International Journal of Statistics and Probability; Vol. 8, No. 6; November 2019
ISSN 1927-7032 E-ISSN 1927-7040
Published by Canadian Center of Science and Education

First Passage and Collective Marks

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Received: September 22, 2019 Accepted: October 11, 2019 Online Published: October 13, 2019

doi:10.5539/ijsp.v8n6p47 URL: https://doi.org/10.5539/ijsp.v8n6p47

Abstract

Probability generating functions for first passage times of Markov chains are found using the method of collective marks. A system of equations is found which can be used to obtain moments of the first passage times. Second passage probabilities are discussed.

Keywords: Markov chains, first passage times, collective marks

AMS Subject Classification: 60J05, 60J22

1. Introduction

Suppose we have a Markov chain with n states labeled $1, 2, \ldots, n$.

Define the random variable X_{ij} to be the number of steps needed to move from state i to state j for the first time. We refer to X_{ij} as the first passage time. Define the first passage probability as $f_{ij}(k) = P(X_{ij} = k)$. There are several ways to compute the first passage probabilities. For example, see (Hunter, 1983) and (Kao, 1996). First passage probabilities are important as they can be used to control processes and determine when to implement parameter changes. First passage times are indicators of changes to a system (e.g. climate change) and act as warning signals that some action may be needed.

Suppose we have a probability mass function for a discrete random variable X that takes on value k with probability p_k for $k = 0, 1, \ldots$. Define the probability generating function for X to be $\psi_X(z) = \sum_{k=0}^{\infty} p_k z^k$. (Alfa, 2014, p. 76) gives an expression for the probability generating function of the first passage probabilities from state i to state j as follows.

$$\psi_{ij}(z) = \frac{P_{ij}(z)}{1 - P_{ij}(z)}$$

where $P_{ij}(z) = \sum_{k=1}^{\infty} p_{ij}^{(k)} z^k$. But this is not a closed form since we need the values $p_{ij}^{(k)}$.

The method of collective marks was originated by (van Dantzig, 1949), and discussed in (Runnenburg, 1965) and (Kleinrock, 1975, chapter 7). The method gives a probabilistic interpretation of a probability generating function $\sum_{k=0}^{\infty} p_k z^k$. Let z be the probability that an item is "marked." Then $p_k z^k$ represents the probability that random variable X takes on the value k and each of the k counts is marked. Summing over all k gives the total probability that all items from a single realization of the random variable X are marked. The method can often simplify computation and explain a system in an understandable way.

In this paper, we use the collective marks method to find the probability generating function for first passage probabilities, in a closed form for a fixed number of states n. We find expressions for moments of the first passage times, by using the system of equations that we develop. We present a method to find probability generating functions of second passage times.

2. Computing First Passage Probabilities

Theorem 2.1 *Let $\psi_{ij}(z)$ be the probability generating function for the first passage random variable from i to j for an n state Markov chain. Then we obtain an equation,*

$$\psi_{ij}(z) = p_{ij} z + \sum_{k:k\neq j} p_{ik} z \psi_{kj}(z).$$

Proof. By the method of collective marks, $\psi_{ij}(z)$ represents the probability that the path starting from i and reaching j for the first time has all of its steps receiving a mark. Here the probability of a step being marked is assumed to be z. The first step may enter state j immediately and this occurs with probability p_{ij}. The probability that the singleton path is marked is z. So $p_{ij}z$ is the probability that the first passage probability consists of 1 step and is marked. Otherwise, the process goes to some other state k with probability p_{ik} and that step is marked with probability z. From the new position k, the process moves to state j eventually with each step being marked with probability generating function $\psi_{kj}(z)$. Summing over all cases gives the result.

Note The equation in our theorem involves the generating functions $\psi_{kj}(z)$ (for all k) and we can get a similar equation for each of these. For fixed j, this will give us a linear system of equations in the variables $\psi_{1j}(z), \ldots, \psi_{nj}(z)$, which can be solved to get any particular first passage generating function desired as a non linear function of z. The coefficients in the system of equations may involve z as well as constants.

