) extended a convolution of bilinear forms to a convolution of random measures, and J.H.J. Park (2016) constructed an algebra of second ordred random measures by using the convolution of M.M. Rao.

In this ariticle, we build two more algebras by using operation introduced from M.M. Rao (2012), namely, O-dot product $\odot$ and the convolution by strict Morse-Transue integral. Section 2 mainly consists of preliminary results and definitions of O-dot product and strict Morse-Transue integral. In section 3 and 4, we build the algebras of random measures by using O-dot product and strict Morse-Transue integral, respectively.

Remark 1 H.S. Chung, D. Skoug and J. Chang (2014) introduced a convolution product on Wiener Space. Their convolution is the convolution of nonrandom functionals on a Wiener Space which is a vector space of nonrandom integrable functionals relative to the Wiener measure which is translation invariant in an infinite dimensional space $\mathbb{R}^\infty$. However, there is no randomness involved in the functions. It is an interesting functional analysis problem, but our interest is in the convolution of stochastic processes and measures of various types, and is distinct from their works.

2. Preliminaries

2.1 Random Measures and Bimeasures

We introduce definitions of random measures and bimeasures.

Definition 2.1 *Let $(G, \mathcal{G})$ be a measurable space, where G is a locally compact abelian group and $\mathcal{G}$ is σ-algebra of G. Let Z be a vector-valued σ-additive set function such that $Z : \mathcal{G} \to L^2(\Omega, \Sigma, P)$, where (Ω, Σ, P) is a probability measure space. Then Z is termed a random measure.*

The domain of a random measure is a σ-algebra $\mathcal{G}$, which is a family of sets $\mathcal{G} \subset \mathcal{P}(G)$ that is closed under complements

and countable unions.

In this article, we focus on the second order random measure. An ouput of Z is a random variable with second moment. If the range of Z is L^p-space, then Z would be a pth order random measure.

Definition 2.2 *Let $(G, \mathcal{G})$ be a measurable space, where G is a locally compact abelian group and $\mathcal{G}$ is σ-algebra of G.*

1. *A mapping $\beta : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ is called a bimeasure if it is separately additive, that is, if $\beta(A, \cdot)$ and $\beta(\cdot, B)$ are (scalar valued) additive measures for every $A, B \in \mathcal{G}$.*

2. *A bimeasure $\beta : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ is said to be σ-additive if it is separately σ-additive.*

3. *A bimeasure $\beta : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ is said to have a bounded or finite Vitali variation if*

$$|\beta| = \sup \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} |\beta(E_i, F_j)| : E_i, F_j \in \mathcal{G} \right\} < \infty,$$

   where all E_i's are disjoint and all F_j's are disjoint.

4. *A bimeasure is called positive definite if*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} \beta(A_i, B_j) \geq 0$$

   for all $a_i, a_j \in \mathbb{C}, A_i, B_j \in \mathcal{G}$ and $1 \leq i, j \leq n$ where $n \in \mathbb{N}$.

A bimeasure is an 'analog' of a covariance function. The following lemma states that there is always a corresponding bimeasure β to Z and vice versa. In this article, if a bimeasure β corresponds to the random measure Z then we denote $Z \sim \beta$.

Lemma 2.3 *From Rao (2012) Let $(G, \mathcal{G}$ be a measurable space, where G is locally compact abelian group and $\mathcal{G}$ is a σ-algebra of bounded Borel sets of G, and $\beta : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ be a bounded bimeasure. Then it is positive definite if and only if there is a probability space (Ω, Σ, P) and a random measure $Z : \mathcal{G} \to L^2(P)$ inducing the bimeasure in the sense that $\beta(A, B) = \langle Z(A), Z(B) \rangle_{L^2(P)}, A, B \in \mathcal{G}$.*

The inner product in $L^2(P)$ is defined by $\langle X, Y \rangle_{L^2} = \int_{\Omega} X \overline{Y} dP$ where $X, Y \in L^2(P)$. Therefore $\langle Z(A), Z(B) \rangle_{L^2(P)} = \int_{\Omega} Z(A) \overline{Z(B)} dP = E[Z(A)\overline{Z(B)}]$.

Lemma 2.4 *From Park (2015) Let $\alpha \in \mathbb{C}$, $Z : \mathcal{G} \to L^2(\Omega, \Sigma, P)$ be a random measure, and β be the corresponding bimeasure of Z, in other words, $\beta(A, B) = E[Z(A)\overline{Z(B)}]$. Then αZ has the corresponding bimeasure $|\alpha|^2 \beta$ (i.e. If $Z \sim \beta$ then $\alpha Z \sim |\alpha|^2 \beta$)*

Proof Let β' be the corresponding bimeasure of αZ. Then $\beta'(A, B) = E[\alpha Z(A)\bar{\alpha}\overline{Z(B)}] = |\alpha|^2 E[Z(A)\overline{Z(B)}] = |\alpha|^2 \beta(A, B)$.

Lemma 2.4 will be useful in section 3.

2.2 Space of Random Measures and Bimeasures

Let $RM(G)$ and $BM(G)$ be the space of random measures and bimeasures, respectively. Then we have the following elementary results from J.H.J. Park (2016).

Theorem 2.5 *From Park (2016)*

1. *$BM(G, +)$ is an abelian group.*

2. *$BM(G, +)$ is a unitary $\mathbb{C}$-module.*

3. *$RM(G, +)$ is an abelian group.*

4. *$RM(G, +)$ is a unitary $\mathbb{C}$-module.*

2.3 O-dot Product

The following proposition motivates the concept.

