

The Beta Log-logistic Weibull Distribution: Model, Properties and Application

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Abstract

We propose and develop the properties of a new generalized distribution called the beta log-logistic Weibull (BLLoGW) distribution. This model contains several new distributions such as beta log-logistic Rayleigh, beta log-logistic exponential, exponentiated log-logistic Weibull, exponentiated log-logistic Rayleigh, exponentiated log-logistic exponential, log-logistic Weibull, log-logistic Rayleigh and log-logistic distributions as special cases. Structural properties of this generalized distribution including series expansion of the probability density function and cumulative distribution function, hazard function, reverse hazard function, quantile function, moments, conditional moments, mean deviations, Bonferroni and Lorenz curves, Rényi entropy and distribution of order statistics are presented. The parameters of the distribution are estimated using maximum likelihood estimation technique. A Monte Carlo simulation study is conducted to examine the bias and mean square error of the maximum likelihood estimates. A real dataset is used to illustrate the applicability and usefulness of the new generalized distribution.

Keywords: generalized distributions, beta distribution, Weibull Distribution, maximum likelihood estimation

1. Introduction

Generalized distributions are of tremendous practical importance and have received considerable attention by many authors in recent years. Eugene, Lee and Famoye (2002) introduced a generalized beta distribution and presented results on the beta normal distribution. The work presented by Jones (2004) is also related to the beta-generated family. Recently, Barreto-Souza, Santos and Cordeiro (2010) introduced the beta generalized exponential distribution where they derived mathematical properties of the distribution. Also, Pescim et al. (2010) studied the beta generalized half-normal distribution. Nadarajah, Cordeiro & Ortega (2012) presented general results on the beta modified Weibull distribution. Famoye, Lee & Olumolade (2005) studied beta Weibull distribution and showed that the lifetime distribution is unimodal and Cordeiro and Nadarajah (2011) extended the beta Weibull distribution by beta exponentiated Weibull distribution. There are several generalizations of the log-logistic distribution including beta log-logistic distribution, presented by Lemonte (2014) and the log-logistic Weibull distribution by Oluyede et al. (2016).

The primary motivation for the development of the beta log-logistic Weibull distribution is the modeling of lifetime data and other data types with a diverse model that takes into consideration not only shape, and scale but also skewness, kurtosis and tail variation. Also, motivated by various applications of log-logistic, Weibull and beta distributions in several areas including reliability, exponential tilting (weighting) in finance and actuarial sciences, as well as economics, where log-logistic distribution plays an important role in income, we construct and develop the statistical properties of this new class of generalized distribution called the beta log-logistic Weibull distribution and apply it to real lifetime data in order to demonstrate the usefulness of the proposed distribution.

In this paper, the results are organized in the following manner. The beta log-logistic Weibull (BLLoGW) distribution, its sub-models, quantile function, hazard and reverse hazard functions are given in section 2. In section 3, moments, moment generating function and conditional moments are presented. Mean deviations, Lorenz and Bonferroni curves are given in section 4. Section 5 contains results on Rényi entropy, density of the order statistics and L-moments. Maximum likelihood estimates of the model parameters are given in section 6. A Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimates are presented in section 7. Section 8 contains an application of the new model to real data set. A short conclusion is given in section 9.

2. The Model Definition

If $G(x)$ denote the cumulative distribution function of a random variable X , then a generalized class of distribution can be defined by

$$F(x) = I_{G(x)}(a, b) = \frac{1}{B(a, b)} \int_0^{G(x)} t^{a-1} (1-t)^{b-1} dt, \tag{1}$$

where the parameters a and b are shape parameters, $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ denote the beta function and $\Gamma(\cdot)$ is the gamma function. The corresponding probability density function (pdf) is given by

$$f(x) = \frac{1}{B(a, b)} [G(x)]^{a-1} [1-G(x)]^{b-1} g(x), \tag{2}$$

where $g(x)$ is the pdf of the baseline cumulative distribution function (cdf) $G(x)$.

2.1 Beta Log-Logistic Weibull Distribution

In this subsection, the model is presented. A series expansion of the BLLoGW cdf is given. The new BLLoGW cdf is given by

$$\begin{aligned} F_{BLLoGW}(x; c, a, b, \alpha, \beta) &= \frac{1}{B(a, b)} \int_0^{G_{LLoGW}(x; c, \alpha, \beta)} t^{a-1} (1-t)^{b-1} dt \\ &= I_{G_{LLoGW}(x; c, \alpha, \beta)}(a, b), \end{aligned} \tag{3}$$

where $G_{LLoGW}(x; c, \alpha, \beta) = 1 - (1+x^c)^{-1} e^{-\alpha x^\beta}$ is the log-logistic Weibull cdf (Oluyede et al., 2016) with parameters c, α and $\beta > 0$. If $|t| < 1$ and $b > 0$ is real non-integer, we apply series representation

$$(1-t)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)j!} t^j, \tag{4}$$

to the BLLoGW cdf to obtain

$$\begin{aligned} F_{BLLoGW}(x; c, a, b, \alpha, \beta) &= \frac{1}{B(a, b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)j!} \int_0^{[1-(1+x^c)^{-1} e^{-\alpha x^\beta}]} t^{a+j-1} dt \\ &= \frac{1}{B(a, b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b) [1 - (1+x^c)^{-1} e^{-\alpha x^\beta}]^{a+j}}{\Gamma(b-j)j!(a+j)} \\ &= \sum_{j=0}^{\infty} w(j, a, b) G_{ELLoGW}(x), \end{aligned}$$

for $x > 0, c > 0, a > 0, b > 0, \alpha > 0, \beta > 0$, where $G_{ELLoGW}(x)$ is the exponentiated log-logistic Weibull cdf with the exponentiated parameter $a+j > 0$. The corresponding pdf from equation (2) is given by

$$\begin{aligned} f_{BLLoGW}(x; c, a, b, \alpha, \beta) &= \frac{1}{B(a, b)} [1 - (1+x^c)^{-1} e^{-\alpha x^\beta}]^{a-1} [(1+x^c)^{-1} e^{-\alpha x^\beta}]^b \\ &\times [cx^{c-1} (1+x^c)^{-1} + \alpha \beta x^{\beta-1}] \end{aligned} \tag{5}$$

for $x > 0, c > 0, a > 0, b > 0, \alpha > 0, \beta > 0$.

2.2 Expansion of Density Function

In this section, the expansion of the BLLoGW pdf is presented. Applying equation (4) to the expression

$$\left(1 - (1+x^c)^{-1} e^{-\alpha x^\beta}\right)^{a-1} = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(a)}{i! \Gamma(a-i)} (1+x^c)^{-i} e^{-\alpha(i)x^\beta}, \tag{6}$$

the BLLoGW pdf can be written as

$$f_{BLLoGW}(x; c, a, b, \alpha, \beta) = \frac{1}{B(a, b)} \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(a)}{i! \Gamma(a-i)} (1+x^c)^{-(i+b)} e^{-\alpha(i+b)x^\beta} [cx^{c-1} (1+x^c)^{-1} + \alpha \beta x^{\beta-1}], \tag{7}$$

for $x > 0, c > 0, a > 0, b > 0, \alpha > 0, \beta > 0$, where parameter α control the scale of the distribution and the parameters c, β, a , and b control the shape, skewness and tail weight of the distribution.

Consequently, the BLLoGW pdf can be written as

$$f_{BLLoGW}(x; c, a, b, \alpha, \beta) = \sum_{i=0}^{\infty} w(i, a, b)g(x; c, (i + b), \alpha(i + b), \beta), \tag{8}$$

where

$$w(i, a, b) = \frac{(-1)^i \Gamma(a)}{i! \Gamma(a - i) B(a, b) (i + b)} \tag{9}$$

and $g(x; c, (i + b), \alpha(i + b), \beta)$ is the pdf of the Burr XII Weibull distribution with parameters $c, (i + b), \alpha(i + b), \beta > 0$. Plots of the BLLoGW pdf for selected values of the model parameters are given in Figure 1. The graphs of the BLLoGW

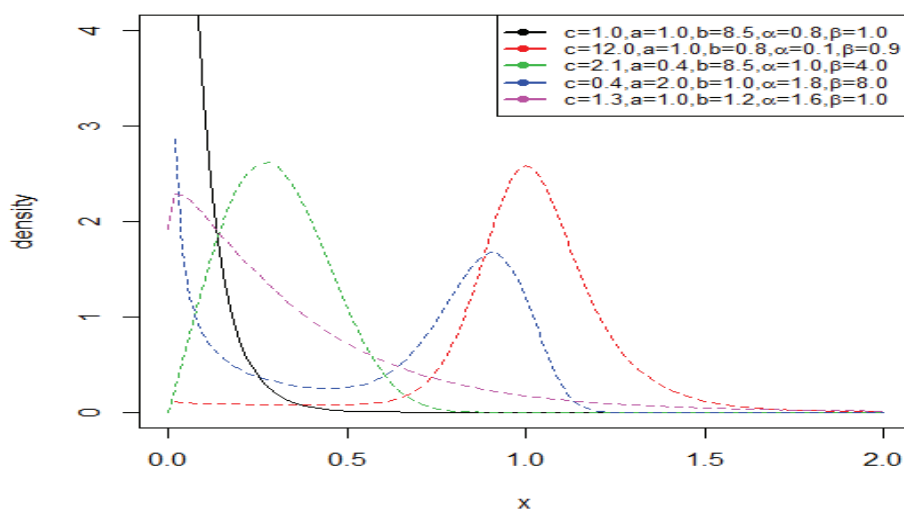


Figure 1. Plots of the BLLoGW Density Function

distribution can be increasing and decreasing L-shaped, uni-modal and decreasing-increasing-decreasing among many potential shapes. The survival or reliability function of the BLLoGW distribution is given by

$$\bar{F}_{BLLoGW}(x; c, a, b, \alpha, \beta) = 1 - I_{G_{LLoGW}(x;c,\alpha,\beta)}(a, b). \tag{10}$$

