Extended Poisson Inverse Weibull Distribution: Theoretical and Computational Aspects

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Abstract

A new extension of the Poisson Inverse Weibull distribution is derived and studied in details. Number of structural mathematical properties are derived. We used the well-known maximum likelihood method for estimating the model parameters. The new model is applied for modeling some real data sets to prove its importance and flexibility empirically.

Keywords: maximum likelihood, zero truncated Poisson distribution, estimation, order statistics, moments, generating function

1. Introduction

A certain continuous random variable (rv) Z is said to have the Inverse Weibull distribution (IWD) with scale parameter $\delta b^{\beta^{-1}} > 0$ and and shape parameter $\beta > 0$ if its probability density function (PDF) is given by

$$\mathbf{h}_{[b,\beta,\delta]}(z) = \mathrm{e}^{-b\delta^{\beta}z^{-\beta}} \mid_{\left(z\geq 0,\delta b^{\beta^{-1}}>0,\beta>0\right)}$$

and the corresponding cumulative distribution function (CDF) is

$$\mathbf{H}_{[b,\beta,\delta]}(z) = b\beta\delta^{\beta}z^{-1-b}\mathrm{e}^{-b\delta^{\beta}z^{-\beta}} \mid_{\left(z\geq 0,\delta b^{\beta^{-1}}>0,\beta>0\right)} .$$

In this work we shall propose a new version (generalization) of the the Topp Leone Inverse Weibull distribution (TL-IWD) via using the discrete zero truncated Poisson distribution (ZTPD). The PDF and CDF of TL-IWD are given by

$$f_{[\theta,b,\beta,\delta]}(x) = 2\beta\theta\delta^{\beta}bx^{-(\beta+1)}\left(1-\mathrm{e}^{-b\delta^{\beta}x^{-\beta}}\right)\left(2-\mathrm{e}^{-b\delta^{\beta}x^{-\beta}}\right)^{\theta-1}\mathrm{e}^{-\theta b\delta^{\beta}x^{-\beta}}, \quad (1)$$

and

$$F_{[\theta,b,\beta,\delta]}(x) = \left(2-\mathrm{e}^{-b\delta^{\beta}x^{-\beta}}\right)^{\theta}\mathrm{e}^{-\theta b\delta^{\beta}x^{-\beta}}, \quad (2)$$

respectively, where β, b, and $\theta > 0$ are shape parameters. The probability mass function (PMF) of ZTPD is given by

$$P\left(N = n|_{\alpha}\right) = \frac{\mathrm{e}^{-\alpha}}{-\mathrm{e}^{-\alpha}+1}\frac{\alpha^{n}}{n!} \mid_{(n\in\mathbf{I}^{+})}. \quad (3)$$

If we have a system has N subsystems functioning independently at a given time t where N has ZTPD with expected value $\mathbf{E}(N)$ and variance $Var(N|\alpha)$ are, respectively, given by

$$\mathbf{E}(N) = \alpha\frac{1}{-\mathrm{e}^{-\alpha}+1},$$

and

$$Var(N|\alpha) = \left(-\mathrm{e}^{-\alpha}+1\right)^{-1}\left(\alpha+\alpha^{2}\right)-\left\{\alpha^{2}\left(-\mathrm{e}^{-\alpha}+1\right)^{-2}\right\} = \left[1+\alpha-\mathbf{E}(N)\right]\mathbf{E}(N).$$

Assume that the failure time of each subsystem has the TLEIW model defined by PDF and CDF in (1) and (2). Let Y_i denote the failure time of the i^{th} subsystem and let

$$X = \min\{Y_1, Y_2, \cdots, Y_N\},$$

then the conditional CDF (CCDF) of X given N is

$$F\left(x|N\right) = 1-\Pr\left(X > x|N\right) = 1-\Pr\left(Y_1 > x\right)^{N} = 1-\left(1-F_{\theta,b,\beta,\delta}(x)\right)^{N}, \quad (4)$$

then, the marginal CDF (MCDF) of X is will be

$$F_{[\mathbf{\Phi}]}(x) = \frac{1 - e^{-\alpha e^{-\theta b\delta^{\beta} x^{-\beta}}\left(2 - e^{-b\delta^{\beta} x^{-\beta}}\right)^{\theta}}}{-e^{-\alpha} + 1}, \tag{5}$$

equation (5) is called the CDF of the PTLIWD, where $\mathbf{\Phi}$=$\alpha, \theta, b, \beta, \delta$. The corresponding PDF of (5) reduces to

$$f_{[\mathbf{\Phi}]}(x) = \frac{2\theta\alpha\beta b\delta^{\beta} x^{-(\beta+1)} e^{-\theta b\delta^{\beta} x^{-\beta}}}{1 - e^{-\alpha}} \left(-e^{-b\delta^{\beta} x^{-\beta}} + 2\right)^{\theta-1} \times \left\{-e^{-b\delta^{\beta} x^{-\beta}} + 1\right\} \underbrace{e^{-\alpha e^{-\theta b\delta^{\beta} x^{-\beta}}\left\{2 - e^{-b\delta^{\beta} x^{-\beta}}\right\}^{\theta}}}_{A_i}, \tag{6}$$

Some useful extensions of the IWD can be found in the statistical literature and cited such as the beta IW distribution (B-IWD) (see Barreto-Souza et al. (2011)) , the Marshall-Olkin IW distribution (MO-IWD) (see Krishna et.al. (2013)), the transmuted IW distribution (T-IWD) (see Mahmoud and Mandouh (2013)), the transmuted exponentiated IW distribution (TE-IWD) (see Elbatal et al. (2014)), the transmuted Marshall-Olkin IW distribution (TMO-IWD) (see Afify et al. (2015)), the transmuted exponentiated generated IW distribution (TEG-IWD) (see Yousof et al. (2015)), the beta exponential IW distribution (BE-IWD) (see Mead et al. (2016)), WIW distribution (W-IWD) (see Afify et al. (2016b)), Kumaraswamy Marshall-Olkin IW distribution (KMO-IWD) (see Afify et al. (2016b)), Kumaraswamy transmuted Marshall-Olkin IW distribution (KTMO-IWD) (see Yousof et al. (2016)), beta transmuted IW distribution (BT-IWD) (see Afify et al. (2016c)), Topp Leone Generated IW distribution (TLG-IWD) (see Yousof et al. (2017)), new five-parameter Fréchet model for extreme values (see Haq et al. (2017)), odd Lindley IW (see Korkmaz et al. (2017)) Odd log-logistic IW distribution (OLL-IWD) (see Yousof et al. (2018a)) and a new statistical model for extreme values (see Chakraborty et al. (2019)), among others.