Proposition 2.6 *From Rao (2012) Let $\beta_i : \mathcal{G}_i \times \mathcal{G}_i \to \mathbb{C}, i = 1, 2$ be a pair of positive definite kernels and $\beta = \beta_1 \cdot \beta_2 :$ $(\mathcal{G}_1 \times \mathcal{G}_1) \times (\mathcal{G}_2 \times \mathcal{G}_2) \to \mathbb{C}$ as their pointwise product. Then β is positive definite. If we let $\mathbb{H}_\beta, \mathbb{H}_{\beta_1}, \mathbb{H}_{\beta_2}$ the corresponding reproducing kernel Hilbert (or Aronszajn) spaces, then $\mathbb{H}_\beta = \mathbb{H}_{\beta_1} \otimes \mathbb{H}_{\beta_2}$, so that $\mathbb{H}_\beta$ is a tensor product of $\mathbb{H}_{\beta_1}$ and $\mathbb{H}_{\beta_2}$.*

The proof of the Proposition 2.6 is detailed in Rao (2012). The Proposition 2.6 leads to the following definition.

Definition 2.7 *From Rao (2012) Let $(G, \mathcal{G})$ be a measurable space and $Z_i : \mathcal{G} \to L_0^2(P)$ be a pair $(i = 1, 2)$ of random measures into $L_0^2(P)$ the Hilbert space of (equivalence classes of) centered (complex) random variables on a probability space (Ω, Σ, P) with covariance bimeasures $\beta_i : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ given by $\beta_i(A, B) = < Z_i(A), Z_i(B) >$ using the inner product notation. Let $\beta = \beta_1 \cdot \beta_2 : (\mathcal{G} \times \mathcal{G}) \times (\mathcal{G} \times \mathcal{G}) \to \mathbb{C}$ be the product, pointwise as in Proposition 2.6.*

Definition 2.7 leads to the definition of O-dot product of random measures Z_1 and Z_2, where Z_1, Z_2 are corresponding random measures of bimeasures β_1, β_2, respectively. This is well illustrated in M.M. Rao (2012). However, we will slightly alter the definition of O-dot product from Rao's text since the problem with pointwise product occurs immediately. In fact, the product $\beta_1 \cdot \beta_2$ does not produce the bimeasure of the same class. The dimension of domain gets larger as we multiply. We will define a new O-dot product with a modification and build an algebra in section 3.

2.4 Convolution of Bimeasures

The integration of bimeasures by M. Morse and W. Transue (1956) is necessary to define the convolution of bimeasures.

Definition 2.8 *If $(\Omega_i, \Sigma_i), i = 1, 2$ are measurable spaces and (Ω, Σ) is their product (so $\Omega = \Omega_1 \times \Omega_2$ and $\Sigma = \Sigma_1 \otimes \Sigma_2$), $f_i : \Omega_i \to \mathbb{C}$ (measurable relative to $\Sigma_i, i = 1, 2$) are given, then the pair (f_1, f_2) is said to be strictly β-integrable where β is a bimeasure on $\Sigma_1 \times \Sigma_2$, provided the following two conditions hold:*

1. *f_1 is $\beta(\cdot, B)$-Lebesgue integrable for each $B \in \Sigma_2$ and f_2 is $\beta(A, \cdot)$-Lebesgue integrable for each $A \in \Sigma_1$ such that $\tilde{\beta}_1(A, F) = \int_F f_2(\omega_2)\beta(A, d\omega_2)$ is σ-additive in $A \in \Sigma_1$ for each $F \in \Sigma_2$ and $\tilde{\beta}_2(E, B) = \int_E f_1(\omega_1)\beta(d\omega_1, B)$ is σ-additve in $B \in \Sigma_2$ for each $E \in \Omega_1$.*

2. *f_1 is $\tilde{\beta}_1(\cdot, F)$-Lebesgue integrable, f_2 is $\tilde{\beta}_2(E, \cdot)$-Lebesgue integrable and*

$$\int_E f_1(\omega_1)\tilde{\beta}_1(d\omega_1, F) = \int_F f_2(\omega_2)\tilde{\beta}_2(E, d\omega_2) \qquad E \in \Sigma_1, F \in \Sigma_2.$$

The common value in above is denoted by $\int_E \int_F (f_1, f_2)d\beta = \int_E f_1(\omega_1)\tilde{\beta}_1(d\omega_1, F) = \int_F f_2(\omega_2)\tilde{\beta}_2(E, d\omega_2)$.

The following proposition also can be considered as a definition. $\mathcal{B}([0, 1])$ is a σ-algebra of Borel subsets of $[0, 1]$.

Proposition 2.9 *From Rao (2012) Let $Z_i : \mathcal{B}([0, 1]) \to L^2(P), i = 1, 2$ are a pair of random measures, and let β_i's are corresponding bimeasures of Z_i's, with finite Vitali variations. Then the convolution of β_1 and β_2 are defined by*

$$(\beta_1 * \beta_2)(A, B) = \int_0^1 \int_0^1 \beta_1(A - x, B - y)\beta_2(dx, dy), A, B \in \mathcal{B}([0, 1]),$$

where the integration is strict Morse-Transue integral. Also, $(\beta_1 * \beta_2)(\cdot, \cdot)$ is a well-defined positive definite bimeasure on $\mathcal{B}([0, 1]) \times \mathcal{B}([0, 1])$ and there is a random measure $Z : \mathcal{B}([0, 1]) \to L^2(P)$ whose bimeasure is $(\beta_1 * \beta_2)(\cdot, \cdot)$.

Existence of such Z is guaranteed by Lemma 2.3. The covolution of bimeasures allows to us define the convolution of random measures, since there is one-to-one correspondence between bimeasures and random measures.

2.5 Wiener Process

A Wiener process is one of examples of stochastic processes.

Definition 2.10 *A process $\{W(t), t \in \mathbb{R}^+\}$ is called a Wiener process with a positive diffusion coefficient σ if*

1. *Each increment $W(s + t) - W(s)$ is $N(0, \sigma^2 t)$*

2. *For every pair of disjoint time intervals $(t_1, t_2], (t_3, t_4]$ with $0 \le t_1 < t_2 \le t_3 < t_4$, the increments $W(t_4) - W(t_3)$ and $W(t_2) - W(t_1)$ are independent random variables, and similarly for n disjoint time intervals, where n is an arbitrary positive integer.*

3. *$W(0) = 0$ and $W(t)$ is continuous as a function of t.*

Let $\{W(t)\}_{t \in \mathbb{R}^+}$ be a Wiener Process and $\mathcal{G}$ be a σ-algebra of Borel subsets of $\mathbb{R}^+$. Pick $A = (t, s) \in \mathcal{G}$, where $0 \le t < s \le \infty$. Then we define a random measure $Z : \mathcal{G} \to L^2(\Omega, \Sigma, P)$ by $Z(A) = W(s) - W(t)$. This is an example of a stochastic process which can be written in terms of a random measure.