2.3 Hazard and Reverse Hazard Functions

The hazard and reverse hazard functions of the BLLoGW distribution are presented in this subsection. The hazard rate and reverse hazard functions of the BLLoGW distribution are given by

$$h_F(x) = \frac{f_{BLLoGW}(x)}{\bar{F}_{BLLoGW}(x)} = \frac{[1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}]^{a-1} [(1 + x^c)^{-1} e^{-\alpha x^\beta}]^b}{B(a, b) \left(1 - I_{[1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}]}(a, b)\right)} [cx^{c-1} (1 + x^c)^{-1} + \alpha \beta x^{\beta-1}], \tag{11}$$

and

$$\tau_F(x) = \frac{f_{BLLoGW}(x)}{F_{BLLoGW}(x)} = \frac{[1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}]^{a-1} [(1 + x^c)^{-1} e^{-\alpha x^\beta}]^b}{B(a, b) I_{[1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}]}(a, b)} [cx^{c-1} (1 + x^c)^{-1} + \alpha \beta x^{\beta-1}], \tag{12}$$

respectively. Graphs of the hazard function are presented in Figure 2. The graphs show different shapes including monotonically decreasing and increasing, bathtub followed by upside-down bathtub and upside-down bathtub shapes. The BLLoGW distribution is flexible to accommodate both monotonic and non-monotonic hazard behaviors that are likely to be encountered when dealing with lifetime and reliability data.

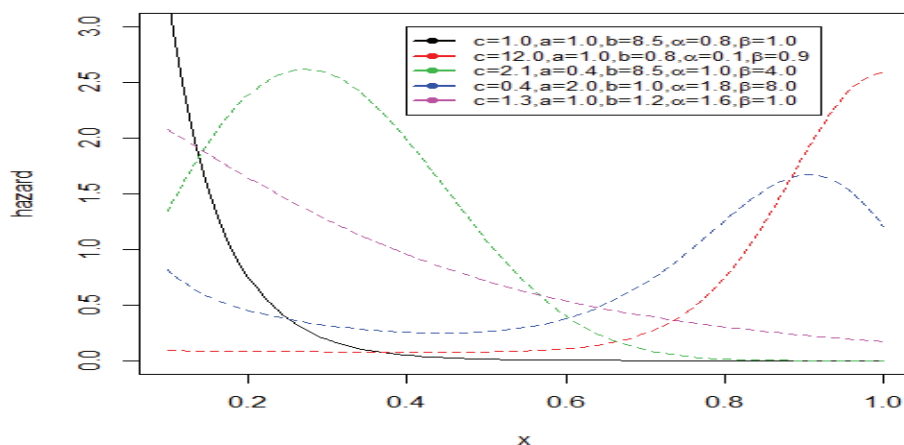


Figure 2. Plots of the BLLoGW Hazard Function

2.4 Some Sub-models

In this subsection, the nested-models of the BLLoGW distribution are presented.

- When $\beta = 1$, we obtain beta log-logistic exponential (BLLoGE) distribution. The cdf of BLLoGE distribution is given by

$$F_{BLLoGE}(x; c, a, b, \alpha) = \frac{1}{B(a, b)} \int_0^{G_{BLLoGE}(x; c, \alpha)} t^{a-1} (1-t)^{b-1} dt, \quad c, a, b, \alpha > 0. \tag{13}$$

- When $\beta = 2$, we obtain beta log-logistic Rayleigh (BLLoGR) distribution. The cdf of BLLoGR distribution is given by

$$F_{BLLoGR}(x; c, a, b, \alpha) = \frac{1}{B(a, b)} \int_0^{G_{BLLoGR}(x; c, \alpha)} t^{a-1} (1-t)^{b-1} dt, \quad c, a, b, \alpha > 0. \tag{14}$$

- When $\alpha \rightarrow 0^+$, we obtain beta log-logistic (BLLoG) distribution. The cdf of BLLoG distribution is given by

$$F_{BLLoG}(x; c, a, b) = \frac{1}{B(a, b)} \int_0^{G_{BLLoG}(x; c)} t^{a-1} (1-t)^{b-1} dt, \quad c, a, b > 0. \tag{15}$$

- When $b = 1$, the BLLoGW distribution becomes the exponentiated log-logistic Weibull (ELLoGW) distribution. The cdf of ELLoGW distribution is given by

$$F_{ELLoGW}(x; c, a, \alpha, \beta) = \left[1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} \right]^a, \quad c, a, \alpha, \beta > 0. \tag{16}$$

- When $b = 1$ and $\beta = 1$, the BLLoGW distribution becomes the exponentiated log-logistic exponential (ELLoGE) distribution and the cdf is given by

$$F_{ELLoGE}(x; c, a, \alpha) = \left[1 - (1 + x^c)^{-1} e^{-\alpha x} \right]^a, \quad c, a, \alpha > 0. \tag{17}$$

- When $\alpha \rightarrow 0^+$ and $b = 1$, the BLLoGW distribution becomes the exponentiated log-logistic (ELLoG) distribution. The cdf of ELLoG distribution is given by

$$F_{ELLoG}(x; c, a) = \left[1 - (1 + x^c)^{-1} \right]^a, \quad c, a > 0. \tag{18}$$

- When $a = 1$, we obtain Lehmann type-II log-logistic Weibull (LELLoGW) distribution. The cdf of LELLoGW distribution is given by

$$F_{LELLoGW}(x; c, b, \alpha, \beta) = 1 - \left[(1 + x^c)^{-1} e^{-\alpha x^\beta} \right]^b, \quad c, b, \alpha, \beta > 0. \tag{19}$$

- When $a = 1$ and $\beta = 1$, we obtain Lehmann type-II log-logistic exponential (LELLOGE) distribution. The cdf is given by

$$F_{LELLOGE}(x; c, b, \alpha) = 1 - \left[(1 + x^c)^{-1} e^{-\alpha x} \right]^b, \quad c, b, \alpha > 0. \tag{20}$$

- When $a = 1$ and $\beta = 2$, we obtain Lehmann type-II log-logistic Rayleigh (LELLOGR) distribution. The cdf is given by

$$F_{LELLOGR}(x; c, b, \alpha) = 1 - \left[(1 + x^c)^{-1} e^{-\alpha x^2} \right]^b, \quad c, b, \alpha > 0. \tag{21}$$

- When $\alpha \rightarrow 0^+$, and $a=1$, we obtain Lehmann type-II log-logistic (LELLOG) distribution. The cdf is given by

$$F_{LELLOG}(x; c, b,) = 1 - \left[(1 + x^c)^{-1} \right]^b, \quad c, a, b > 0.$$

- When $a = b = 1$, the BLLoGW cdf reduces to the log-logistic Weibull (LLoGW) distribution. The cdf is given by

$$F_{LLoGW}(x; c, \alpha, \beta) = 1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}, \quad c, \alpha, \beta > 0. \tag{22}$$

- When $a = b = \beta = 1$, the BLLoGW cdf reduces to the log-logistic exponential (LLoGE) cdf and is given by

$$F_{LLoGE}(x; c, \alpha) = 1 - (1 + x^c)^{-1} e^{-\alpha x}, \quad c, \alpha > 0. \tag{23}$$

- When $a = b = 1$ and $\alpha \rightarrow 0^+$, the BLLoGW cdf reduces to the log-logistic (LLoG) cdf and is given by

$$F_{LLoG}(x; c) = 1 - (1 + x^c)^{-1}, \quad c > 0. \tag{24}$$

2.5 Quantile Function

To obtain the quantile function of the EBW distribution, we can invert the following equation: $F_{BLLoGW}(x) = u, 0 \leq u \leq 1$, that is,

$$I_{1-(1+x^c)^{-1} e^{-\alpha x^\beta}}(a, b) = u, \quad \text{and} \quad 1 - (1 + x^c)^{-1} e^{-\alpha x^\beta} = I_u^{-1}(a, b).$$

Consequently, we obtain the quantile function of the BLLoGW distribution by solving the equation (25) using numerical methods.

$$\log(1 + x^c) + \alpha x^\beta + \log(1 - I_u^{-1}(a, b)) = 0, \tag{25}$$

Consequently, random number can be generated based on equation (25). The quantile for selected values of the BLLoGW distribution parameters are listed in Table 1.