Expanding A_i via the power series, we have

$$A_i = \Sigma_{\tau=0}^{\infty} \frac{(-1)^{\tau} \alpha^{\tau} e^{-\tau\theta b\delta^{\beta} x^{-\beta}}}{\tau! \left(2 - e^{-b\delta^{\beta} x^{-\beta}}\right)^{-\theta\tau}}, \tag{7}$$

inserting (7) into (6), we get

$$f_{[\mathbf{\Phi}]}(x) = \Sigma_{\tau=0}^{\infty} \theta\beta\delta^{\beta}\alpha^{1+\tau} \frac{(-1)^{\tau} x^{-\beta-1} e^{-[(1+\tau)\theta]b\delta^{\beta} x^{-\beta}}}{\tau! 2^{-(1+\tau)\theta} (-e^{-\alpha} + 1)} \times \left\{1 - e^{-b\delta^{\beta} x^{-\beta}}\right\} \left\{1 - 2^{-1} e^{-b\delta^{\beta} x^{-\beta}}\right\}^{(1+\tau)\theta-1}. \tag{8}$$

Consider the power series

$$(-q + 1)^{b-1} \,|_{|a|<1 \text{ and } b>0 \text{ realnon-integer}} = \Sigma_{w=0}^{\infty} \left\{ \left[(-q)^{w} \, \Gamma(b)\right] / \left[w! \, \Gamma(b - w)\right] \right\}, \tag{9}$$

Using (9) we get

$$f_{[\mathbf{\Phi}]}(x) = \Sigma_{\tau,w=0}^{\infty} \left[a_{\tau,w} \, \mathbf{h}_{[(1+\tau)\theta+w]b,\beta,\delta}(x) - a_{\tau,w}^{\bigstar} \, \mathbf{h}_{[(1+\tau)\theta+w+1]b,\beta,\delta}(x)\right], \tag{10}$$

where

$$a_{\tau,w} = \frac{\theta\alpha^{1+\tau} 2^{(1+\tau)\theta-w} (-1)^{\tau+w}}{\tau! \,(-e^{-\alpha} + 1) \left[(1+\tau)\theta + w\right] b} \binom{-1 + (1+\tau)\theta}{w},$$

$$a_{\tau,w}^{\bigstar} = \frac{\theta\alpha^{1+\tau} 2^{(1+\tau)\theta-w} (-1)^{\tau+w}}{\tau! \,(-e^{-\alpha} + 1) \left[1 + (1+\tau)\theta + w\right] b} \binom{-1 + (1+\tau)\theta}{w},$$

and

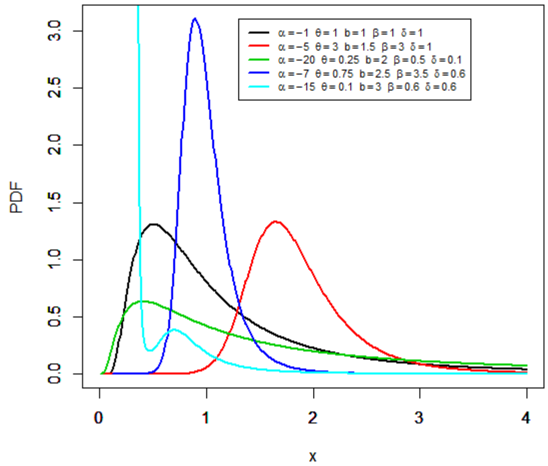
$$\mathbf{h}_{[(1+\tau)\theta+w]b,\beta,\delta}(x) = \beta \left[(1+\tau)\theta + w\right] b\delta^{\beta} x^{-(\beta+1)} \exp\left\{-\left[(1+\tau)\theta + w\right] b\delta^{\beta} x^{-\beta}\right\},$$

is the IWD with scale parameter $\delta \left\{\left[(1+\tau)\theta + w\right] b\right\}^{\beta^{-1}}$ and shape parameter β and

$$\mathbf{h}_{[(1+\tau)\theta+w+1]b,\beta,\delta}(x) = \beta \left[(1+\tau)\theta + w + 1\right] b\delta^{\beta} x^{-(\beta+1)} \exp\left\{-\left[(1+\tau)\theta + w + 1\right] b\delta^{\beta} x^{-\beta}\right\},$$

is the IWD with scale parameter $\delta \left\{\left[(1+\tau)\theta + w + 1\right] b\right\}^{\beta^{-1}}$ and shape parameter β. Via integrating (10), we get

$$F_{[\mathbf{\Phi}]}(x) = \Sigma_{\tau,w=0}^{\infty} \left[a_{\tau,w} \, \mathbf{H}_{[(1+\tau)\theta+w]b,\beta,\delta}(x) - a_{\tau,w}^{\bigstar} \, \mathbf{H}_{[(1+\tau)\theta+w+1]b,\beta,\delta}(x)\right],$$



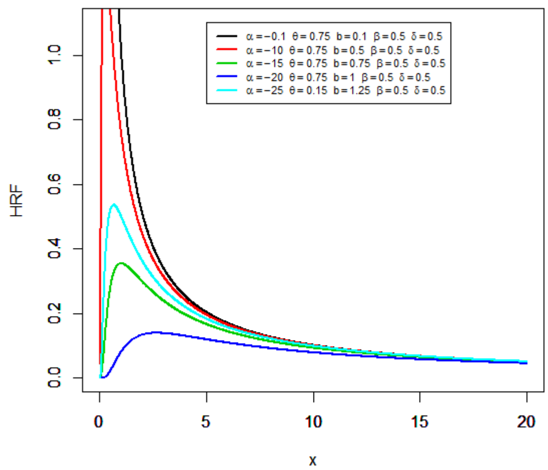


Figure 1. Plots of PTLIWD PDF (left) and hazard rate function (HRF) (right)

where $\mathbf{H}_{[(1+\tau)\theta+w]b,\beta,\delta}(x)$ is the CDF of the IW model with scale parameter $\delta\left\{\left[(1+\tau)\theta+w\right]b\right\}^{\beta^{-1}}$ and shape parameter β and $\mathbf{H}_{[(1+\tau)\theta+w+1]b,\beta,\delta}(x)$ is the CDF of the IW model with scale parameter $\delta\left\{\left[(1+\tau)\theta+w+1\right]b\right\}^{\beta^{-1}}$ and shape parameter β.