Definition 2.11 *Given a centered $L^2(P)$-stochastic process $\{X(t) : t \in \mathbb{R}^+\}$, its covariance function, or kernel is given by $C(t, s) = Cov(X(t), X(s))$.*

Lemma 2.12 *From Park (2015) For the Wiener Process $\{W(t) : t \in \mathbb{R}^+\}$, the covariance funtion is $Cov(W(s), W(t)) = \sigma^2 \min\{s, t\}$ for $s, t \geq 0$.*

Proof. Recall $E[W(t)] = 0$, and $E[W(t)]^2 = \sigma^2 t$. Then for $0 \leq s < t$,

$$
\begin{aligned}
Cov(W(s), W(t)) &= E[W(s)W(t)] \\
&= E[W(s)(W(t) - W(s) + W(s))] \\
&= E[W(s)(W(t) - W(s))] + E[W(s)W(s)] \\
&\quad \text{Since } W(s) \text{ and } W(t) - W(s) \text{ are independent} \\
&= E[W(s)]E[W(t) - W(s)] + E[W^2(s)] \\
&= 0 + E[W^2(s)] \\
&= \sigma^2 s.
\end{aligned}
$$

Similarly, for $0 \leq t < s$, we obtain $Cov(W(s), W(t)) = \sigma^2 t$, which leads to the formula

$$Cov(W(s), W(t)) = \sigma^2 \min\{s, t\} \text{ for } s, t \geq 0$$

3. O-dot Product and Convolution of Bimeasures

3.1 O-dot Product

The product $\beta = \beta_1 \cdot \beta_2$ in Definition 2.7 does not have the same domain as β_1, β_2, therefore β is not in the same class as β_1, β_2. Recall the diagonal set of $\mathbb{R} \times \mathbb{R}$ is $\{(x, x) \in \mathbb{R} \times \mathbb{R} | x \in \mathbb{R}\}$, which is isomorphic to $\mathbb{R}$. We extend this idea to $(\mathcal{G} \times \mathcal{G}) \times (\mathcal{G} \times \mathcal{G})$. Let the diagonal set of $(\mathcal{G} \times \mathcal{G}) \times (\mathcal{G} \times \mathcal{G})$ be $\{(A \times B, A \times B) | A \times B \in \mathcal{G} \times \mathcal{G}\}$. Now, consider the following definition. Let $\tilde{\beta} : (\mathcal{G} \times \mathcal{G}) \times (\mathcal{G} \times \mathcal{G}) \to \mathbb{C}$ be such that

$$
\tilde{\beta} = \begin{cases} \beta & \text{on diagonal set of } (\mathcal{G} \times \mathcal{G}) \times (\mathcal{G} \times \mathcal{G}) \\ 0 & \text{otherwise.} \end{cases}
$$

$\tilde{\beta}$ is defined on isomorphic copy of $\mathcal{G} \times \mathcal{G}$. We rewrite

$$\tilde{\beta}(A, B) = \beta(A \times B, A \times B).$$

Now we assert that the $\tilde{\beta}$ is positive definite and σ-additive.

Lemma 3.1 *Suppose $\tilde{\beta}$ is defined as above.*

1. *$\tilde{\beta}$ is positive definite*

2. *$\tilde{\beta} : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ is separately σ-additive.*

Proof

1. $\tilde{\beta}$ is positive definite since β is positive definite by Proposition 2.6.

2. We consider the disjoint set (A, C) and (B, D). Let $\tilde{\beta}$ be the pointwise product of $\beta_1, \beta_2 \in BM(G)$, and $A, B, C, D \in \mathcal{G}$, where $A \cap C = \varnothing, B \cap D = \varnothing$. Then

$$
\begin{aligned}
&\tilde{\beta}((A \cup C), (B \cup D)) \\
&= \beta((A \cup C) \times (B \cup D), (A \cup C) \times (B \cup D)) \\
&= \beta((A \times B) \cup (A \times D) \cup (C \times B) \cup (C \times D), \\
&\quad (A \times B) \cup (A \times D) \cup (C \times B) \cup (C \times D)) \\
&= \beta(A \times B, A \times B) + \beta(A \times B, A \times D) + \beta(A \times B, C \times B) + \beta(A \times B, C \times D) \\
&\quad + \beta(A \times D, A \times B) + \beta(A \times D, A \times D) + \beta(A \times D, C \times B) + \beta(A \times D, C \times D) \\
&\quad + \beta(C \times B, A \times B) + \beta(C \times B, A \times D) + \beta(C \times B, C \times B) + \beta(C \times B, C \times D) \\
&\quad + \beta(C \times D, A \times B) + \beta(C \times D, A \times D) + \beta(C \times D, C \times B) + \beta(C \times D, C \times D) \\
&\quad \text{Note that } \beta(A \times B, A \times D) = 0 \text{ by the definition of } \tilde{\beta} \\
&= \beta(A \times B, A \times B) + \beta(A \times D, A \times D) + \beta(C \times B, C \times B) + \beta(C \times D, C \times D) \\
&= \tilde{\beta}(A, B) + \tilde{\beta}(A, D) + \tilde{\beta}(C, B) + \tilde{\beta}(C, D)
\end{aligned}
$$

Since β is σ-additive, $\tilde{\beta}$ is also σ-additive. $\quad\square$

We have now well-defined a product of two positive definite bimeasures. We also define O-dot product of random measures Z_1 and Z_2, which are corresponding random measures of β_1, β_2 by Lemma 2.3.