Table 1. BLLoGW quantile for selected values

u	(c, a, b, α, β)				
	(0.8,0.5,5.0,7.0,4.0)	(1.0,3.0,2.5,5.0,1.8)	(1.2,2.0,3.0,3.0,0.6)	(2.2,4.5,1.0,3.5,5.5)	(0.9,10.0,12.0,0.9,4.0)
0.1	0.0157	0.2224	0.7348	0.1195	0.5233
0.2	0.0379	0.5026	0.7865	0.1567	0.5909
0.3	0.0656	0.6677	0.8205	0.1885	0.6400
0.4	0.1005	0.7299	0.8473	0.2193	0.6821
0.5	0.1442	0.7685	0.8707	0.2514	0.7215
0.6	0.2009	0.7983	0.8924	0.2868	0.7609
0.7	0.2784	0.8247	0.9139	0.3287	0.8033
0.8	0.3945	0.8509	0.9373	0.3831	0.8527
0.9	0.6077	0.8822	0.9667	0.4686	0.9210

3. Moments and Conditional Moments

In this section, we present moments and conditional moments of the BLLoGW distribution.

Moments

Moments are used to understand different characteristics of a distribution such as skewness, kurtosis, central tendency and dispersion.

Theorem: The r^{th} non-central moment of the BLLoGW distribution is given by

$$E(X^r) = \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+m} \Gamma(a) [\alpha(i+b)]^m}{B(a,b) \Gamma(a-i) i! m! (i+b)} (i+b) B\left(i+b - \frac{r}{c} - \frac{m\beta}{c}, \frac{r+m\beta+c}{c}\right) + \frac{\alpha(i+b)\beta}{c} B\left(i+b - \frac{r}{c} - \frac{m\beta}{c} - \frac{\beta}{c}, \frac{r+m\beta+\beta}{c}\right).$$

Proof: Now, we apply the expansion $e^x = \sum_{p=0}^{\infty} \frac{x^p}{p!}$ to obtain

$$\begin{aligned} E(X^r) &= \int_0^{\infty} x^r f_{BLLoGW}(x) dx \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(a)}{i! \Gamma(a-i) B(a,b) (i+b)} \int_0^{\infty} x^r (1+x^c)^{-(i+b)} e^{-\alpha(i+b)x^\beta} \\ &\times [c x^{c-1} (1+x^c)^{-1} (i+b) + \alpha(i+b)\beta x^{\beta-1}] dx \\ &= \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+m} \Gamma(a) [\alpha(i+b)]^m}{B(a,b) \Gamma(a-i) i! m! (i+b)} \int_0^{\infty} x^{r+m\beta} (1+x^c)^{-(i+b)} \\ &\times [c x^{c-1} (1+x^c)^{-1} (i+b) + \alpha(i+b)\beta x^{\beta-1}] dx \\ &= \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+m} \Gamma(a) [\alpha(i+b)]^m}{B(a,b) \Gamma(a-i) i! m! (i+b)} \left[c(i+b) \int_0^{\infty} x^{r+m\beta+c-1} (1+x^c)^{-(i+b+1)} dx \right. \\ &\left. + \alpha(i+b)\beta \int_0^{\infty} x^{r+m\beta+\beta-1} (1+x^c)^{-(i+b)} dx \right]. \end{aligned}$$

Let $y = (1+x^c)^{-1}$, then $x = \left(\frac{1-y}{y}\right)^{\frac{1}{c}}$ and $dx = -\frac{y^{-2}(1-y)^{\frac{1}{c}-1} y^{1-\frac{1}{c}}}{c} dy$, so that

$$\begin{aligned} E(X^r) &= \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+m} \Gamma(a) [\alpha(i+b)]^m}{B(a,b) \Gamma(a-i) i! m! (i+b)} \left[c(i+b) \int_0^1 \left[\left(\frac{1-y}{y}\right)^{\frac{1}{c}}\right]^{r+m\beta+c-1} y^{i+b+1} \frac{y^{-2}(1-y)^{\frac{1}{c}-1} y^{1-\frac{1}{c}}}{c} dy \right. \\ &\left. + \alpha(i+b)\beta \int_0^1 \left[\left(\frac{1-y}{y}\right)^{\frac{1}{c}}\right]^{r+m\beta+\beta-1} y^{i+b} \frac{y^{-2}(1-y)^{\frac{1}{c}-1} y^{1-\frac{1}{c}}}{c} dy \right] \\ &= \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+m} \Gamma(a) [\alpha(i+b)]^m}{B(a,b) \Gamma(a-i) i! m! (i+b)} \left[(i+b) \int_0^1 y^{i+b-\frac{r+m\beta+c}{c}-1} (1-y)^{\frac{r+m\beta+c}{c}-1} dy \right. \\ &\left. + \frac{\alpha(i+b)\beta}{c} \int_0^1 y^{i+b-\frac{r+m\beta+\beta}{c}-1} (1-y)^{\frac{r+m\beta+\beta}{c}-1} dy \right] \\ &= \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+m} \Gamma(a) [\alpha(i+b)]^m}{B(a,b) \Gamma(a-i) i! m! (i+b)} (i+b) B\left(i+b - \frac{r}{c} - \frac{m\beta}{c}, \frac{r+m\beta+c}{c}\right) \\ &\left. + \frac{\alpha(i+b)\beta}{c} B\left(i+b - \frac{r}{c} - \frac{m\beta}{c} - \frac{\beta}{c}, \frac{r+m\beta+\beta}{c}\right), \right. \end{aligned}$$

where $B(a,b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt$ is the beta function.

The first six moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) for some selected parameters values of the BLLoGW distribution are given in Table 2. The variance (σ^2), (SD (σ)), CV, CS and CK are given by

$$\sigma^2 = \mu'_2 - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu'_2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1},$$

$$CS = \frac{E[(X - \mu)^3]}{[E(X - \mu)^2]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}}, \quad \text{and} \quad CK = \frac{E[(X - \mu)^4]}{[E(X - \mu)^2]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2},$$

respectively.

Table 2. BLLoGW Moments for selected parameter values

Moments	(c, a, b, α, β)				
	(8.0,1.0,1.0,2.2,0.9)	(6.5,1.0,1.0,4.5,0.8)	(5.0,1.0,1.0,2.5,0.6)	(4.0,1.0,1.0,2.0,0.9)	(3.0,1.0,1.0,5.0,0.3)
EX	0.3774	0.1685	0.2562	0.3992	0.0273
EX2	0.2507	0.0677	0.1693	0.2839	0.0114
EX3	0.2094	0.0408	0.1568	0.2686	0.0107
EX4	0.1989	0.0309	0.1794	0.3130	0.0169
EX5	0.2070	0.0274	0.2479	0.4393	0.0468
EX6	0.2331	0.0272	0.4229	0.7419	0.2651
SD	0.3291	0.1982	0.3219	0.3528	0.1029
CV	0.8722	1.1759	1.2562	0.8838	3.7734
CS	0.9274	2.0684	1.8105	1.2734	8.9839
CK	3.0807	8.1493	6.7484	5.1151	140.6982

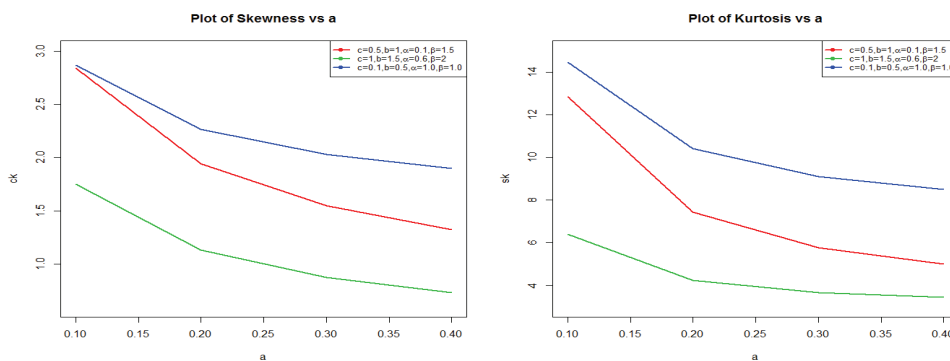


Figure 3. Plot of Skewness and Kurtosis for selected parameter values of the BLLoGW distribution

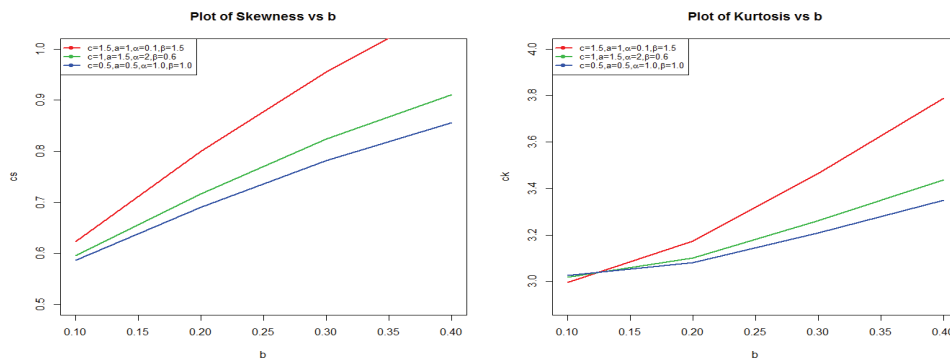


Figure 4. Plot of Skewness and Kurtosis for selected parameter values of the BLLoGW distribution

Plots of skewness and kurtosis of the BLLoGW distribution as a function of different model shape parameters are presented in Figures 3, 4, 5, and 6. The plots shows the dependence of the kurtosis and skewness measures of the shape parameter a, b, c and β , respectively. Note that plot of the skewness and kurtosis decreases as the parameters a and β increase, while the plot of the skewness and kurtosis increase as parameters b and c increase.