The PTLIWD is a suitable for fitting the unimodal and right skewed data sets (see Figure 1(left panel)). The PTLIWD provide adequate fits as compared to other IWDs in both applications with smallest values of AI_C and BI_C.

2. Mathematical Properties

2.1 Moments

The r^{th} non central moment of X is given by

$$\mu'_r = \mathbf{E}(X^r) = \int_{-\infty}^{\infty} f_{[\mathbf{\Phi}]}(x)\, x^r dx,$$

then we have

$$\mu'_r = \delta^r \Gamma\left(1 - r\beta^{-1}\right) \sum_{\tau,w=0}^{\infty} \left(\begin{array}{c} a_{\tau,w} \left\{\left[(1+\tau)\theta+w\right]b\right\}^{r\beta^{-1}} \\ -a^{\bigstar}_{\tau,w} \left\{\left[(1+\tau)\theta+w+1\right]b\right\}^{r\beta^{-1}} \end{array} \right), \forall\, \beta > r, \tag{11}$$

where

$$\Gamma(1+\xi)\,|_{(\xi\in\mathbb{R}^{+})} = \xi! = \Pi_{w=0}^{\xi-1}(\xi - w) = \int_{0}^{\infty} \mathrm{e}^{-t} x^{\xi}\, dt.$$

Setting $r = 1, 2, 3$ and 4 in (11), we have

$$\mu_1' = \delta\Gamma\left(1-\beta^{-1}\right)\sum_{\tau,w=0}^{\infty}\left(\begin{array}{c} a_{\tau,w}\{[(1+\tau)\theta+w]\,b\}^{\beta^{-1}} \\ -a^{\bigstar}_{\tau,w}\{[(1+\tau)\theta+w+1]\,b\}^{\beta^{-1}}\end{array}\right),\ \forall\,\beta>1,$$

which is the mean of X,

$$\mu_2' = \delta^2\Gamma(1-2/\beta)\sum_{\tau,w=0}^{\infty}\left(\begin{array}{c} a_{\tau,w}\{[(1+\tau)\theta+w]\,b\}^{2\beta^{-1}} \\ -a^{\bigstar}_{\tau,w}\{[(1+\tau)\theta+w+1]\,b\}^{2\beta^{-1}}\end{array}\right),\ \forall\,\beta>2,$$

$$\mu_3' = \delta^3\Gamma(1-3/\beta)\sum_{\tau,w=0}^{\infty}\left(\begin{array}{c} a_{\tau,w}\{[(1+\tau)\theta+w]\,b\}^{3\beta^{-1}} \\ -a^{\bigstar}_{\tau,w}\{[(1+\tau)\theta+w+1]\,b\}^{3\beta^{-1}}\end{array}\right),\ \forall\,\beta>3,$$

and

$$\mu_4' = \delta^4\Gamma(1-4/\beta)\sum_{\tau,w=0}^{\infty}\left(\begin{array}{c} a_{\tau,w}\{[(1+\tau)\theta+w]\,b\}^{4\beta^{-1}} \\ -a^{\bigstar}_{\tau,w}\{[(1+\tau)\theta+w+1]\,b\}^{4\beta^{-1}}\end{array}\right),\ \forall\,\beta>4.$$

2.2 Incomplete Moments (IM)

The s^{th} IM $[I_s(t)]$ of X can be derived (10) as

$$\int_{-\infty}^{t} x^s f_{[\mathbf{\Phi}]}(x)\,dx = I_s(t).$$

Then

$$I_s(t) = \delta^s\sum_{\tau,w=0}^{\infty}\left[\begin{array}{c}\left(\begin{array}{c} a_{\tau,w}\{[(1+\tau)\theta+w]\,b\}^{s\beta^{-1}}\times \\ \gamma\left(1-s\beta^{-1},[(1+\tau)\theta+w]\,b\,(\delta/t)^{\beta}\right)\end{array}\right) \\ -\left(\begin{array}{c} a^{\bigstar}_{\tau,w}\{[(1+\tau)\theta+w+1]\,b\}^{s\beta^{-1}}\times \\ \gamma\left(1-s\beta^{-1},[(1+\tau)\theta+w+1]\,b\,(\delta/t)^{\beta}\right)\end{array}\right)\end{array}\right],\ \forall\,\beta>s, \tag{12}$$

where

$$\begin{aligned}\gamma(\xi,\phi)\,|_{(\xi\neq 0.-1,-2,\ldots)} &= \int_0^{\phi} t^{\xi-1}\exp(-t)\,dt \\ &= \{{}_1\mathbf{F}_1[\xi;\xi+1;-\phi]\}\,\frac{\phi^{\xi}}{\xi} \\ &= \sum_{d=0}^{\infty}\frac{(-1)^d}{(\xi+d)\,d!}\phi^{\xi+d}.\end{aligned}$$

2.3 Moments Generating Function (MGF)

The MGF $M_X(t) = \mathbf{E}\left(e^{tX}\right)$ of X can be calculated from equation (10) as

$$M_X(t) = \delta^r\Gamma\left(1-r\beta^{-1}\right)\sum_{\tau,w,r=0}^{\infty}\left(\begin{array}{c} a_{\tau,w,r}\{[(1+\tau)\theta+w]\,b\}^{s\beta^{-1}} \\ -a^{\bigstar}_{\tau,w,r}\{[(1+\tau)\theta+w+1]\,b\}^{r\beta^{-1}}\end{array}\right),\ \forall\,\beta>r,$$

where

$$a_{\tau,w,r} = a_{\tau,w}t^r\,(r!)^{-1},$$

and

$$a^{\bigstar}_{\tau,w,r} = a^{\bigstar}_{\tau,w}t^r\,(r!)^{-1}.$$

2.4 Probability Weighted Moments (PWMs)

The (s,r)th PWM of a rv X following the PTLIWD is derived by

$$\rho_{r,s} = \mathbf{E}\left\{X^s\, F_{\mathbf{\Phi}}(X)^r\right\} = \int_{-\infty}^{\infty} x^s\, F_{[\mathbf{\Phi}]}\left(x\right)^r\; f_{[\mathbf{\Phi}]}\left(x\right)\, dx.$$