Definition 3.2 *Suppose the $\beta = \beta_1 \cdot \beta_2 : (\mathcal{G} \times \mathcal{G}) \times (\mathcal{G} \times \mathcal{G}) \to \mathbb{C}$ as in Definition 2.7. Let $\tilde{\beta} = \beta$ on the diagonal of $(\mathcal{G} \times \mathcal{G}) \times (\mathcal{G} \times \mathcal{G})$, and 0 otherwise. Define O-dot product $\odot$ of bimeasures by $\tilde{\beta} = \beta_1 \odot \beta_2$. Therefore, $\tilde{\beta} : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ is defined by $\tilde{\beta}(A, B) = \beta_1 \odot \beta_2(A, B) = \beta_1(A, B) \cdot \beta_2(A, B)$, where $A \times B \in \mathcal{G} \times \mathcal{G}$. Moreover, there exist a reproducing kernel Hilbert space, $\mathbb{H}$ of $\tilde{\beta}$ and a random measure Z such that $\tilde{\beta}(A, B) = E[Z(A)\overline{Z(B)}]$. If Z_1, Z_2 and $\mathbb{H}_1, \mathbb{H}_2$ are the corresponding random measures and reproducing kernel Hilbert spaces for bimeasures β_1, β_2, then define the O-dot product of Z_1 and Z_2 as $Z = Z_1 \odot Z_2$, whose bimeasure is $\tilde{\beta}$.*

Remark 2 Note that there is a slight change from the definition of O-dot product in Rao (2012). In this article, we have restricted domain of the product bimeasure β so it can have the same domain of β_1, β_2.

3.2 Structure of $BM(G, \odot)$ and $RM(G, \odot)$

We construct an algebra of random measures under O-dot product finally. If $Z = Z_1 \odot Z_2$ and β_1, β_2 are corresponding bimeasures of Z_1, Z_2, respectively, then we will denote β as the corresponding bimeasure of Z and $\beta = \beta_1 \odot \beta_2$. Note that we omit the tilde notation ($\tilde{}$) above β for convenience of writing notations.

Theorem 3.3 *$BM(G, \odot)$ is a ring.*

Proof i) $BM(G, +)$ is an abelian group by Theorem 2.5. ii) We show the associative property. Let $A \times B \in \mathcal{G} \times \mathcal{G}$. $\beta_1 \odot (\beta_2 \odot \beta_3)(A, B) = \beta_1(A, B) \cdot (\beta_2 \odot \beta_3(A, B)) = \beta_1(A, B) \cdot (\beta_2(A, B) \cdot \beta_3(A, B)) = \beta_1(A, B) \cdot \beta_2(A, B) \cdot \beta_3(A, B) = (\beta_1 \odot \beta_2(A, B)) \cdot \beta_3(A, B) = (\beta_1 \odot \beta_2) \odot \beta_3(A, B)$ iii) We show the distributive property. $\beta_1 \odot (\beta_2 + \beta_3)(A, B) = \beta_1(A, B) \cdot (\beta_2 + \beta_3)(A, B) = \beta_1(A, B) \cdot (\beta_2(A, B) + \beta_3(A, B)) = \beta_1(A, B) \cdot \beta_2(A, B) + \beta_1(A, B) \cdot \beta_3(A, B) = (\beta_1 \odot \beta_2)(A, B) + (\beta_1 \odot \beta_3)(A, B)$.

Remark 3 The multiplicative identity of $BM(G)$ is not trivial. One can think of a bimeasure $\delta(A, B) = 1$ for all $A, B \in \mathcal{G}$. However, this δ will not have the additive property of bimeasure.

Theorem 3.4 *$BM(G, \odot)$ is an algebra over $\mathbb{C}$.*

Proof i) $BM(G)$ is a unitary $\mathbb{C}$-module by Theorem 2.5. ii) Let $a \in \mathbb{C}$. Then $a(\beta_1 \odot \beta_2)(A, B) = a(\beta_1(A, B) \cdot \beta_2(A, B)) = a\beta_1(A, B) \cdot \beta_2(A, B) = (a\beta_1 \odot \beta_2)(A, B) = \beta_1(A, B) \cdot a\beta_2(A, B) = (\beta_1 \odot a\beta_2)(A, B)$.

Theorem 3.5 *$RM(G, \odot)$ is a ring.*

Proof Let $Z_1, Z_2, Z_3 \in RM(G)$, and $\beta_1, \beta_2, \beta_3$ be their corresponding bimeasures. (i.e. $Z_1 \sim \beta_1, Z_2 \sim \beta_2, Z_3 \sim \beta_3$). i) $RM(G, +)$ is an abelian group by Theorem 2.5. ii) Note $Z_1 \odot Z_2 \sim \beta_1 \odot \beta_2$ by definition. $(Z_1 \odot Z_2) \odot Z_3 \sim (\beta_1 \odot \beta_2) \odot \beta_3 = \beta_1 \odot (\beta_2 \odot \beta_3) \sim Z_1 \odot (Z_2 \odot Z_3)$. iii) $Z_1 \odot (Z_2 + Z_3) \sim \beta_1 \odot (\beta_2 + \beta_3) = \beta_1 \odot \beta_2 + \beta_1 \odot \beta_3 \sim Z_1 \odot Z_2 + Z_1 \odot Z_3$. iv) $Z_1 \odot Z_2 \sim \beta_1 \odot \beta_2 = \beta_2 \odot \beta_1 \sim Z_2 \odot Z_1$.

Theorem 3.6 *$RM(G, \odot)$ is an algebra over $\mathbb{C}$.*

Proof Let $Z_1, Z_2 \in RM(G)$ and $k \in \mathbb{C}$. Then i) $RM(G, +)$ is a unitary $\mathbb{C}$-module by Theorem 2.5. ii) Recall $kZ \sim |k|^2\beta$ by Lemma 2.4. $k(Z_1 \odot Z_2) \sim |k|^2(\beta_1 \odot \beta_2) = |k|^2\beta_1 \odot \beta_2 = \beta_1 \odot |k|^2\beta_2 \sim kZ_1 \odot Z_2$ and $\sim Z_1 \odot kZ_2$, respectively.