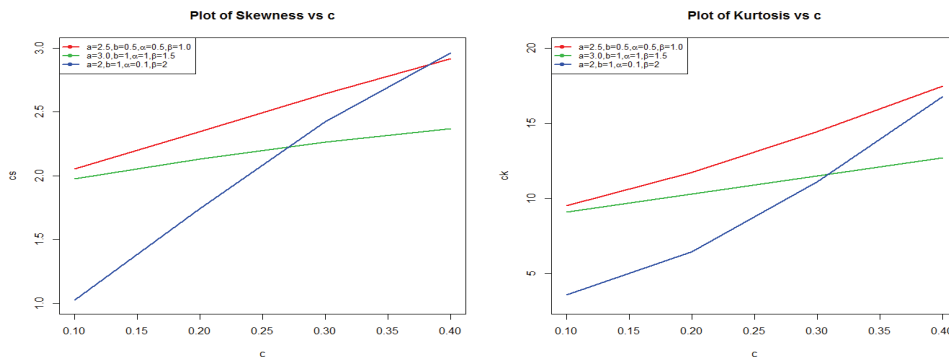


Figure 5. Plot of Skewness and Kurtosis for selected parameter values of the BLLoGW distribution

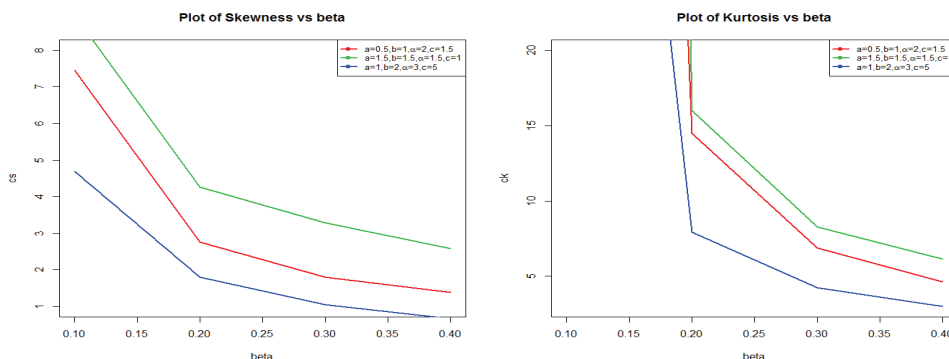


Figure 6. Plot of Skewness and Kurtosis for selected parameter values of the BLLoGW distribution

3.2 Conditional Moments

The r^{th} conditional moment of the BLLoGW distribution is given by

$$\begin{aligned}
 E(X^r|X > t) &= \frac{1}{\bar{F}_{BLLoGW}(t)} \int_t^\infty x^r f_{BLLoGW}(x) dx \\
 &= \frac{1}{\bar{F}_{BLLoGW}(t)} \int_t^\infty x^r \sum_{i=0}^\infty \frac{(-1)^i \Gamma(a)}{B(a, b)(i+b)! \Gamma(a-i)} (1+x^c)^{-(i+b)} e^{-\alpha(i+b)x^\beta} \\
 &\quad \times \left[cx^{c-1} (1+x^c)^{-1} (i+b) + \alpha(i+b)\beta x^{\beta-1} \right] dx \\
 &= \frac{1}{\bar{F}_{BLLoGW}(t)} \sum_{i=0}^\infty \sum_{m=0}^\infty \frac{(-1)^{i+m} [\alpha(i+b)]^m \Gamma(a)}{B(a, b)(i+b)! m! \Gamma(a-i)} \\
 &\quad \times (i+b) B_{[1+t^c]}^{-1} \left(i+b - \frac{r}{c} - \frac{m\beta}{c}, \frac{r+m\beta+c}{c} \right) \\
 &\quad + \frac{\alpha(i+b)\beta}{c} B_{[1+t^c]}^{-1} \left(i+b - \frac{r}{c} - \frac{m\beta}{c} - \frac{\beta}{c}, \frac{r+m\beta+\beta}{c} \right),
 \end{aligned}$$

where $B_y(a, b) = \int_0^y t^{a-1} (1-t)^{b-1} dt$ is the incomplete beta function. The mean residual life function $E(X - t|X > t)$ can be readily obtained.

4. Mean Deviations, Bonferroni and Lorenz Curves

The amount of scatter in a population can be measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median.

Bonferroni and Lorenz curves are widely used tool for analyzing and visualizing income inequality. Lorenz curve, $L(p)$ can be regarded as the proportion of total income volume accumulated by those units with income lower than or equal to the volume x , and Bonferroni curve, $B(p)$ is the scaled conditional mean curve, that is, ratio of group mean income of the population. Bonferroni and Lorenz curves have applications not only in economics for the study income and poverty,

but also in other fields such as reliability, demography, insurance and medicine. Mean deviations, Lorenz and Bonferroni curves for the BLLoGW distribution are presented in this section.

4.1 Mean Deviations

The mean deviation about the mean μ and the mean deviation about the median M are defined by

$$\delta_{\mu}(X) = \int_0^{\infty} |x - \mu| f(x) dx = 2\mu F(x)(\mu) - 2\mu + 2 \int_{\mu}^{\infty} x f(x) dx, \tag{26}$$

and

$$\delta_M(X) = \int_0^{\infty} |x - M| f(x) dx = -\mu + 2 \int_M^{\infty} x f(x) dx. \tag{27}$$

respectively. The mean deviation about the mean μ for BLLoGW distribution is given by

$$\delta_1(X) = 2\mu F_{BLLoGW}(\mu) - 2\mu + 2T(\mu),$$

where

$$\begin{aligned} T(\mu) &= \int_{\mu}^{\infty} x f_{BLLoGW}(x) dx \\ &= \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+m} [\alpha(i+b)]^m \Gamma(a)}{B(a,b)(i+b)! m! \Gamma(a-i)} (i+b) B_{[1+\mu^c]^{-1}} \left(i+b - \frac{1}{c} - \frac{m\beta}{c}, \frac{1+m\beta+c}{c} \right) \\ &+ \frac{\alpha(i+b)\beta}{c} B_{[1+\mu^c]^{-1}} \left(i+b - \frac{1}{c} - \frac{m\beta}{c} - \frac{\beta}{c}, \frac{1+m\beta+\beta}{c} \right). \end{aligned}$$

The mean deviation about the median M for BLLoGW distribution is given by

$$\delta_2(X) = -\mu + 2T(M),$$

where

$$\begin{aligned} T(M) &= \int_M^{\infty} x f_{BLLoGW}(x) dx \\ &= \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+m} [\alpha(i+b)]^m \Gamma(a)}{B(a,b)(i+b)! m! \Gamma(a-i)} (i+b) B_{[1+M^c]^{-1}} \left(i+b - \frac{1}{c} - \frac{m\beta}{c}, \frac{1+m\beta+c}{c} \right) \\ &+ \frac{\alpha(i+b)\beta}{c} B_{[1+M^c]^{-1}} \left(i+b - \frac{1}{c} - \frac{m\beta}{c} - \frac{\beta}{c}, \frac{1+m\beta+\beta}{c} \right). \end{aligned}$$

4.2 Bonferroni and Lorenz Curves

Lorenz and Bonferroni curves for the BLLoGW distribution are presented in this subsection. Bonferroni curve for the BLLoGW distribution is given by

$$B(p) = \frac{1}{p\mu} \int_0^q x f_{BLLoGW}(x) dx = \frac{1}{p\mu} [\mu - T(q)],$$

where

$$\begin{aligned} T(q) = \int_q^{\infty} x f_{BLLoGW}(x) dx &= \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{i+m} [\alpha(i+b)]^m \Gamma(a)}{B(a,b)(i+b)! m! \Gamma(a-i)} \\ &\times (i+b) B_{[1+q^c]^{-1}} \left(i+b - \frac{1}{c} - \frac{m\beta}{c}, \frac{1+m\beta+c}{c} \right) \\ &+ \frac{\alpha(i+b)\beta}{c} B_{[1+q^c]^{-1}} \left(i+b - \frac{1}{c} - \frac{m\beta}{c} - \frac{\beta}{c}, \frac{1+m\beta+\beta}{c} \right). \end{aligned}$$

Lorenz curves for the BLLoGW distribution is given by

$$L(p) = \frac{1}{\mu} \int_0^q x f_{BLLoGW}(x) dx = \frac{1}{\mu} [\mu - T(q)],$$

where $T(q)$ is given above.

5. Rényi Entropy and Order Statistics

The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. Order statistics play an important role in probability and statistics, particularly in reliability and lifetime data analysis. Rényi entropy and distribution of order statistics from the BLLoGW distribution are presented in this section.