Using (5) and (6), we have

$$f_{[\mathbf{\Phi}]}\left(x\right) F_{[\mathbf{\Phi}]}\left(x\right)^r = \sum_{\tau,w=0}^{\infty}\left[v_{\tau,w}\, \mathbf{h}_{[(1+\tau)\theta+w]b,\delta,\beta}(x) - v^{\star}_{\tau,w}\, \mathbf{h}_{[(1+\tau)\theta+w+1]b,\delta,\beta}(x)\right],$$

where

$$v_{\tau,w} = \sum_{d=0}^{\infty}\binom{r}{d}\frac{\theta\alpha^{1+\tau}\left(-1\right)^{\tau+w+d}\left(d+1\right)^{\tau} 2^{(1+\tau)\theta-w}}{\tau!\left(-\mathrm{e}^{-\alpha}+1\right)^{r+1}\left[\left(1+\tau\right)\theta+w\right]b}\binom{-1+\left(1+\tau\right)\theta}{w}$$

and

$$v^{\star}_{\tau,w} = \sum_{d=0}^{\infty}\binom{r}{d}\frac{\theta\alpha^{1+\tau}\left(-1\right)^{\tau+w+d}\left(d+1\right)^{\tau} 2^{(1+\tau)\theta-w}}{\tau!\left(-\mathrm{e}^{-\alpha}+1\right)^{r+1}\left[\left(1+\tau\right)\theta+w+1\right]b}\binom{-1+\left(1+\tau\right)\theta}{w}.$$

The (s,r)th PWM will be

$$\rho_{r,s} = \delta^s\Gamma\left(1-s\beta^{-1}\right)\sum_{\tau,w=0}^{\infty}\left(\begin{array}{c} v_{\tau,w}\left\{\left[\left(1+\tau\right)\theta+w\right]b\right\}^{s\beta^{-1}} \\ -v^{\star}_{\tau,w}\left\{\left[\left(1+\tau\right)\theta+w+1\right]b\right\}^{s\beta^{-1}}\end{array}\right),\ \forall\,\beta > s,$$

2.5 Residual Life and Reversed Residual Life Functions (MRL) & (MRRL)

The n^{th} MRL given as

$$m_n(t) = \mathbf{E}\left[\left(X-t\right)^n\,|_{(X>t,\ n=1,2,\ldots)}\right],$$

Then^{th} MRL of X can derived as

$$m_n(t) = \frac{\int_t^{\infty}(x-t)^n dF_{[\mathbf{\Phi}]}\left(x\right)}{1-F_{\mathbf{\Phi}}\left(t\right)},$$

then

$$m_n(t) = \frac{\delta^n}{1-F_{\mathbf{\Phi}}(t)}\sum_{r=0}^{n}\sum_{\tau,w=0}^{\infty}\left[\begin{array}{c}\left(\begin{array}{c}\frac{p_{\tau,w}}{\{[(1+\tau)\theta+w]b\}^{\frac{-n}{\beta}}}\times \\ \Gamma\left(1-n/\beta,\left[\left(1+\tau\right)\theta+w\right]b\left(\delta/t\right)^{\beta}\right)\end{array}\right) \\ -\left(\begin{array}{c}\frac{p^{\star}_{\tau,w}}{\{[(1+\tau)\theta+w+1]b\}^{\frac{-n}{\beta}}}\times \\ \Gamma\left(1-n/\beta,\left[\left(1+\tau\right)\theta+w+1\right]b\left(\delta/t\right)^{\beta}\right)\end{array}\right)\end{array}\right],\ \forall\,\beta > n,$$

where

$$\begin{aligned}\Gamma\left(\xi,\phi\right)|_{(\phi>0)} &= \int_0^{\phi} t^{\xi-1}\exp\left(-t\right)dt \\ &\sim \frac{\phi^{\xi-1}}{\mathrm{e}^{\phi}}\left[1+\frac{\xi-1}{\phi}+\frac{\left(\xi-1\right)\left(\xi-2\right)}{\phi^2}+\ldots\right],\end{aligned}$$

$$p_{\tau,w} = a_{\tau,w}\binom{n}{r}\left(-t\right)^{n-r},$$

and

$$p^{\star}_{\tau,w} = a^{\star}_{\tau,w}\binom{n}{r}\left(-t\right)^{n-r}.$$

The n^{th} MRRL given as

$$M_n(t) = \mathbf{E}\left[\left(t-X\right)^n\,|_{X\le t,n=1,2,\ldots}^{t>0}\right]$$

we obtain

$$M_n(t) = \frac{\int_0^t(t-x)^n dF_{[\mathbf{\Phi}]}\left(x\right)}{F_{[\mathbf{\Phi}]}\left(x\right)}.$$

Then, the n^{th} MRRL can written as

$$M_n(t) = \frac{\delta^n}{F_{\mathbf{\Phi}}(t)} \sum_{r=0}^{n} \sum_{\tau,w=0}^{\infty} \left[\begin{array}{c} \left(\begin{array}{c} \frac{q_{\tau,w}}{\{[(1+\tau)\theta+w]b\}^{\frac{-n}{\beta}}} \times \\ \gamma\left(1-n/\beta, [(1+\tau)\theta+w]\, b\,(\delta/t)^{\beta}\right) \end{array} \right) \\ -\left(\begin{array}{c} \frac{q^{\bigstar}_{\tau,w}}{\{[(1+\tau)\theta+w+1]b\}^{\frac{-n}{\beta}}} \times \\ \gamma\left(1-n/\beta, [(1+\tau)\theta+w+1]\, b\,(\delta/t)^{\beta}\right) \end{array} \right) \end{array} \right], \ \forall\, \beta > n,$$

where

$$q_{\tau,w} = a_{\tau,w}\,(-1)^r\, t^{n-r} \text{ and } q^{\bigstar}_{\tau,w} = a^{\bigstar}_{\tau,w}\,(-1)^r\, t^{n-r}.$$