3.3 O-dot Product of Wiener Processes

In this section, O-dot product of bimeasures of Wiener processes is considered as an explicit example.

Lemma 3.7 *Suppose $Z_W : \mathcal{G} \to L^2(P)$ is a random measure that represesnts Wiener Process, where $\mathcal{G}$ is a σ-algebra of Borel subsets of $\mathbb{R}^+$. Suppose β_W is the corresponding bimeasure of Z_W (i.e. β_W is a scalar bimeasure induced from a Wiener process). If $Z = Z_W \odot Z_W : \mathcal{G} \to L^2(P)$, then Z has the bimeasure $\beta = \beta_W \odot \beta_W : \mathcal{G} \times \mathcal{G} \to \mathbb{C}$ such that $\beta(A, B) = \sigma^4\mu(A \cap B)^2$, where σ is a positive diffusion coefficient of the Wiener Process.*

Proof By the definition of β,

$$\begin{aligned} \beta(A, B) &= \beta_W \odot \beta_W(A, B) \\ &= \beta_W(A, B) \cdot \beta_W(A, B) \\ &= \sigma^2\mu(A \cap B) \cdot \sigma^2\mu(A \cap B) \\ &= \sigma^4\mu(A \cap B)^2. \quad \square \end{aligned}$$

$Z \odot Z$ has the covariance bimeasure $\beta = \sigma^4(\mu(A \cap B))^2$, where Z is a random measure of Wiener process. There exist a unique Gaussian Process corresponding to a given bimeasure. However, $Z \odot Z$ itself will not be the random measure of Wiener process.

4. Convolution by Strict Morse-Transue Integral

In this section, we will illustrate the random measure algebra with the convolution by Morse-Transue Integral as in

Proposition 2.9. The definition of convolution by Morse-Transue Integral is well-defined in Rao (2012). The following lemma shows the convolution is commutative.

Lemma 4.1 *The convolution product of positive definite bimeasures is commutative.*

Proof

$$
\begin{aligned}
\beta_1 * \beta_2(A, B) &= \int_0^1 \int_0^1 \beta_1(A - x, B - y)d\beta_2(x, y) \\
&= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \chi_{A,B}(a + x, b + y)d\beta_1(a, b)d\beta_2(x, y) \\
&= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \chi_{A,B}(a + x, b + y)d\beta_2(x, y)d\beta_1(a, b) \\
&= \int_0^1 \int_0^1 \beta_2(A - a, B - b)d\beta_1(a, b) \\
&= \beta_2 * \beta_1(A, B) \quad \square
\end{aligned}
$$

4.1 Structure of $BM([0, 1])$

Let's denote $BM([0, 1])$ as a set of positive definite bimeasures $\beta : \mathcal{B}([0, 1]) \times \mathcal{B}([0, 1]) \to \mathbb{C}$, where $\mathcal{B}([0, 1])$ is a σ-algebra of Borel subsets of $[0, 1]$, and denote $RM([0, 1])$ as a set of random measures $Z : \mathcal{B}([0, 1]) \to L^2(\Omega, \Sigma, P)$. We investigate the algebraic structure of $BM([0, 1])$ and $RM([0, 1])$ with the convolution.

Theorem 4.2 *The set of bimeasure $BM([0, 1])$ with the convolution by Morse-Transue Integral is a ring with identity.*

Proof It is trivial to show that $BM([0, 1])$ is a group under addition. We show that $BM([0, 1])$ is a monoid space under convolution $*$ and has the distributive property.

First, we want to show $(\beta_1 * \beta_2) * \beta_3 = \beta_1 * (\beta_2 * \beta_3)$. We use Fubini's Theorem for bimeasures and commutative properties.

$$
\begin{aligned}
((\beta_1 * \beta_2) * \beta_3)(A, B) &= \int_0^1 \int_0^1 \beta_1 * \beta_2(A - x, B - y)\beta_3(dx, dy) \\
&= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \beta_1(A - x - a, B - y - b)d\beta_2(a, b)d\beta_3(x, y) \\
&= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \beta_1(A - x - a, B - y - b)d\beta_3(x, y)d\beta_2(a, b) \\
&= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \beta_3(A - x - a, B - y - b)d\beta_1(x, y)d\beta_2(a, b) \\
&= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \beta_3(A - x - a, B - y - b)d\beta_2(a, b)d\beta_1(x, y) \\
&= \int_0^1 \int_0^1 \beta_3 * \beta_2(A - x, B - y)d\beta_1(x, y) \\
&= ((\beta_3 * \beta_2) * \beta_1)(A, B) \\
&= (\beta_1 * (\beta_3 * \beta_2))(A, B) = (\beta_1 * (\beta_2 * \beta_3))(A, B)
\end{aligned}
$$

Simliarly, one can show $(\beta_1 * (\beta_3 * \beta_2))(A, B) = (\beta_1 * (\beta_2 * \beta_3))(A, B)$.

$BM([0, 1])$ has a unit $\delta_0(\cdot, \cdot)$, where $\delta_0(\cdot, \cdot)$ is defined by

$$
\delta_0(A, B) = \begin{cases} 1 & \text{if } 0 \in A \text{ and } 0 \in B \\ 0 & \text{otherwise} \end{cases}
$$

Observe that δ_0 is a bimeasure since it has the σ-additive property $\delta_0(A, \cup_{n \in I} B_n) = \Sigma_{n \in I} \delta_0(A, B_n)$. Also $\beta * \delta_0(A, B) = \int_0^1 \int_0^1 \beta(A - x, B - y)d\delta_0(dx, dy) = \beta(A, B)$, and $\delta_0 * \beta(A, B) = \int_0^1 \int_0^1 \delta_0(A - a, B - b)d\beta(a, b) = \beta(A, B)$.