5.1 Rényi Entropy

Rényi entropy is defined as

$$I_R(v) = \frac{1}{1-v} \log \left(\int_0^\infty [f_{BLLoGW}(x; c, a, b, \alpha, \beta)]^v dx \right), v \neq 1, v > 0. \tag{28}$$

Note that,

$$\begin{aligned} f_{BLLoGW}^v(x) &= \frac{1}{B(a, b)^v} [1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}]^{av-v} [(1 + x^c)^{-1} e^{-\alpha x^\beta}]^{bv} [cx^{c-1} (1 + x^c)^{-1} + \alpha\beta x^{\beta-1}]^v \\ &= \frac{1}{B(a, b)^v} \sum_{i,j=0}^\infty \frac{(-1)^{i+j} \Gamma(av - v + 1) [\alpha(bv + i)]^j}{\Gamma(av - v + 1 - i)! j!} (1 + x^c)^{-(bv+i)} x^{j\beta} [cx^{c-1} (1 + x^c)^{-1} + \alpha\beta x^{\beta-1}]^v. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^\infty f_{BLLoGW}^v(x) dx &= \frac{1}{B(a, b)^v} \sum_{i,j=0}^\infty \frac{(-1)^{i+j} \Gamma(av - v + 1) [\alpha(bv + i)]^j}{\Gamma(av - v + 1 - i)! j!} \\ &\quad \times \int_0^\infty x^{j\beta} (1 + x^c)^{-(bv+i)} [cx^{c-1} (1 + x^c)^{-1} + \alpha\beta x^{\beta-1}]^v dx. \end{aligned}$$

Using the binomial expansion,

$$[cx^{c-1} (1 + x^c)^{-1} + \alpha\beta x^{\beta-1}]^v = \sum_{k=0}^\infty \binom{v}{k} [cx^{c-1} (1 + x^c)^{-1}]^{v-k} [\alpha\beta x^{\beta-1}]^k, \tag{29}$$

we have

$$\begin{aligned} \int_0^\infty f_{BLLoGW}^v(x) dx &= \frac{1}{B(a, b)^v} \sum_{i,j,k=0}^\infty \frac{(-1)^{i+j} \Gamma(av - v + 1) [\alpha(bv + i)]^j}{\Gamma(av - v + 1 - i)! j!} \binom{v}{k} \\ &\quad \times (\alpha\beta)^k (c)^{v-k} \int_0^\infty x^{j\beta + \beta k + cv - ck - v} (1 + x^c)^{-v+k-bv-i} dx. \end{aligned}$$

Let $y = (1 + x^c)^{-1}$, then

$$\begin{aligned} \int_0^\infty f_{BLLoGW}^v(x) dx &= \frac{1}{B(a, b)^v} \sum_{i,j,k=0}^\infty \frac{(-1)^{i+j} \Gamma(av - v + 1) [\alpha(bv + i)]^j}{\Gamma(av - v + 1 - i)! j!} \binom{v}{k} (\alpha\beta)^k (c)^{v-k-1} \\ &\quad \times \int_0^1 y^{bv-k+i+v-\frac{j\beta-\beta k-cv+ck+v}{c}-\frac{1}{c}-1} (1-y)^{\frac{j\beta+\beta k+cv-ck-v}{c}+\frac{1}{c}-1} dy \\ &= \frac{1}{B(a, b)^v} \sum_{i,j,k=0}^\infty \frac{(-1)^{i+j} \Gamma(av - v + 1) [\alpha(bv + i)]^j}{\Gamma(av - v + 1 - i)! j!} \binom{v}{k} (\alpha\beta)^k (c)^{v-k-1} \\ &\quad \times B\left(bv - k + i + v - \frac{j\beta - \beta k - cv + ck + v}{c} - \frac{1}{c}, \frac{j\beta + \beta k + cv - ck - v}{c} + \frac{1}{c}\right). \end{aligned}$$

As a result, Rényi entropy is given by

$$\begin{aligned} I_R(v) &= \frac{1}{1-v} \log \left[\frac{1}{B(a, b)^v} \sum_{i,j,k=0}^\infty \frac{(-1)^{i+j} \Gamma(av - v + 1) [\alpha(bv + i)]^j}{\Gamma(av - v + 1 - i)! j!} \right. \\ &\quad \times \left. \binom{v}{k} (\alpha\beta)^k (c)^{v-k-1} B\left(bv - k + i + v - \frac{j\beta - \beta k - cv + ck + v}{c} - \frac{1}{c}, \frac{j\beta + \beta k + cv - ck - v}{c} + \frac{1}{c}\right) \right] \tag{30} \end{aligned}$$

for $v > 0, v \neq 1$.

5.2 Order Statistics

Suppose that X_1, X_2, \dots, X_n denote continuous independent BLLoGW random variables with common cdf and pdf. The pdf of the k^{th} order statistic for a random sample of size n from BLLoGW distribution is given by

$$\begin{aligned}
 g_{k:n}(x) &= \frac{n!f(x)}{(k-1)!(n-k)!} [F(x)]^{k-1} [1-F(x)]^{n-k} \\
 &= \frac{n!f(x)}{(k-1)!(n-k)!} \sum_{p=0}^{n-k} (-1)^p \binom{n-k}{p} \left[\frac{1}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b) [1 - (1+x^c)^{-1} e^{-\alpha x^\beta}]^{a+j}}{\Gamma(b-j) j!(a+j)} \right]^{k+p-1}.
 \end{aligned}
 \tag{31}$$

Note that,

$$\begin{aligned}
 [F(x)]^{k+p-1} &= \left[\frac{1}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b) [1 - (1+x^c)^{-1} e^{-\alpha x^\beta}]^{a+j}}{\Gamma(b-j) j!(a+j)} \right]^{k+p-1} \\
 &= \frac{(\Gamma(b))^{p+k-1}}{(B(a,b))^{p+k-1}} \sum_{j=0}^{\infty} d_{a+j,p+k-1} [1 - (1+x^c)^{-1} e^{-\alpha x^\beta}]^j
 \end{aligned}
 \tag{32}$$

by using the identity $(\sum_{j=0}^{\infty} a_j x^j)^m = \sum_{j=0}^{\infty} c_{j,m} x^j$ for n positive integer (see Gradshteyn and Ryzhik (2000)), where

$$a_j = \frac{(-1)^j}{\Gamma(b-j) j!(a+j)},
 \tag{33}$$

$d_{a+j,k+p-1} = ((a+1)c_0)^{-1} \sum_{l=1}^{a+j} [(n-i+1)l - a - j + l] c_l d_{a+j-1,k+p-1}$ and $d_0 = c_{k+p-1}^0$. Now, using the series representation

$$(1 - (1+x^c)^{-1} e^{-\alpha x^\beta})^{k+p-1} = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(k+p)}{m! \Gamma(k+p-m)} (1+x^c)^{-m} e^{-\alpha(m)x^\beta},$$

we have

$$[F(x)]^{k+p-1} = \frac{(\Gamma(b))^{k+p-1}}{(B(a,b))^{k+p-1}} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(k+p)}{m! \Gamma(k+p-m)} (1+x^c)^{-m} e^{-\alpha(m)x^\beta} d_{a+j,k+p-1}.$$

The distribution of the k^{th} order statistics is given by:

$$\begin{aligned}
 g_{k:n}(x) &= \frac{n!(\Gamma(b))^{k+p-1} \Gamma(a)}{(k-1)!(n-k)!(B(a,b))^{k+p}} \sum_{p=0}^{n-k} \sum_{i,m=0}^{\infty} \frac{(-1)^{p+m+i} \Gamma(k+p)}{(i+b)\Gamma(a-i)\Gamma(p+k-m)m!i!} \\
 &\times \binom{n-k}{p} d_{a+j,k+p-1} (1+x^c)^{-(i+b+m)} e^{-\alpha(i+b+m)x^\beta} [cx^{c-1} (1+x^c)^{-1} (i+b) + \alpha(i+b)\beta x^{\beta-1}].
 \end{aligned}
 \tag{34}$$

The t^{th} moment of the distribution of the i^{th} order statistics of BLLoGW distribution can be derived from the result of Barakat and Abdelkader (2004), that is,

$$E[X_{i:n}^t] = t \sum_{p=n+i-1}^n (-1)^{p-n+i-1} \binom{p-1}{n-i} \binom{n}{p} \int_0^\infty x^{t-1} [1-F(x)]^p dx.
 \tag{35}$$

Note that

$$\begin{aligned}
 \int_0^\infty x^{t-1} [1-F_{BLLoGW}(x)]^p dx &= \sum_{l=0}^p (-1)^l \binom{p}{l} \int_0^\infty x^{t-1} [F_{BLLoGW}(x)]^l dx \\
 &= \sum_{l=0}^p (-1)^l \binom{p}{l} \int_0^\infty \frac{x^{t-1}}{(B(a,b))^l} \left[\sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b) [1 - (1+x^c)^{-1} e^{-\alpha x^\beta}]^{a+j}}{\Gamma(b-j) j!(a+j)} \right]^l dx.
 \end{aligned}
 \tag{36}$$