2.6 Order Statistics

The PDF of i^{th} order statistic, say $X_{i:n}$, can be

$$f^{(i:n)}_{\mathbf{\Phi}}(x) = \sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j}}{\mathrm{B}(i, n-i+1)} f_{[\mathbf{\Phi}]}(x)\ \ F_{[\mathbf{\Phi}]}(x)^{j+i-1}, \tag{13}$$

where $\mathrm{B}(i, n-i+1)$ is the beta function. using (5), (6) and (13) we get

$$f_{[\mathbf{\Phi}]}(x)\ F_{[\mathbf{\Phi}]}(x)^{j+i-1} = \sum_{w,\tau=0}^{\infty} \left[\zeta_{w,\tau}\, \mathbf{h}_{[\theta(1+w)+\tau]b}(x) - \zeta^{\bigstar}_{w,\tau}\, \mathbf{h}_{[j+(1+i)\theta+1]b}(x)\right],$$

where

$$\zeta_{w,\tau} = \sum_{d=0}^{\infty} \frac{\theta\alpha^{1+w}(-1)^{w+\tau+d}(d+1)^w\, 2^{(1+w)\theta-\tau}}{w!\,(-\mathrm{e}^{-\alpha}+1)^{j+i}\,[(1+w)\,\theta+\tau]\,b} \binom{-1+\theta\,(1+w)}{\tau} \binom{j+i-1}{d},$$

and

$$\zeta^{\bigstar}_{w,\tau} = \sum_{d=0}^{\infty} \frac{\theta\alpha^{1+w}(-1)^{w+\tau+d}(d+1)^w\, 2^{(1+w)\theta-\tau}}{w!\,(-\mathrm{e}^{-\alpha}+1)^{j+i}\,[1+\theta\,(1+w)+\tau]\,b} \binom{-1+\theta\,(1+w)}{\tau} \binom{j+i-1}{d}.$$

The PDF of $X_{i:n}$ will be

$$f^{(i:n)}_{\mathbf{\Phi}}(x) = \sum_{j=0}^{n-i} \sum_{w,\tau=0}^{\infty} \left[\zeta_{w,\tau}\, \mathbf{h}_{[\theta(1+w)+\tau]b}(x) - \zeta^{\bigstar}_{w,\tau}\, \mathbf{h}_{[j+(1+i)\theta+1]b}(x)\right] \frac{(-1)^j \binom{n-i}{j}}{\mathrm{B}(i, n-1+i)}.$$

The moments of $X_{i:n}$ will be

$$\begin{aligned} \mathbf{E}\left(X^q_{i:n}\right) &= \delta^q \Gamma\left(1-q\beta^{-1}\right) \sum_{j=0}^{n-i} \sum_{w,\tau=0}^{\infty} \frac{(-1)^j}{\mathrm{B}(i, n-i+1)} \binom{n-i}{j} \\ &\quad \times \left(\begin{array}{c} \zeta_{w,\tau}\, \{[j+(1+i)\,\theta]\, b\}^{\frac{q}{\beta}} \\ -\zeta^{\bigstar}_{w,\tau}\, \{[j+(1+i)\,\theta+1]\, b\}^{\frac{q}{\beta}} \end{array} \right), \ \forall\, \beta > q, \end{aligned}$$

3. Maximum Likelihood Estimation

The log-likelihood function

$$\begin{aligned} \ell(\mathbf{\Phi}) &= n\log 2 + n\log\theta + n\log\alpha + n\log b + n\log\beta - n\log\left(-\mathrm{e}^{-\alpha}+1\right) + n\beta\log\delta \\ &\quad + (\theta-1)\sum_{i=1}^{n} \log\left(-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+2\right) + \sum_{i=1}^{n} \log\left(-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+1\right) \\ &\quad -\alpha\sum_{i=1}^{n} \left(-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+2\right)^{\theta} \mathrm{e}^{-\theta b\delta^{\beta}x_i^{-\beta}}. \end{aligned}$$

The above $\ell(\mathbf{\Phi})$ can be maximized numerically by using `R` (`optim`). The components of the score vector

$$\partial\ell(\mathbf{\Upsilon})/\partial\mathbf{\Upsilon} = \left(\partial\ell(\mathbf{\Phi})/\partial\alpha, \partial\ell(\mathbf{\Phi})/\partial\theta, \partial\ell(\mathbf{\Phi})/\partial b, \partial\ell(\mathbf{\Phi})/\partial\beta, \partial\ell(\mathbf{\Phi})/\partial\delta\right)^{\intercal} = \mathbf{U}(\mathbf{\Phi}),$$

are

$$\partial\ell(\mathbf{\Phi})/\partial\alpha = \frac{n}{\alpha} - \frac{n\mathrm{e}^{-\alpha}}{-\mathrm{e}^{-\alpha}+1} - \sum_{i=1}^{n}\left(-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+2\right)^{\theta}\mathrm{e}^{-\theta b\delta^{\beta}x_i^{-\beta}},$$

$$\partial\ell(\mathbf{\Phi})/\partial\theta = \frac{n}{\theta} + \sum_{i=1}^{n}\log\left(-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+2\right) + \alpha\sum_{i=1}^{n}-b\delta^{\beta}x_i^{-\beta}\left(-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+2\right)^{\theta}\mathrm{e}^{-\theta b\delta^{\beta}x_i^{-\beta}},$$

$$\begin{aligned}\partial\ell(\mathbf{\Phi})/\partial b \quad = \quad & \frac{n}{b} + (\theta-1)\sum_{i=1}^{n}\frac{\delta^{\beta}x_i^{-\beta}\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}}{-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+2} + \sum_{i=1}^{n}\frac{\delta^{\beta}x_i^{-\beta}\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}}{-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+1} \\ & -\alpha\theta\delta^{\beta}\sum_{i=1}^{n}x_i^{-\beta}\mathrm{e}^{-\theta b\delta^{\beta}x_i^{-\beta}}\left(-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+2\right)^{\theta}\left[\frac{\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}}{-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+2}-1\right],\end{aligned}$$