For the multiplicative distributive property, we have

$$
\begin{aligned}
\beta_1 * (\beta_2 + \beta_3)(A, B) &= \int_0^1 \int_0^1 \beta_1(A - x, B - y)d(\beta_2 + \beta_3)(x, y) \\
&= \int_0^1 \int_0^1 \beta_1(A - x, B - y)d\beta_2 + \int_0^1 \int_0^1 \beta_1(A - x, B - y)d\beta_3 \\
&= (\beta_1 * \beta_2 + \beta_1 * \beta_3)(A, B)
\end{aligned}
$$

Therefore, $BM[0, 1]$ is a ring. $\square$

We extend the structure to $\mathbb{C}$-algebra.

Theorem 4.3 $BM([0, 1], *)$ *is a $\mathbb{C}$-algebra.*

Proof $BM([0, 1], +)$ is a $\mathbb{C}$-module. We have the compatibility with scalars, that is $(a\beta_1) * (b\beta_2)(A, B) = \int_0^1 \int_0^1 a\beta_1(A - x, B - y)d(b\beta_2)(x, y) = ab \int_0^1 \int_0^1 \beta_1(A - x, B - y)d\beta_2(x, y) = ab\beta_1 * \beta_2(A, B)$. $\square$

4.2 Structure of $RM([0, 1], *)$

We move on to the structure of random measure algebra, which is our main interest.

Definition 4.4 *Given the Borel measurable space* $([0, 1], \mathcal{B}([0, 1]))$, *let* $Z_i : \mathcal{B}([0, 1]) \rightarrow L_0^2(P)$ *be a pair* $(i = 1, 2)$ *of random measures into the Hilbert space of (equivalence classes of) centered (complex) random variables on a probability space* (Ω, Σ, P) *with covariance bimeasures* $\beta_i : \mathcal{B}([0, 1]) \times \mathcal{B}([0, 1]) \rightarrow \mathbb{C}$ *given by* $\beta_i(A, B) = < Z_i(A), Z_i(B) >$ *using the inner product notation. Let* $\beta = \beta_1 * \beta_2 : \mathcal{B}([0, 1]) \times \mathcal{B}([0, 1]) \rightarrow \mathbb{C}$ *be the convolution product given in Proposition 2.9. If* $Z : \mathcal{B}([0, 1]) \rightarrow L_0^2(P)$ *is the induced random measure by* β, *then denote by* $Z = Z_1 * Z_2 : \mathcal{B}([0, 1]) \rightarrow L_0^2(P)$, *the random measure. It is well defined as convolution of* Z_1 *and* Z_2.

The convolution of the bimeasures is the key to define the convolution of random measures, since a positive definite bimeasure induces a random measure, and vice versa, by Lemma 2.3.

Theorem 4.5 $RM([0, 1], *)$ *is a ring with identity.*

Proof i) $RM([0, 1], *)$ is an abelian group. ii) $Z_1 * (Z_2 + Z_3) \sim \beta_1 * (\beta_2 + \beta_3) = \beta_1 * \beta_2 + \beta_1 + \beta_3 \sim Z_1 * Z_2 + Z_1 * Z_3$. iii) $(Z_1 + Z_2) * Z_3 \sim (\beta_1 + \beta_2) * \beta_3 = \beta_1 * \beta_3 + \beta_1 * \beta_3 \sim Z_1 * Z_3 + Z_2 * Z_3$. iv) $(Z_1 * Z_2) * Z_3 \sim (\beta_1 * \beta_2) * \beta_3 = \beta_1 * (\beta_2 * \beta_3) \sim Z_1 * (Z_2 * Z_3)$. v) $Z_1 * Z_2 \sim Z_2 * Z_1$ for commutativity. vi) Define a random measure $Z_0(E) = 1$ if $0 \in E, 1$ otherwise. Z_0 is finitely additive since $Z_0(\cup E_n) = \Sigma Z_0(E_n) = 1$ if $0 \in E_i$ for any i, 0 otherwise. Let δ_0 be correspoding bimeasure of Z_0. Then $\delta_0(A, B) = E[Z(A)Z(B)] = 1$ if $0 \in A$ and $0 \in B$, 0 otherwise, which is the exact definition of Dirac δ-function. (i.e $Z_0 \sim \delta_0$. $Z_1 * Z_0 \sim \beta_1 * \delta_0 = \beta_1 \sim Z_1$. Similarly, $Z_0 * Z_1 \sim Z_1$. $\square$

Theorem 4.6 $RM([0, 1], *)$ *is a* $\mathbb{C}$-*algebra.*

Proof i) $RM([0, 1])$ is a $\mathbb{C}$-module. ii) $k(Z_1 * Z_2) \sim |k|^2(\beta_1 * \beta_2) = |k|^2\beta_1 * \beta_2 = \beta_1 * |k|^2\beta_2 \sim kZ_1 * Z_2$ and $\sim Z_1 * kZ_2$. $\square$

Finally, an algebra of random measure is constructed.

4.3 Wiener Process

Let $\{W_t\}_{t \in [0,1]}$ be a Wiener Process, and Z be the associated random measure, that is $Z : \mathcal{B}([0, 1]) \rightarrow L^2(\Omega, \Sigma, P)$ such that $Z([t_i, t_{i+1}]) = W_{t_i} - W_{t_{i+1}}$. Let β be the bimeasure of Z, that is $\beta(A, B) = E[Z(A)\overline{Z(B)}]$. We explicitly compute $\beta * \beta(A, B)$ if $A = [t_1, t_2], B = [s_1, s_2]$ with $s_1 < t_1 < s_2 < t_2$.