That is,

$$\begin{aligned}
 [F_{BLLoGW}(x)]^l &= \left[\frac{1}{B(a, b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b) [1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}]^{a+j}}{\Gamma(b-j) j! (a+j)} \right]^l \\
 &= \frac{\Gamma(b)^l}{B(a, b)^l} [1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}]^{al} \left[\sum_{j=0}^{\infty} \frac{(-1)^j [1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}]^j}{\Gamma(b-j) j! (a+j)} \right]^l.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \int_0^\infty x^{t-1} [1 - F_{BLLoGW}(x)]^p dx &= \sum_{l=0}^p \frac{(-1)^l \binom{p}{l} \Gamma(b)^l}{[B(a, b)]^l} \int_0^\infty x^{t-1} [1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}]^{al} \\
 &\quad \times \left[\sum_{j=0}^{\infty} a_j [1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}]^j \right]^l dx,
 \end{aligned}$$

where

$$a_j = \frac{(-1)^j}{\Gamma(b-j) j! (a+j)}. \tag{37}$$

Using the identity $(\sum_{k=0}^\infty a_k x^k)^n = \sum_{k=0}^\infty c_{k,n} x^k$ for n positive integer (see Gradshteyn and Ryzshik (2000)), we have

$$\left[\sum_{j=0}^{\infty} a_j [1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}]^j \right]^l = \sum_{j=0}^{\infty} c_{j,l} [1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}]^j, \tag{38}$$

where $c_{0,l} = a_0^l$ and $c_{j,l} = (ja_0)^{-1} \sum_{q=1}^j (lq - j + q) a_q c_{j-q,l}$. Therefore,

$$\begin{aligned}
 \int_0^\infty x^{t-1} [1 - F_{BLLoGW}(x)]^p dx &= \sum_{l=0}^p \sum_{j=0}^{\infty} \frac{(-1)^l \binom{p}{l} \Gamma(b)^l}{[B(a, b)]^l} c_{j,l} \\
 &\quad \times \int_0^\infty x^{t-1} [1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}]^{al+j} dx.
 \end{aligned}$$

Note that

$$[1 - (1 + x^c)^{-1} e^{-\alpha x^\beta}]^{al+j} = \sum_{v=0}^{\infty} \frac{(-1)^v \Gamma(al+j+1)}{\Gamma(al+j+1-v) v!} [(1 + x^c)^{-1} e^{-\alpha x^\beta}]^v. \tag{39}$$

Using the series expansion above, we have

$$\begin{aligned}
 \int_0^\infty x^{t-1} [1 - F_{BLLoGW}(x)]^p dx &= \sum_{l=0}^p \sum_{j,v=0}^{\infty} \frac{(-1)^{l+v} \binom{p}{l} \Gamma(b)^l \Gamma(al+j+1)}{B(a, b)^l \Gamma(al+j+1-v) v!} c_{j,l} \int_0^\infty x^{t-1} [(1 + x^c)^{-1} e^{-\alpha x^\beta}]^v dx \\
 &= \sum_{l=0}^p \sum_{j,v,r=0}^{\infty} \frac{(-1)^{l+v+r} \binom{p}{l} \Gamma(b)^l \Gamma(al+j+1) (\alpha)^r (v)^r}{B(a, b)^l \Gamma(al+j+1-v) v! r! c} c_{j,l} \int_0^1 y^{v-\frac{t}{c}-\frac{r\beta}{c}-1} (1-y)^{\frac{t+r\beta}{c}-1} dy \\
 &= \sum_{l=0}^p \sum_{j,v,r=0}^{\infty} \frac{(-1)^{l+v+r} \binom{p}{l} \Gamma(b)^l \Gamma(al+j+1) (\alpha)^r (v)^r}{B(a, b)^l \Gamma(al+j+1-v) v! r! c} c_{j,l} B\left(v - \frac{t}{c} - \frac{r\beta}{c}, \frac{t+r\beta}{c}\right). \tag{40}
 \end{aligned}$$

Therefore, the t^{th} moment of the distribution of the i^{th} order statistic from BLLoGW distribution is given by

$$\begin{aligned}
 E[X_{i:n}^t] &= t \sum_{p=n+i-1}^n (-1)^{p-n+i-1} \binom{p-1}{n-i} \binom{n}{p} \int_0^\infty x^{t-1} [1 - F_{BLLoGW}(x)]^p dx \\
 &= t \sum_{p=n+i-1}^n \sum_{l=0}^p \sum_{j,v,r=0}^{\infty} \frac{(-1)^{p-n+i+l+v+r-1} \binom{p}{l} \binom{p-1}{n-i} \binom{n}{p} \Gamma(b)^l \Gamma(al+j+1) (\alpha)^r (v)^r}{B(a, b)^l \Gamma(al+j+1-v) v! r! c} c_{j,l} \\
 &\quad \times B\left(v - \frac{t}{c} - \frac{r\beta}{c}, \frac{t+r\beta}{c}\right), \quad \text{for } c > t + r\beta. \tag{41}
 \end{aligned}$$

6. Estimation and Inference

Let x_1, x_2, \dots, x_n denote a random sample of size n from BLLoGW distribution and $\varphi = (c, a, b, \alpha, \beta)^T$ the vector of parameters. The log-likelihood function of the parameter vector can be written as

$$\begin{aligned} \ell = \log L &= (a - 1) \sum_{i=1}^n \ln \left(1 - (1 + x_i^c)^{-1} e^{-\alpha x_i^\beta} \right) - (b - 1) \sum_{i=1}^n \ln \left((1 + x_i^c)^{-1} e^{-\alpha x_i^\beta} \right) \\ &- \ln(B(a, b)) - \sum_{i=1}^n \ln(1 + x_i^c) - \alpha \sum_{i=1}^n x_i^\beta + \sum_{i=1}^n \ln \left(c x_i^{c-1} (1 + x_i^c)^{-1} + \alpha \beta x_i^{\beta-1} \right). \end{aligned}$$

The associated score vector is

$$U_n(\varphi) = \left(\frac{\partial \ell}{\partial c}, \frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta} \right)^T, \tag{42}$$

where the elements are given by

$$\begin{aligned} \frac{\partial \ell}{\partial c} &= (a - 1) \sum_{i=1}^n \frac{(1 + x_i^c)^{-2} e^{-\alpha x_i^\beta} x_i^c \ln(x_i)}{\left(1 - (1 + x_i^c)^{-1} e^{-\alpha x_i^\beta} \right)} - (b - 1) \sum_{i=1}^n \frac{(1 + x_i^c)^{-2} e^{-\alpha x_i^\beta} x_i^c \ln(x_i)}{\left(1 + x_i^c \right)^{-1} e^{-\alpha x_i^\beta}} \\ &- \sum_{i=1}^n \frac{x_i^c \ln(x_i)}{\left(1 + x_i^c \right)} + \sum_{i=1}^n \frac{x_i^{c-1} + c x_i^{c-1} \ln(x_i) + c x_i^{c-1} x_i^c \ln(x_i)}{c x_i^{c-1} \left(1 + x_i^c \right)^{-1} + \alpha \beta x_i^{\beta-1}}, \end{aligned} \tag{43}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} &= (a - 1) \sum_{i=1}^n \frac{\left(1 + x_i^c \right)^{-1} e^{-\alpha x_i^\beta} \alpha x_i^\beta \ln(x_i)}{\left(1 - \left(1 + x_i^c \right)^{-1} e^{-\alpha x_i^\beta} \right)} + (b - 1) \sum_{i=1}^n \frac{\left(1 + x_i^c \right)^{-1} e^{-\alpha x_i^\beta} \alpha x_i^\beta \ln(x_i)}{\left(1 + x_i^c \right)^{-1} e^{-\alpha x_i^\beta}} \\ &- \alpha \sum_{i=1}^n x_i^\beta \ln(x_i) + \sum_{i=1}^n \frac{\alpha x_i^{\beta-1} (1 + \beta \ln(x_i))}{c x_i^{c-1} \left(1 + x_i^c \right)^{-1} + \alpha \beta x_i^{\beta-1}}, \end{aligned} \tag{44}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha} &= (a - 1) \sum_{i=1}^n \frac{\left(1 + x_i^c \right)^{-1} x_i^\beta e^{-\alpha x_i^\beta}}{\left(1 - \left(1 + x_i^c \right)^{-1} e^{-\alpha x_i^\beta} \right)} + (b - 1) \sum_{i=1}^n \frac{\left(1 + x_i^c \right)^{-1} x_i^\beta e^{-\alpha x_i^\beta}}{\left(1 + x_i^c \right)^{-1} e^{-\alpha x_i^\beta}} \\ &- \sum_{i=1}^n x_i^\beta + \sum_{i=1}^n \frac{\beta x_i^{\beta-1}}{c x_i^{c-1} \left(1 + x_i^c \right)^{-1} + \alpha \beta x_i^{\beta-1}}, \end{aligned}$$

$$\frac{\partial \ell}{\partial a} = -(\psi(a) - \psi(a + b)) + \sum_{i=1}^n \left(1 - (1 + x_i^c)^{-1} e^{-\alpha x_i^\beta} \right), \tag{45}$$

and

$$\frac{\partial \ell}{\partial b} = -(\psi(b) - \psi(a + b)) + \sum_{i=1}^n \left((1 + x_i^c)^{-1} e^{-\alpha x_i^\beta} \right). \tag{46}$$

respectively. Solving the nonlinear system equations, $U_n(\varphi) = \mathbf{0}$ yields the maximum likelihood estimates. These equations can be solved numerically via iterative methods such as Newton-Raphson technique using statistical software.