$$\begin{aligned}\partial\ell(\mathbf{\Phi})/\partial\beta \quad = \quad & \frac{n}{\beta} + n\log\delta + (\theta-1)\sum_{i=1}^{n}\frac{-s_i\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}}{-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+2} + \sum_{i=1}^{n}\frac{-s_i\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}}{-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+1} \\ & -\alpha\sum_{i=1}^{n}\left[\left(-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+2\right)^{\theta}m_i - \theta s_i\left(-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+2\right)^{\theta-1}\mathrm{e}^{-\theta b\delta^{\beta}x_i^{-\beta}}\right]\end{aligned}$$

and

$$\begin{aligned}\partial\ell(\mathbf{\Phi})/\partial\delta \quad = \quad & \frac{n\beta}{\delta} + (\theta-1)\beta b\delta^{\beta-1}\sum_{i=1}^{n}\frac{x_i^{-\beta}\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}}{-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+2} + \beta b\delta^{\beta-1}\sum_{i=1}^{n}\frac{x_i^{-\beta}\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}}{-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+1} \\ & -\alpha\theta b\beta\delta^{\beta-1}\sum_{i=1}^{n}x_i^{-\beta}\mathrm{e}^{-\theta b\delta^{\beta}x_i^{-\beta}}\left(-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+2\right)^{\theta}\left[\frac{\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}}{-\mathrm{e}^{-b\delta^{\beta}x_i^{-\beta}}+2}-1\right]\end{aligned}$$

where

$$s_i = -b\left(\frac{\delta}{x_i}\right)^{\beta}\mathrm{e}^{-b\left(\frac{\delta}{x_i}\right)^{\beta}}\log\left(\frac{\delta}{x_i}\right) \text{ and } m_i = -\theta b\left(\frac{\delta}{x_i}\right)^{\beta}\mathrm{e}^{-\theta b\left(\frac{\delta}{x_i}\right)^{\beta}}\log\left(\frac{\delta}{x_i}\right)$$

4. Data Analysis

4.1 Modeling Survival Times

The 1^{st} data, the data consists of 72 observations and represents the survival times for pigs of Guinea injected with different doses of tubercle bacilli, the data are{24, 146, 99, 38, 48, 52, 67, 68, 70, 146, 175, 175, 72, 58, 127, 129, 70, 87, 91, 95, 58, 59, 60, 60, 12, 211, 131,61, 62, 75, 84, 85, 233, 24, 32, 15, 22, 32, 38, 43, 96, 98, 263, 297, 341, 76, 63, 65, 60, 60, 109, 110, 121, 143, 65, 76, 81, 83, 258, 258, 44, 33, 54, 54, 55, 56, 57, 34, 73, 53, 341 and 376}. We compare the proposed PTLIWD with other related models namely: the KumMO-IW, MOKum-IW, MO-IW, Kum-IW, B-IW, MO Invere Exponential (MO-IE), MO Invere Rayleigh (MO-IR), Exponentiated IW (E-IW) and IW distributions. All PDFs used in the application are available in the literature. Table 1 list the values of $\mathrm{AI_C}$ and $\mathrm{BI_C}$, while the MLEs and their corresponding standard errors (SEs) in parentheses are listed in Tables 2. All results are obtained using the R program. Fitted PDF, PP Plot, and Kaplan-Meier survival plot and estimated HRF for the 1^{st} data are displayed in Figure 3, however Figure 2 gives the total time test (TTT) for 1^{st} data set, this plot indicates that the empirical HRFs of the data set is upside down then increasing.

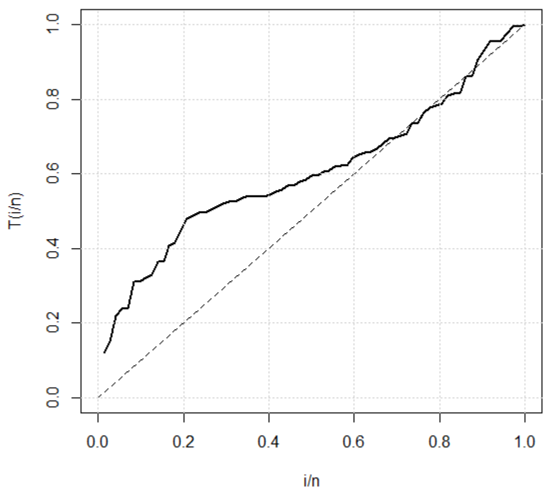


Figure 2. TTT plots for 1^{st} data set

Table 1. The AI_C and BI_C statistics for the survival times for Guinea pigs

Model	AI_C	BI_C
PTLIWD	**764.8**	**776.2**
E-IWD	786.5	793.3
Kum-IWD	788.5	797.6
ZB-IWD	787.2	794.1
B-IWD	788.6	797.7
KumMO-IED	790.7	799.8
MOKum-IWD	794.2	805.6
IWD	795.3	799.9
MO-IWD	796.1	802.9
KumMO-IRD	808.2	817.3

Table 2. MLEs and their standard errors (in parentheses) for the survival times

$\text{PTLIWD}_{(\alpha,\theta,b,\beta,\delta)}$	−7.109	2.506	1.218	0.863	11.02
	(0.000)	(0.000)	(0.219)	(0.0255)	(1.87)
$\text{EIWD}_{(b,\beta,\delta)}$			8.2723	0.6207	336.3679
			(7.953)	(0.208)	(374.803)
$\text{Kum-IWD}_{(a,b,\beta,\delta)}$		45.7326	8.2723	0.6207	0.7111
		(0.092)	(0.979)	(0.003)	(0.013)
$\text{ZB-IWD}_{(a,\beta,\delta)}$		26.048		1.537	6.638
		(0.597)		(0.008)	(0.007)
$\text{B-IWD}_{(a,b,\beta,\delta)}$		19.9786	20.1331	0.322	24.5032
		(7.246)	(7.26)	(0.00115)	(0.087)
$\text{KumMO-IED}_{(a,b,\alpha,\delta)}$	8.8727	68.1393	2.6258	–	0.1758
	(1.174)	(0.020)	(0.512)	–	(0.000)
$\text{MOKum-IWD}_{(\alpha,a,b,\beta,\delta)}$	0.449	22.880	1.376	2.666	0.449
	(0.021)	(3.338)	(0.087)	(0.869)	(0.021)
$\text{IWD}_{(\beta,\delta)}$				1.4148	54.1888
				(0.00271)	(0.111)
$\text{MO-IWD}_{(\alpha,\beta,\delta)}$	14.9816			1.7855	13.991
	(4.6305)			(0.193)	(2.964)
$\text{KumMO-IRD}_{(a,b,\alpha,\delta)}$	9.993	58.4697	0.6389		1.6788
	(1.972)	(0.105)	(0.098)		(0.001)