Theorem 2.2 *Let $\psi_{13}(z)$ be the probability generating function for the first passage random variable from* 1 *to* 3 *for an* 3 *state Markov chain. Then*

$$\psi_{13}(z) = \frac{p_{13}z + (p_{12}p_{23} - p_{13}p_{22})z^2}{1 - (p_{11} + p_{22})z + (p_{11}p_{22} - p_{12}p_{21})z^2}.$$

Proof. From Theorem 2.1, we have

$$\begin{aligned}\psi_{13}(z) &= p_{11}z\psi_{13}(z) + p_{12}z\psi_{23}(z) + p_{13}z \\ \psi_{23}(z) &= p_{21}z\psi_{13}(z) + p_{22}z\psi_{23}(z) + p_{23}z\end{aligned}$$

Solving this system of two equations in two unknowns gives our result.

Note

(a) A similar result holds for any pair, not just $(1, 3)$.

(b) Our method manages to obtain a closed form for the probability generating function of the first passage times for 3 state Markov chains

(c) Theorem 2.2 can be extended to a larger number number of states as we still essentially get a linear system to solve.

(d) Although the system of equations is linear in the $\psi_{ij}(z)$ unknowns, the coefficients involve the variable z, and the resulting expressions are nonlinear functions of z.

Example 2.1

Consider the Markov transition matrix $P = \begin{bmatrix} .3 & .4 & .3 \\ .3 & .3 & .4 \\ .5 & .4 & .1 \end{bmatrix}$ We will compute first passage probability generating functions for $\psi_{13}(z)$, $\psi_{23}(z)$, and $\psi_{33}(z)$. For the first two we use theorem 2, with appropriate changes for $\psi_{23}(z)$, and for the third, we get a separate equation. According to Theorem 2, the probability generating function for the first passage probabilities from state 1 to state 3 is given by

$$\psi_{13}(z) = \frac{.3z + (.4 * .4 - .3 * .3)z^2}{1 - (.3 + .3)z + (.3 * .3 - .4 * .3)z^2} = \frac{.3z + .07z^2}{1 - .6z - .03z^2}$$

We use the “series” command in MAPLE to find the Taylor expansion and get results.

$$\psi_{13}(z) = .3z + .25z^2 + .159z^3 + .1029z^4 + .06651z^5 + 0.04299z^6 + 0.02779z^7 + \ldots$$

This result agrees with other methods.

Similarly, from Theorem 2, we find

$$\psi_{23}(z) = \frac{.4z + (.3 * .3 - .4 * .3)z^2}{1 - (.3 + .3)z + (.3 * .3 - .3 * .4)z^2} = \frac{.4z - .03z^2}{1 - .6z - .03z^2}$$

Finally,

$$\begin{aligned}\psi_{33}(z) &= p_{33}z + p_{31}z\psi_{13}(z) + \psi_{32}z\psi_{23}(z) = .1z + .5z\psi_{13}(z) + .4z\psi_{23}(z)\\ &= \frac{.1z - .06z^2 - .003z^3 + .15z^2 + .035z^3 + .16z^2 - .012z^3}{1 - .6z - .03z^2}\\ &= \frac{.1z + .25z^2 + .02z^3}{1 - .6z - .03z^2}\end{aligned}$$

3. Moments of First Passage Times

One can easily find expressions for the moments of first passage probabilities via a system of equations (Hunter, 1983). However, we will use our system of equations to give an alternative method. Theorem 2.2 gives an expression for $\psi_{ij}(z)$ so we can find the moments of the first passage probabilities by simply taking derivatives and evaluating the expressions at $z = 1$, making any additional computations needed. But this explicitly requires solving for $\psi_{ij}(z)$ which can be a somewhat burdensome task as the coefficients of the linear system involve the variable z.

Theorem 2.1 gives an equation for $\psi_{ij}(z)$ involving the probability generating function of first passage times from i to j and since we have similar expressions for $\psi_{kj}(z)$ (for $k \neq j$), we have a system of equations that we can work with. We can take the derivative of the SYSTEM of equations, and then substitute $z = 1$ into the system to create a much more tractable system of equations. Of course, $\psi_{ij}(1) = 1$ and $\psi'_{ij}(1) = \mu_{ij}$ where $\mu_{ij} = E(X_{ij})$, where X_{ij} is the number of steps needed to reach state j from state i for the first time. Also, $\psi^{(2)}_{ij}(1) = E(X_{ij}(X_{ij} - 1))$.