We maximized the likelihood function using NLMixed in SAS as well as the function nlm in R (R Development Core Team (2011)). These functions were applied and executed for wide range of initial values. This process often results or lead to more than one maximum, however, in these cases, we take the MLEs corresponding to the largest value of the maxima. In a few cases, no maximum was identified for the selected initial values. In these cases, a new initial value was tried in order to obtain a maximum.

The issues of existence and uniqueness of the MLEs are theoretical interest and has been studied by several authors for different distributions including Seregina(2010), Santos Silva and Tenreyros (2010), Zhou (2009), and Xia, Mi, and

Zhou (2009). At this point we are not able to address the theoretical aspects (existence, uniqueness) of the MLE of the parameters of the BLLoGW distribution.

6.1 Asymptotic Confidence Interval

The Fisher information of the BLLoGW distribution is a 5×5 symmetric matrix given by $\mathbf{I}(\varphi) = [\mathbf{I}_{\theta_i, \theta_j}]_{5 \times 5} = E(-\frac{\partial^2 \log L}{\partial \theta_i \partial \theta_j})$, $i, j = 1, 2, 3, 4, 5$. The total Fisher information matrix $n\mathbf{I}(\varphi)$ can be approximated by $\mathbf{I}_n(\hat{\varphi}) \approx \left[-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j} \Big|_{\varphi=\hat{\varphi}} \right]_{5 \times 5}$, for $i, j = 1, 2, 3, 4, 5$. Let $\hat{\varphi} = (\hat{a}, \hat{b}, \hat{c}, \hat{\alpha}, \hat{\beta})$ be the maximum likelihood estimates of $\varphi=(a, b, c, \alpha, \beta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, are met, we have $\sqrt{n}(\hat{\varphi}-\varphi) \xrightarrow{d} N_5(\mathbf{0}, \mathbf{I}^{-1}(\varphi))$, where $I(\varphi)$ is the expected Fisher information matrix. The multivariate normal distribution with mean vector $\mathbf{0}$ and covariance matrix $I(\varphi)$ can be used to construct confidence intervals for the BLLoGW distribution parameters. A large sample $100(1 - \xi)\%$ two-sided confidence intervals for BLLoGW distribution parameters: a, b, c, α and β are given by: $\hat{a} \pm Z_{\xi/2} \sqrt{I_{aa}^{-1}(\hat{\varphi})}$, $\hat{b} \pm Z_{\xi/2} \sqrt{I_{bb}^{-1}(\hat{\varphi})}$, $\hat{\alpha} \pm Z_{\xi/2} \sqrt{I_{\alpha\alpha}^{-1}(\hat{\varphi})}$ $\hat{\beta} \pm Z_{\xi/2} \sqrt{I_{\beta\beta}^{-1}(\hat{\varphi})}$, and $\hat{c} \pm Z_{\xi/2} \sqrt{I_{cc}^{-1}(\hat{\varphi})}$, where $Z_{\xi/2}$ is the $1 - \frac{\xi}{2}$ quantile of the standard normal distribution.

6.2 Likelihood Ratio Test

The likelihood ratio (LR) test can be used to compare the fit of the BLLoGW distribution with its nested-models for a given data set. For example, to test $a = b = c = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{c}, \hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta})) - \ln(L(1, 1, 1, \tilde{\alpha}, \tilde{\beta}))]$, where $\hat{c}, \hat{a}, \hat{b}, \hat{\alpha}$, and $\hat{\beta}$ are the unrestricted estimates, and $\tilde{\alpha}$ and $\tilde{\beta}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi_d^2$, where χ_d^2 denote the upper $100d\%$ point of the χ^2 distribution with 3 degrees of freedom.

7. Simulation Study

The performance and accuracy of MLEs of the BLLoGW distribution parameters is examined by conducting a simulation study in this section.

The simulation study is repeated $N = 2000$ times with each sample size $n = 35, 50, 70, 100, 200, 400, 800, 1600$ and true parameters values $I : a = 1, b = 1, c = 1, \alpha = 0.02, \beta = 2.2$ and $II : a = 1, b = 1, c = 0.8, \alpha = 0.5, \beta = 3$. In this simulation study, we computed three quantities: the mean estimate, root mean square error and average bias. The mean estimate, average bias (Abias) and root mean square error (RMSE) of the MLE $\hat{\varphi}$ for the parameter $\varphi = a, b, c, \alpha, \beta$ are given by $Mean = \frac{\sum_{i=1}^N \hat{\varphi}_i}{N}$, $ABias(\hat{\varphi}) = \frac{\sum_{i=1}^N (\hat{\varphi}_i - \varphi)}{N}$ and $RMSE = \sqrt{\frac{\sum_{i=1}^N (\hat{\varphi}_i - \varphi)^2}{N}}$, respectively. The mean MLEs of the BLLoGW distribution parameters along with their respective root mean squared errors (RMSE) and average bias for different sample sizes are listed in Table 3.

Table 3. Monte Carlo Simulation Results: Mean, Average Bias and RMSE

Parameter	n	I			II		
		Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
<i>a</i>	35	9.9150	8.9150	22.6950	10.3933	9.3933	23.4362
	50	8.4664	7.4664	22.0104	7.8071	6.8071	17.7589
	70	6.3712	5.3712	17.5209	6.7053	5.7053	14.6949
	100	5.0748	4.0748	14.4317	6.0698	5.0698	12.3993
	200	2.5801	1.5801	6.9832	4.4065	3.4065	9.3691
	400	1.9886	0.9886	5.1553	3.0216	2.0216	6.1737
	800	1.3256	0.3256	1.7279	1.9382	0.9381	3.7781
	1600	1.1287	0.1287	0.5203	1.4826	0.4826	2.4887
<i>b</i>	35	9.3545	8.3545	22.6797	9.8245	8.8245	23.3862
	50	8.0962	7.0962	21.6772	7.2515	6.2515	17.6077
	70	6.1140	5.1140	17.1819	6.1587	5.1587	14.2823
	100	4.9397	3.9397	14.1993	5.4726	4.4726	11.8467
	200	2.5441	1.5441	6.8594	3.8884	2.8884	8.6754
	400	1.9800	0.9800	5.1388	2.6128	1.6158	5.5992
	800	1.3172	0.3172	1.7139	1.6353	0.6353	3.3627
	1600	1.1241	0.1241	0.5129	1.2164	0.2164	2.1924
<i>c</i>	35	2.1987	1.9187	2.0646	1.9752	1.1752	2.0525
	50	1.3183	0.3183	1.6161	1.9222	1.1222	2.0422
	70	1.2193	0.2196	1.0583	1.7558	0.9558	1.0843
	100	1.2619	0.2619	0.9383	1.6413	0.8413	1.0019
	200	1.1557	0.1557	0.6914	1.5667	0.7667	0.8545
	400	1.0669	0.0669	0.5177	1.1015	0.3015	0.7588
	800	1.0251	0.0258	0.3651	1.0963	0.2963	0.6174
	1600	1.0067	0.0067	0.2502	1.0517	0.2517	0.4857
α	35	0.7919	0.2919	1.8947	0.7929	0.2929	1.8015
	50	0.1567	0.1367	1.0336	0.7773	0.2773	1.7688
	70	0.1248	0.1048	1.1052	0.7159	0.2159	1.6068
	100	0.0708	0.0508	0.4115	0.7270	0.2270	1.4274
	200	0.0410	0.0210	0.1406	0.6552	0.1552	0.9203
	400	0.0275	0.0075	0.0393	0.6354	0.1354	0.6805
	800	0.0244	0.0044	0.0243	0.5654	0.0654	0.4141
	1600	0.0221	0.0021	0.0139	0.5429	0.0429	0.2917
β	35	3.8670	0.8670	2.9265	3.8255	0.8256	2.8539
	50	2.6502	0.4502	1.3907	3.6269	0.6289	2.5258
	70	2.6205	0.4205	1.2397	3.4538	0.4538	1.9566
	100	2.6051	0.4051	1.2056	3.1443	0.1443	0.3926
	200	2.5591	0.3591	0.9352	2.9959	-0.0041	0.8291
	400	2.3593	0.1593	0.6060	2.9615	-0.0385	0.6049
	800	2.2581	0.0581	0.4069	2.9874	-0.0126	0.4349
	1600	2.2215	0.0215	0.2725	2.9699	-0.0301	0.3207

From the results in the Table 3, it can be verified that the mean estimates of the parameters converges to the true parameter and the RMSEs decay toward zero as the sample size *n* increases. We also notice that for all the parametric values, the biases decrease as the sample size *n* increases.