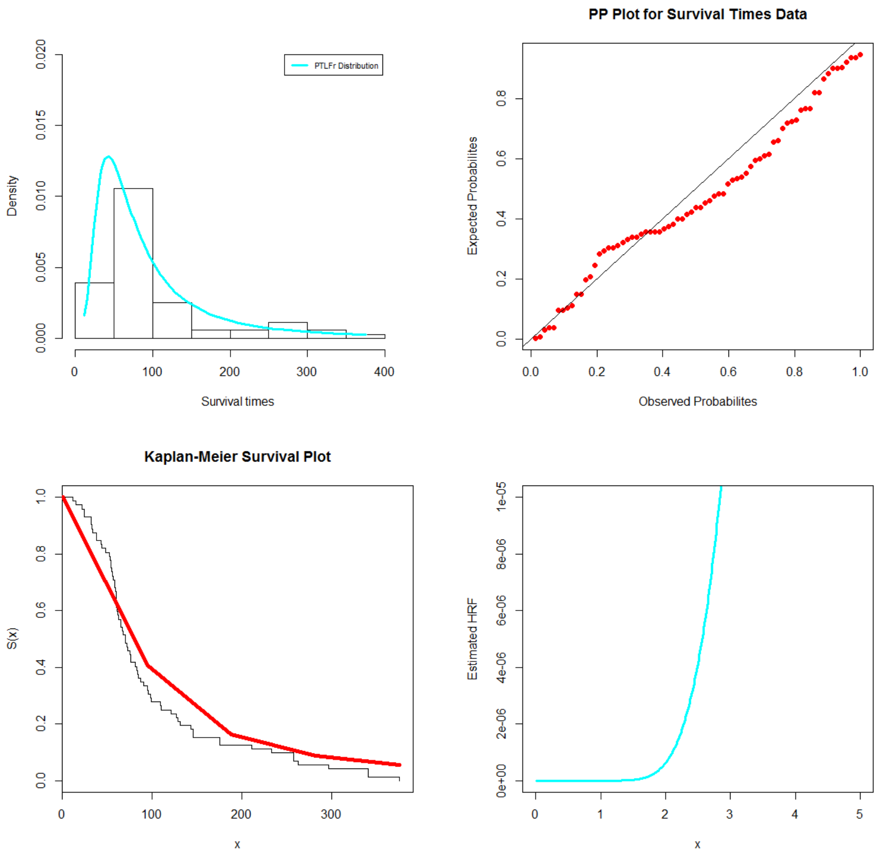


Figure 3. Fitted PDF, PP Plot, and Kaplan-Meier survival plot and estimated HRF for the 1^{st} data

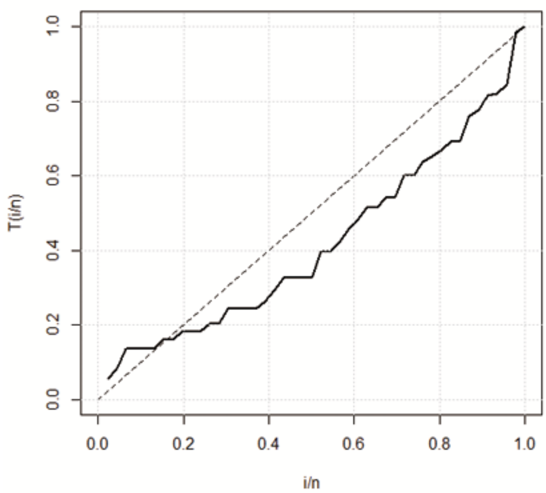


Figure 4. TTT plots for 1^{st} data set

4.2 Modeling Repair Times

The 2^{nd} data called the repair times data (see Krishna et al. (2013)), this data set are: 0.20, 0.50, 3.3, 5.4, 5.4, 1.5,4.0, 1.0,2.5, 2.7, 0.5, 0.60, 0.7, 0.80, 3.0, 3.3, 0.70, 1.0, 1.30, 2.0, 0.8, 0.6, 0.7, 0.3, 0.50, 2.0, 2.2, 1.50, 3.0, 1.5, 1.5, 5.0, 7.50, 8.80, 1.0, 1.0, 1.1,0.5, 9.0, 4.0, 4.50, 4.70, 7.0, 10.3, 22.0 and 24.5. We compare the proposed PTLIWD with other related models: Topp Leone Generated IW (TLG-IW) , IW, Kum-IW, Exponentiated IW (E-IW), B-IW, transmuted IW (T-IW), MOIW and Mcdonald IW (Mac-IW) distributions. Tables 3 list the values of AI_C and BI_C , while the MLEs and their SEs of the model parameters are listed in Tables 4. Fitted PDF, PP Plot, and Kaplan-Meier survival plot and estimated HRF for the 1^{st} data are displayed in Figure 5, however Figure 4 gives the TTT for 2^{nd} data set, this plot indicates that the empirical HRFs of the data set isupside down shaped.