Example 3.1 We use the same 3×3 transition matrix as in Example 2.1

The system of equations from Theorem 1 is

$$\begin{aligned}\psi_{13}(z) &= .3z\psi_{13}(z) + .4z\psi_{23}(z) + .3z\\ \psi_{23}(z) &= .3z\psi_{13}(z) + .3z\psi_{23}(z) + .4z\end{aligned}$$

Also $\psi_{33}(z) = .1z + .5z\psi_{13}(z) + .4z\psi_{23}(z)$.

Taking derivatives gives

$$\begin{aligned}\psi'_{13}(z) &= .3\psi_{13}(z) + .3z\psi'_{13}(z) + .4\psi_{23}(z) + .4z\psi'_{23}(z) + .3\\ \psi'_{23}(z) &= .3\psi_{13}(z) + .3z\psi'_{13}(z) + .3\psi_{23}(z) + .3z\psi'_{23}(z) + .4\end{aligned}$$

and

$\psi'_{33}(z) = .1 + .5\psi_{13}(z) + .5z\psi'_{13}(z) + .4\psi_{23}(z) + .4z\psi'_{23}(z)$

Evaluating at $z = 1$ gives

$$\begin{aligned}\mu_{13} &= .3 + .3\mu_{13} + .4 + .4\mu_{23} + .3 = 1 + .3\mu_{13} + .4\mu_{23}\\ \mu_{23} &= .3 + .3\mu_{13} + .3 + .3\mu_{23} + .4 = 1 + .3\mu_{13} + .3\mu_{23}\end{aligned}$$

and

$\mu_{33} = .1 + .5 + .5\mu_{13} + .4 + .4\mu_{23} = 1 + .5\mu_{13} + .4\mu_{23}$

Solving these gives $\mu_{13} = 2.97$, $\mu_{23} = 2.70$ and $\mu_{33} = 3.57$.

4. Second Passage Times

Theorem 4.1 *Let Y_{ij} be the random variable representing the number of steps needed to move from i to j for the second time. Then the probability generating function for Y_{ij} is $\psi_{ij}(z)\psi_{jj}(z)$.*

Proof $Y_{ij} = X_{ij} + X_{jj}$ where X_{ij} is the first passage random variable, so Y_{ij} is just the convolution of two independent random variables. Since the pgf of a convolution is the product of the pgf's of each part, the result follows.

Example 4.1 We will compute the second passage time from state 1 to state 3 in the Markov chain with transition matrix $P = \begin{bmatrix} .3 & .4 & .3\\ .3 & .3 & .4\\ .5 & .4 & .1\end{bmatrix}$ We earlier calculated

$\psi_{13}(z) = \dfrac{.3z + .07z^2}{1 - .6z - .03z^2}$ and $\psi_{33}(z) = \dfrac{.1z + .25z^2 + .02z^3}{1 - .6z - .03z^2}$ so

$\psi_{second}(z) = \dfrac{(.3z + .07z^2)(.1z + .25z^2 + .02z^3)}{(1 - .6z - .03z^2)^2}$. If we expand this (using MAPLE) into a Taylor series, we get

$\psi_{second}(z) = 0.03z^2 + .118z^3 + .1561z^4 + .1522z^5 + .1316z^6 + .1065z^7 + \ldots$

Thus, for example, the probability of moving from 1 to 3 for the second time on step 4 is 0.1561.

In a similar manner,we can obtain higher order passage probabilities.

5. Discussion

The use of collective marks is not absolutely necessary to obtain Theorem 2.1, but it does make the proof simpler than other methods. The closed form result of Theorem 2.2 appears to be new. If there are a large number of states, the expressions of Theorem 2.2 would quickly become much more complex. Second passage (or higher order) passage times can be studied by expanding the Markov transition matrix to contain the information about how many passages have occurred but it is much easier to simply view a second passage time as the convolution of two single passage times, as presented here.

Acknowledgements

We acknowledge funding and support from NSERC (Natural Sciences and Engineering Research Council of Canada).

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