International Journal of Economics and Finance; Vol. 7, No. 5; 2015
ISSN 1916-971X E-ISSN 1916-9728
Published by Canadian Center of Science and Education

Asset Pricing Kernels in an OLG Economy with Housing

Hyeon Park[1]

[1] School of Business, Manhattan College, New York, USA

Correspondence: Hyeon Park, Department of Economics & Finance, Manhattan College, Riverdale, NY, 10471, USA. Tel: 1-718-862-7462. E-mail: hyeon.park@manhattan.edu

Received: January 22, 2015 Accepted: March 11, 2015 Online Published: April 25, 2015

doi:10.5539/ijef.v7n5p71 URL: http://dx.doi.org/10.5539/ijef.v7n5p71

Abstract

This paper introduces an asset pricing model with housing and explores properties of an asset pricing kernel using housing service as a numeraire in an OLG economy. When there is non-separable utility over consumption goods and housing services, the model generates not only the new type of risk known as composition risk, but also a new type of pricing kernel. This paper derives this novel pricing kernel using housing service as the numeraire and studies its property by exploring its relationship with the standard pricing kernel, which uses consumption as the numeraire. A simulation result for steady state asset prices exhibits a certain interrelationship among many consumer parameters in asset pricing with housing.

Keywords: asset pricing kernel, housing, OLG model, non-separable utility

1. Introduction

This paper introduces a new type of asset pricing kernel using *housing service* as a numeraire in an OLG economy and develops its property related with the usual pricing kernel where consumption is used as numeraire. Housing has several important implications in economics and finance. First, housing (or broadly defined as real estate) may be by far the largest component of household total wealth (Ivan, 2012) and its expenditure comprises a substantial share in household expenditure as well (Piazzesi et al., 2007). Second, the value of housing capital stock is larger than that of business capital (Tibaijuka, 2009). Third, significant fluctuations in housing price would imply significant fluctuations in wealth (Leung, 2004). Moreover, as shown in several area of economics literature including asset pricing and lifecycle models, housing plays such important roles as a consumption good, asset, and collateral device. As an asset, housing capital or housing stock yields capital gains and rental earnings, similar to the cum-dividend return of usual stocks. As a consumption good, economic agents enjoy housing services in any time, by renting and consuming a home from housing stocks. In this paper we focus on both of these roles from an asset pricing perspective (Note 1).

The usual assumption of standard asset pricing models is summarized by a tenet of an infinitely-lived representative agent (ILA), with time-separable utility, who discounts future expected utilities by time preference. ILA implies an allocation decision made by a single infinitely-lived agent who acts on behalf of both the present and all future generations. The standard consumption-based asset pricing framework is derived from this baseline assumption. It is well-known that for the case of power utility function, this framework returns a relatively simple form of asset pricing kernel, the value of which is completely determined by consumption growth (Note 2). This is because the pricing kernel is derived from intertemporal marginal rate of substitution (IMRS) between current and future utility arguments. In a simple model of asset allocation via consumption and saving, it is the marginal utility from current and future consumption through which investors rebalance their asset holdings. The main implication is that, when consumption growth rate is high, the IMRS gets lower due to a decreasing marginal utility and thus the future asset value in the high state should be lower. In this sense, the standard C-CAPM may be called *consumption risk* approach (Piazzesi et al., 2007).

However, this consumption risk approach with standard asset pricing kernels has not been very successful to explain some empirical features of asset returns, noted as *puzzles*, such as the equity premium puzzle (Note 3) (Mehra & Prescott, 1985) and risk free rate puzzle (Weil, 1989). The main implication of the equity premium puzzle is that under standard risk aversion parameter, the model generates so little premium because consumption is too smooth. In other words, to match the fact in data, an unreasonable value of risk aversion is required (Note 4).

Several remedies have been proposed to solve the puzzle, by redefining each of the elements in constructing the standard model (Note 5): (i) preference, (ii) market structure, (iii) type of agents. Regarding preference, habit formation by Campbell and Cochrane (1999) (Note 6) and non-expected utility by Epstein and Zin (1989) were suggested. Regarding market structure, incomplete market structure may be considered although it needs very persistent income shocks and counter-cyclical consumption variance to match the data. Constantinides and Duffie (1996) suggest heterogeneity by introducing uninsured idiosyncratic risks into the economy. We may even think about behavioral approaches or beliefs. Staying away from these, Piazzesi et al. (2007), by introducing housing explicitly as an asset and upgrading the separable utility specification to non-separable one, replace the old consumption risk with the *composition risk* (Note 7). They find that the presence of composition risk helps generate a sizeable and volatile equity premium observed in the data.