Theorem 4.7 *Let* $\{W_t\}_{t \in [0,1]}$ *be a Wiener process, and* Z *be the associated random measure, that is* $Z : \mathcal{B}([0, 1]) \rightarrow L^2(\Omega, \Sigma, P)$ *such that* $Z([t_i, t_{i+1}]) = W_{t_i} - W_{t_{i+1}}$. *Let* β *be the bimeasure of* Z, *that is* $\beta(A, B) = E[Z(A)\overline{Z(B)}]$ *with* $A = [t_1, t_2], B = [s_1, s_2]$ *with* $s_1 < t_1 < s_2 < t_2$. *Then*

$$
\begin{aligned}
\beta * \beta(A, B) \quad &= -\frac{s_1^2 t_1^2}{4} + \frac{s_2 t_1^3}{4} - \frac{t_1^4}{12} + \frac{s_2}{3} + \frac{t_1 s_2}{2} + \frac{t_1^2 s_2}{2} + \frac{t_1^3 s_2}{6} - \frac{s_2^2}{2} - \frac{t_1 s_2^2}{2} - \frac{t_1^2 s_2^2}{4} \\
&+ \frac{t_1 s_2^3}{6} - \frac{s^4}{12} + \frac{t_2}{3} + \frac{s_1 t_2}{2} + \frac{s_1^2 t_2}{2} + \frac{s_1^3 t_2}{6} + \frac{s_2 t_2}{2} + \frac{s_2^2 t_2}{2} + \frac{s_2^3 t_2}{6} - \frac{t_2^2}{2} \\
&- \frac{s_1 t_2^2}{2} - \frac{s_1^2 t_2^2}{4} - \frac{s_2 t_2^2}{2} - \frac{s_2^2 t_2^2}{4} + \frac{t_2^3}{3} + \frac{s_1 t_2^3}{6} + \frac{s_2 t_2^3}{6} - \frac{t_2^4}{12}.
\end{aligned}
$$

Proof We are to compute

$$\beta_1 * \beta_2(A, B) = \int_0^1 \int_0^1 \beta_1(A - x, B - y)\beta_2(dx, dy), \qquad \text{with } A, B \in \mathcal{B}([0, 1]),$$

where β_1, β_2 are both bimeasures of random measures from Wiener process. We investigate $\beta(A - x, B - y)$ by,

$$
\begin{aligned}
\beta(A - x, B - y) \quad &= E[Z(A - x)Z(B - y)] \text{ where } Z(\cdot) \text{ is as above} \\
&= E[(X(t_2 - x) - X(t_1 - x)) \cdot (X(s_2 - y) - X(s_1 - y))] \\
&= E[X(t_2 - x)X(s_2 - y)] - E[X(t_2 - x)X(s_1 - y)] \\
&\quad - E[X(t_1 - x)X(s_2 - y)] + E[X(t_1 - x)X(s_1 - y)] \\
&= \min\{t_2 - x, s_2 - y\} - \min\{t_2 - x, s_1 - y\} \\
&\quad - \min\{t_1 - x, s_2 - y\} + \min\{t_1 - x, s_1 - y\} \\
&\text{by Lemma 2.12.}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\beta_1 * \beta_2(A, B) \quad &= \int_0^1 \int_0^1 \beta_1(A - x, B - y)\beta_2(dx, dy) \\
&= \int_0^1 \int_0^1 \min\{t_2 - x, s_2 - y\} - \min\{t_2 - x, s_1 - y\} \\
&\quad - \min\{t_1 - x, s_2 - y\} + \min\{t_1 - x, s_1 - y\}d\beta_2(x, y) \\
&= \int_0^1 \int_0^1 \min\{t_2 - x, s_2 - y\}d\beta_2(x, y) \\
&\quad - \int_0^1 \int_0^1 \min\{t_2 - x, s_1 - y\}d\beta_2(x, y) \\
&\quad - \int_0^1 \int_0^1 \min\{t_1 - x, s_2 - y\}d\beta_2(x, y) \\
&\quad + \int_0^1 \int_0^1 \min\{t_1 - x, s_1 - y\}d\beta_2(x, y)
\end{aligned}
$$

Observe that

$$\min\{t_2 - x, s_2 - y\} = \begin{cases} t_2 - x & \text{if } x - (t_2 - s_2) \geq y \\ s_2 - y & \text{if } x - (t_2 - s_2) < y \end{cases}$$

$$\min\{t_2 - x, s_1 - y\} = \begin{cases} t_2 - x & \text{if } x - (t_2 - s_1) \geq y \\ s_1 - y & \text{if } x - (t_2 - s_1) < y \end{cases}$$

$$\min\{t_1 - x, s_2 - y\} = \begin{cases} t_1 - x & \text{if } x + s_2 - t_1 \geq y \\ s_2 - y & \text{if } x + s_2 - t_1 < y \end{cases}$$

$$\min\{t_1 - x, s_1 - y\} = \begin{cases} t_1 - x & \text{if } x - (t_1 - s_1) \geq y \\ s_1 - y & \text{if } x - (t_1 - s_1) < y \end{cases}$$

Also $d\beta(x, y) = d(\min(x, y)) = x dx dy$ if $x \leq y$, and $= y dx dy$ if $x > y$. We will now calculate each of the integrals explicitly.

$$\int_0^1 \int_0^1 \min\{t_2 - x, s_2 - y\} d\beta_2(x, y)$$
$$= \int_0^1 \int_0^y x(s_2 - y) dx dy + \int_0^{t_2 - s_2} \int_y^{t_2 - s_2} y(s_2 - y) dx dy$$
$$+ \int_{t_2 - s_2}^1 \int_{x - (t_2 - s_2)}^x y(s_2 - y) dy dx + \int_{t_2 - s_2}^1 \int_0^{x - (t_2 - s_2)} y(t_2 - x) dy dx$$
$$= -\tfrac{1}{8} + \tfrac{s_2}{6}$$
$$- \tfrac{1}{12}(s_2 - t_2)^3(3s_2 - t_2)$$
$$+ \tfrac{1}{6}(s_2^4 - s_2^3(4t_2 + 1) + 6s_2^2 t_2^2 + s_2(-4t_2^3 + 3t_2^2 - 3t_2 + 2) + t_2(t_2^3 - 2t_2^2 + 3t_2 - 2))$$
$$+ \tfrac{1}{24}(s_2 - t_2 + 1)^3(s_2 + 3t_2 - 3)$$