Table 3. The AI_C and BI_C statistics for the repair times data

1-9 Model	AI_C	BI_C
PTLIWD	**146.05**	**155.19**
TLG-IWD	207.2	214.5
IWD	207.4	215.0
Kum-IWD	207.4	214.6
E-IWD	207.4	214.9
B-IWD	207.4	214.7
T-IWD	207.8	215.3
MO-IWD	207.9	214.7
Mac-IWD	207.8	216.9

Table 4. MLEs and their standard errors (in parentheses) for the repair times

Model	Estimates				
$PTLIWD_{(\alpha,\theta,b,\beta,\delta)}$	−1.63	0.0073	6.806	1.1227	10.535
	(2.01)	(0.0055)	(18.455)	(0.141)	(25.431)
$TLG\text{-}IWD_{(a,b,\beta,\delta)}$	0.1405	2.1672		0.8958	4.9552
	(0.2299)	(20.072)		(0.1675)	(51.257)
$IWD_{(\beta,\delta)}$				1.0128	1.1297
				(0.1129)	(0.1740)
$Kum\text{-}IWD_{(a,b,\beta,\delta)}$	1.1619	3.8034		0.5401	4.0226
	(7.452)	(4.604)		(0.2753)	(47.459)
$E\text{-}IWD_{(a,\beta,\delta)}$	0.9881			1.0125	1.1433
	(23.679)			(0.1129)	(27.057)
$B\text{-}IWD_{(a,b,\beta,\delta)}$	2.3521	5.8362		0.4147	3.4905
	(8.581)	(14.877)		(0.5619)	(13.461)
$T\text{-}IWD_{(a,\beta,\delta)}$	−0.6364			1.0853	0.7747
	(0.1173)			(0.1226)	(0.3633)
$MO\text{-}IWD_{(a,\beta,\delta)}$	4.9168			1.3384	0.5066
	(6.1834)			(0.2574)	(0.3068)
$Mac\text{-}IWD_{(a,b,\alpha,\beta,\delta)}$	0.0125	96.427	0.8957	12.281	10.570
	(0.0108)	(354.85)	(0.1297)	(45.502)	(35.124)

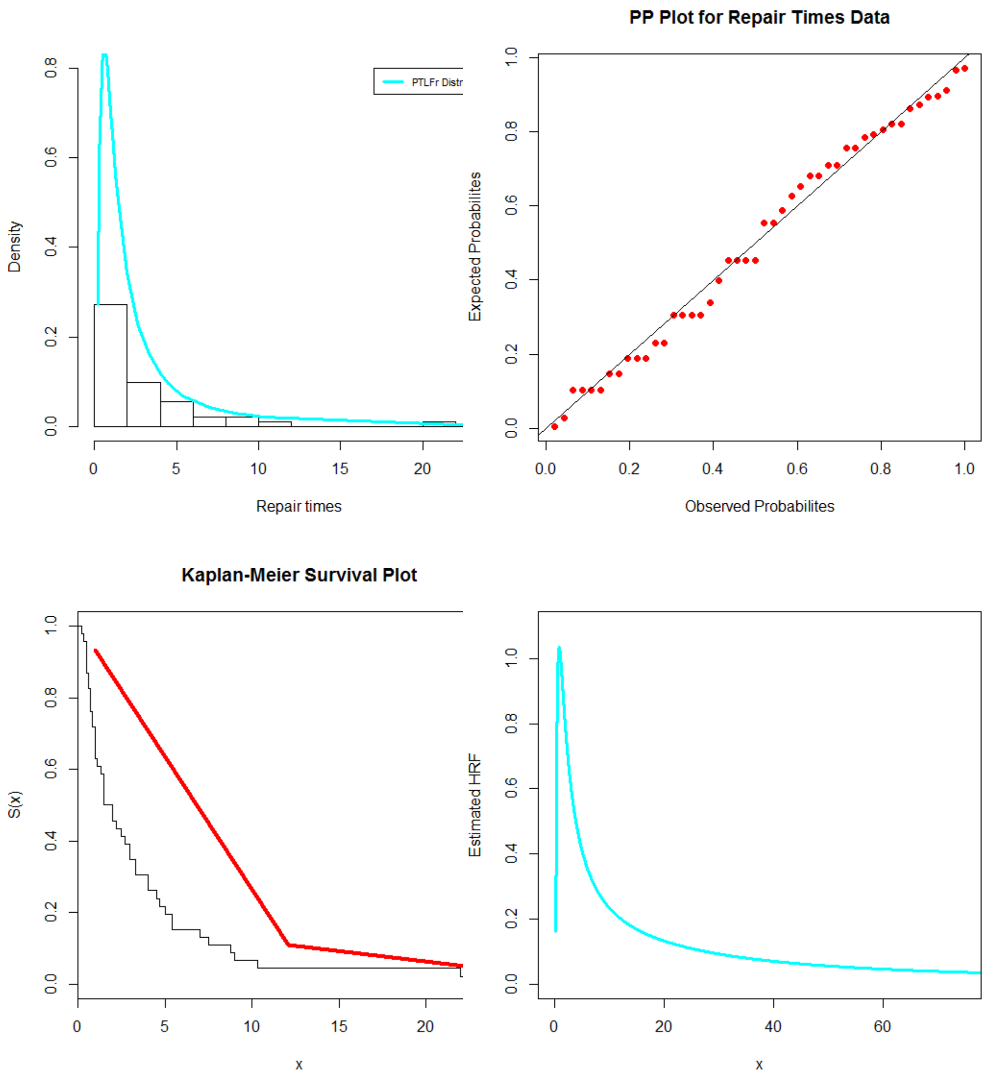


Figure 5. Fitted PDF, PP Plot, and Kaplan-Meier survival plot and estimated HRF for the 2^{nd} data

Based on values displayed in Tables 1 and 3 we conclude that: the PTLIWD provide sufficient fits as compared to other IW extensions with the smallest values of AI_C and BI_C in both applications. From application 1, the PTLIWD is much better than the MO-IWD, MO-IED, Kum-IWD, B-IWD, MO-IRD, E-IWD and IWD. From application 2, the PTLIWD is much better than the IWD, TLG-IWD , Kum-IWD, E-IWD, T-IWD, B-IWD, MO-WD and Mac-IWD. The PTLIWD is suitable model for modeling right skewed data sets.

5. Conclusions

A new extension of the Poisson Inverse Weibull distribution is derived and studied in details. Number of structural mathematical properties are derived. We used the well-known maximum likelihood method for estimating the model parameters. The new model is applied for modeling some real data sets to prove its importance and flexibility empirically. We also conclude that:

1-The PTLIWD provide sufficient fits as compared to other IW extensions with the smallest values of AI_C and BI_C in both applications. From application 1, the PTLIWD is much better than the MO-IWD, B-IWD, Kum-IWD, MO-IRD, MO-IED, E-IWD and IWD.

2- From application 2, the PTLIWD is much better than the IWD, TLG-IWD , Kum-IWD, E-IWD, T-IWD, B-IWD, MO-WD and Mac-IWD.

3-The PTLIWD is suitable model for modeling right skewed data sets.

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