8. Application

In this section, we present an example to illustrate the flexibility of the BLLoGW distribution and its sub-models for data fitting. We compared the fit of the BLLoGW distribution with the fits of the nested BLLoG, ELLoG, LLoG distributions and the non-nested beta Weibull log-logistic (BWLLoG) distribution. The pdf of the BWLLoG distribution (Makubate et al. (2018)) is given by

$$\begin{aligned}
 g_{BWLLoG}(x; a, b, c, \alpha, \beta) &= \frac{\alpha\beta c}{B(a, b)} x^{c-1} (1+x^c)^{-2} \frac{[1 - (1+x^c)^{-1}]^{\beta-1}}{[(1+x^c)^{-1}]^{\beta+1}} \\
 &\times \exp\{-ab[(1+x^c)-1]^\beta\} [1 - \exp\{-\alpha[(1+x^c)-1]^\beta\}]^{\alpha-1}. \tag{47}
 \end{aligned}$$

for $a, b, c, \alpha, \beta > 0$.

8.1 Stress-Rupture Life of Kevlar 49/Epoxy Strands Data

This data reported by Cooray and Ananda (2008) consist of observations that represent the stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all have failed, so that

the complete dataset with the exact times of failure is recorded and shown in Table ???. Andrews and Herzberg (1985) and Barlow, Toland and Freeman (1984) gives the recorded failure times in hours. Initial values for BLLoGW model in R code are $a = 0.8, b = 1.0, c = 0.2, \alpha = 1.0, \beta = 2.0$.

Table 4. Failure times data of kevlar 49/epoxy strands with pressure at 90%

0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.04	0.05	0.06	0.07	0.07	0.08	0.09
0.09	0.10	0.10	0.11	0.11	0.12	0.13	0.18	0.19	0.20	0.23	0.24	0.24	0.29
0.34	0.35	0.36	0.38	0.40	0.42	0.43	0.52	0.54	0.56	0.60	0.60	0.63	0.65
0.67	0.68	0.72	0.72	0.72	0.73	0.79	0.79	0.80	0.80	0.83	0.85	0.90	0.92
0.95	0.99	1.00	1.01	1.02	1.03	1.05	1.10	1.10	1.11	1.15	1.18	1.20	1.29
1.31	1.33	1.34	1.40	1.43	1.45	1.50	1.51	1.52	1.53	1.54	1.54	1.55	1.58
4.20	4.69	7.89											

The asymptotic covariance matrix of the MLE's for the BLLoGW distribution $\mathbf{I}_n^{-1}(\hat{\phi})$ is given by:

$$\begin{pmatrix} 232.886258 & 202.793181 & -4.46670232 & -5.09291312 & -4.69378563 \\ 202.793181 & 178.012075 & -3.83400390 & -4.61840501 & -3.87711698 \\ -4.46670232 & -3.83400390 & 0.08883694 & 0.09040585 & 0.09748933 \\ -5.09291312 & -4.61840501 & 0.09040585 & 0.13669012 & -0.07204885 \\ -4.693786 & 3.87711698 & 0.09748933 & -0.07204885 & 0.14308952 \end{pmatrix} \tag{48}$$

and the approximate 95% two-sided confidence interval for the parameters a, b, c, α and β are given by:

$$6.46809 \pm 29.9107982, 5.07658 \pm 26.1505485, 0.22372 \pm 0.5841883, 0.244637 \pm 0.7246439,$$

and 0.93981 ± 0.7414126 , respectively.

Table 5 gives the maximum likelihood estimates (MLE), standard errors (in parentheses), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (AICC) and Hannan-Quinn information criterion (HQIC) of BLLoGW, BLLoG, ELLoG, LLoG and BWLLoG distributions. The values of the goodness-of-fit statistics W^* and A^* are also presented. The smaller the values of these statistics the better the fit. For the probability plot, we plotted $F(y_{(j)}; \hat{c}, \hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b})$ against $\frac{j - 0.375}{n + 0.25}$, $j = 1, 2, \dots, n$, where $y_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS = \sum_{j=1}^n \left[F(y_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2.$$

Table 5. Estimation of Models for Failure Times of Kevlar Dataset

Model	Estimates					Statistics						
	\hat{a}	\hat{b}	\hat{c}	$\hat{\alpha}$	$\hat{\beta}$	$-2 \log L$	AIC	AICC	BIC	HQIC	W^*	A^*
BLLoGW	6.468097 (15.26061)	5.07658 (13.34212)	0.22372 (0.29806)	0.24463 (0.36972)	0.93981 (0.37827)	204.0771	214.0771	214.7087	227.1527	219.3705	0.1333	0.8042
BLLoG	0.581650 (0.036581)	1.091929 (0.133559)	1.295956 (0.108778)	-	-	462.1078	468.1078	468.3552	475.9531	471.2838	0.4176	2.2683
ELLoG	1.211659 (0.096551)	0.570392 (0.034775)	-	-	-	587.683	591.683	3591.8055	596.9133	593.8004	0.4388	2.3777
LLoG	1.2265 (0.1057)	-	-	-	-	237.589	239.589	239.6294	242.2041	240.6476	0.4760	2.5805
BWLLoG	0.70773 (0.21792)	0.15311 (0.15378)	1.45779 (84.60684)	5.74192 (6.12430)	0.72356 (41.99367)	204.8205	214.8205	215.4521	227.8961	220.1139	0.4673	3.3332

The LR statistics for testing the hypothesis H_0 : BLLoG against H_a : BLLoGW, H_0 : ELLoG against H_a : BLLoGW and H_0 : LLoG against H_a : BLLoGW are 258.0299 (p-value < 0.0001), 374.6059 (p-value < 0.0001), 33.5119 (p-value < 0.0001), respectively. Since the p-values are small, we reject the null hypothesis and conclude that the BLLoGW distribution is significantly different from BLLoG, ELLoG and LLoG distributions. The BLLoGW distribution is the better model since it has the smallest value for each of the statistics: AIC, AICC, BIC and HIQC. The values of the goodness-of-fit statistics W^* and A^* gives clear and convincing evidence that the BLLoGW distribution fits better for kevlar 49/epoxy strands data. Plots of the estimated pdf of the BLLoGW, BLLoG, ELLoG, LLoG and BWLLoG distributions fitted to the dataset are given in Figure 7. The Figure showed that BLLoGW distribution is good lifetime distribution when compared with the three sub-models and the non-nested BWLLoG distribution. Also, the Probability Plots of the compared distributions using the failure times of the kevlar data set is shown in Figure 8. The value of SS from the probability plots is smallest for the BLLoGW distribution.

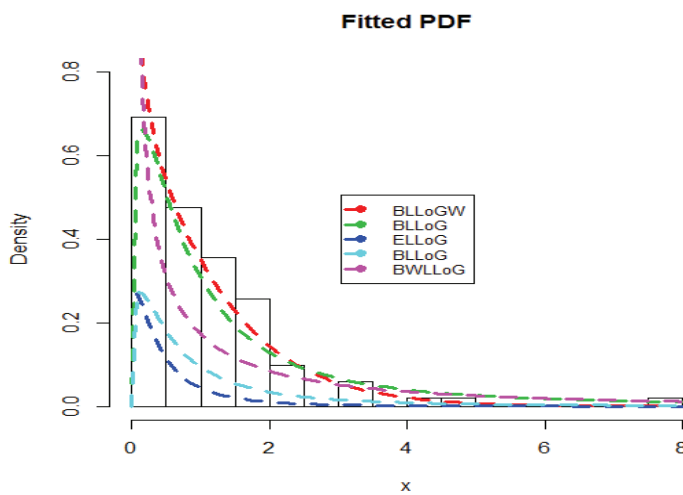


Figure 7. Fitted PDF for Failure Times of Kevlar Data Set

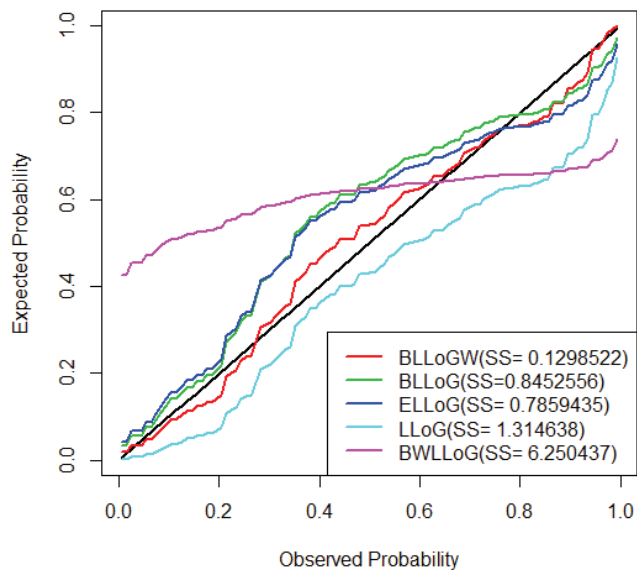


Figure 8. Probability Plots for Failure Times of Kevlar Data Set

9. Concluding Remarks

A new and generalized distribution called the beta log-logistic Weibull (BLLoGW) distribution has been proposed and studied. The BLLoGW distribution has several new and well known distributions as special cases. The BLLoGW distribution possesses hazard function with very flexible behavior. We also obtain closed form expressions for the moments, conditional moments, mean deviations, Bonferroni and Lorenz curves, distribution of order statistics and Rényi entropy. Maximum likelihood estimation technique was used to estimate the model parameters. Finally, the BLLoGW distribution is fitted to a real dataset in order to illustrate its applicability and usefulness.

Disclosure Statement

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