Similarly, other integrals can be obtained. Thus,

$$\int_0^1 \int_0^1 \min\{t_2 - x, s_1 - y\} d\beta_2(x, y)$$
$$= \int_0^1 \int_0^y x(s_1 - y) dx dy + \int_0^{t_2 - s_1} \int_y^{t_2 - s_1} y(s_1 - y) dx dy$$
$$+ \int_{t_2 - s_1}^1 \int_{x - (t_2 - s_1)}^x y(s_1 - y) dy dx + \int_{t_2 - s_1}^1 \int_0^{x - (t_2 - s_1)} y(t_2 - x) dy dx$$
$$= -\tfrac{1}{8} + \tfrac{s_1}{6}$$
$$- \tfrac{1}{12}(s_1 - t_2)^3(3s_1 - t_2)$$
$$+ \tfrac{1}{6}(s_1^4 - s_1^3(4t_2 + 1) + 6s_1^2 t_2^2 + s_1(-4t_2^3 + 3t_2^2 - 3t_2 + 2) + t_2(t_2^3 - 2t_2^2 + 3t_2 - 2))$$
$$+ \tfrac{1}{24}(s_1 + 3t_2 - 3)(s_1 - t_2 + 1)^3,$$

and

$$\int_0^1 \int_0^1 \min\{t_1 - x, s_2 - y\} d\beta_2(x, y)$$
$$= \int_{s_2 - t_1}^1 \int_0^{y - s_2 + t_1} x(s_2 - y) dx dy + \int_0^{1 - s_2 + t_1} \int_x^{x + s_2 - t_1} x(t_1 - x) dy dx$$
$$+ \int_{1 - s_2 + t_1}^1 \int_x^1 x(t_1 - x) dy dx + \int_0^1 \int_0^x y(t_1 - x) dy dx$$
$$= \tfrac{1}{24}(1 + t_1 - s_2)^3(-3 + t_1 + 3s_2) + \tfrac{1}{6}(1 + t_1 - s_2)^2(-t_1 + s_2)(-2 + t_1 + 2s_2)$$
$$+ \tfrac{1}{12}(t_1 - s_2)^2(-6 + t_1^2 + 2t_1(-1 + s_2) + 8s_2 - 3s_2^2)$$
$$+ \tfrac{t_1}{6} - \tfrac{1}{8}.$$

The last integral is

$$\int_0^1 \int_0^1 \min\{t_1 - x, s_1 - y\} d\beta_2(x, y)$$
$$= \int_0^1 \int_0^y x(s_1 - y) dx dy + \int_0^{t_1 - s_1} \int_y^{t_1 - s_1} y(s_1 - y) dx dy$$
$$+ \int_{t_1 - s_1}^1 \int_{x - (t_1 - s_1)}^x y(s_1 - y) dy dx + \int_{t_1 - s_1}^1 \int_0^{x - (t_1 - s_1)} y(t_1 - x) dy dx$$
$$= -\tfrac{1}{8} + \tfrac{s_1}{6}$$
$$- \tfrac{1}{12}(s_1 - t_1)^3(3s_1 - t_1)$$
$$+ \tfrac{1}{6}(s_1^4 - s_1^3(4t_1 + 1) + 6s_1^2 t_1^2 + s_1(-4t_1^3 + 3t_1^2 - 3t_1 + 2) + t_1(t_1^3 - 2t_1^2 + 3t_1 - 2))$$
$$+ \tfrac{1}{24}(s_1 + 3t_1 - 3)(s_1 - t_1 + 1)^3$$

We put these calculated integrals together.

$$
\begin{aligned}
\beta_1 * \beta_2(A, B) \quad &= \int_0^1 \int_0^1 \beta_1(A - x, B - y)\beta_2(dx, dy) \\
&= \int_0^1 \int_0^1 \min\{t_2 - x, s_2 - y\}d\beta_2(x, y) \\
&\quad - \int_0^1 \int_0^1 \min\{t_2 - x, s_1 - y\}d\beta_2(x, y) \\
&\quad - \int_0^1 \int_0^1 \min\{t_1 - x, s_2 - y\}d\beta_2(x, y) \\
&\quad + \int_0^1 \int_0^1 \min\{t_1 - x, s_1 - y\}d\beta_2(x, y) \\
&= -1 + \frac{s_1}{3} - \frac{s_1^2}{2} - \frac{s_1^3}{3} - \frac{s_1^4}{12} + \frac{t_1}{3} + \frac{s_1 t_1}{2} + \frac{s_1^2 t_1}{2} + \frac{s_1^3 t_1}{6} - \frac{t_1^2}{2} - \frac{s_1 t_1^2}{2} \\
&\quad - \frac{s_1^2 t_1^2}{4} + \frac{s_2 t_1^3}{4} - \frac{t_1^4}{12} + \frac{s_2}{3} + \frac{t_1 s_2}{2} + \frac{t_1^2 s_2}{2} + \frac{t_1^3 s_2}{6} - \frac{s_2^2}{2} - \frac{t_1 s_2^2}{2} - \frac{t_1^2 s_2^2}{4} \\
&\quad + \frac{t_1 s_2^3}{6} - \frac{s_2^4}{12} + \frac{t_2}{3} + \frac{s_1 t_2}{2} + \frac{s_1^2 t_2}{2} + \frac{s_1^3 t_2}{6} + \frac{s_2 t_2}{2} + \frac{s_2^2 t_2}{2} + \frac{s_2^3 t_2}{6} - \frac{t_2^2}{2} \\
&\quad - \frac{s_1 t_2^2}{2} - \frac{s_1^2 t_2^2}{4} - \frac{s_2 t_2^2}{2} - \frac{s_2^2 t_2^2}{4} + \frac{t_2^3}{3} + \frac{s_1 t_2^3}{6} + \frac{s_2 t_2^3}{6} - \frac{t_2^4}{12} \quad \square
\end{aligned}
$$

There exists a unique (centered) Gaussian Process corresponding to a given covariance function and hence for that we just computed, since it is well known that a Gaussian process is determined by its mean and covariance functions. However, with such a complex representation, it is not trivial to express the exact representation of the Gasussian Process related to the covariance function above, although there exists a unique (Gaussian) process, which will not be the (traditional) Wiener process.

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