Other than these, all about ILA approach, recent development focuses on asset pricing modeling based on the overlapping generations model (OLG). With OLG, allocation decisions are made simultaneously by different age groups. Different generations face different incomes and different risks regarding future consumptions. This is how OLG models perform differently from ILA when intertemporal allocations are linked to asset markets: the lifecycle feature in OLG is said to play a crucial role in improving the consumption based asset pricing, according to Bohm et al. (2008). Some documents include Constantinides et al. (2002) for imperfect credit market for the young, and Storesletten, Telmer and Yaron (2007) for idiosyncratic risks related with age (Note 8). Many authors argue a general advantage of OLG over ILA in other areas, too. Bohm, Kikuchi and Vachadze (2008) analyze two-period OLG vs. ILA with productivity growth and find that the OLG model improves the fit in the following three elements of asset pricing criteria: risk premium, equity return, and bond return (Note 9). Then we may ask the following: what is not answered with the above models? Among current literature, we may argue that one of them should be OLG with housing. We may wonder if we can still get better prediction on those criteria with OLG. Before an empirical work to answer this, this paper tries to analyze the theoretical building block of OLG with housing assets to explain the puzzle and others. Specifically this paper aims to analyze the asset pricing kernels with housing for non-separable utility functions in an OLG environment. When the model assumes non-separable utility specification, it would have not only the new type of risk (composition risk) but also a new type of pricing kernel. In this paper we focus on this new type of pricing kernel using housing as a numeraire. Its derivation and relationship with the usual kernel are studied.

1.1 Other Related Literature

There is some research on the consumption-based asset pricing models with housing. All the papers in the following list assume the infinitely lived agents (ILA). Lustig and Nieuwerburgh (2010) occupy assumption with state contingent heterogeneity. They specifically introduce an incomplete market due to housing collateral constraint, which keeps the household from defaulting. Davis and Martin (2009), working with home production with housing, estimate the intratemporal elasticity of substitution (ES) between housing and consumption. Similarly, Bansal, Tallarini, and Yaron (2008) estimate the intratemporal elasticity of substitution using composite wealth return (Note 10).

As mentioned in Introduction, Piazzesi, Schneider and Tuzel (2007) introduce composition risk relating change in asset prices to expenditure share and show improved performance by the composition risk factor. Fillat (2008) furthermore analyzes long run risk from composition risk, showing that presence of housing increases the price of the long run risk. Kwan, Leung and Dong (2014) compare eight different consumption-based asset pricing models (with or without housing) and show that not the existence of housing in the consumption-based models but how it is introduced matters in improving the models' performance. They find that recursive utility model and variation with housing outperform alternative models in predicting stock return, while a collateral constraint model outperforms in predicting housing return.

Regarding the traditional consumption risk in asset pricing and returns, but without housing, we may consider Lettau and Ludvigson (2001) and Parker and Julliard (2005). Models with modified consumption risk can be summarized by the external habit model (many works including Campbell & Cochrane, 1999) and the recursive preferences model with total wealth (Bansal & Yaron, 2004). Other than these, we may consider such models as ones with deviation from rational expectations (Brunnermeier & Julliard, 2007), ones with robust control (Hansen & Sargent, 2010) and ones with ambiguity aversion (Ulrich, 2010).

2. The Model

We want to study two-period OLG Model of two goods and three assets, where agents receive wage income only in the first period but consume in both periods. To incorporate the new asset pricing kernel with housing, we assume two types of consumption: usual consumption and housing service (Note 11). The three assets are (1) discounted bond, (2) claims to the housing consumption for the following period, and (3) claims to the nonhousing consumption for the following period.

2.1 Environment

Time is discrete. Consider an OLG economy, where each generation lives for two consecutive periods. They supply one unit of labor, in efficiency term e_t, and receive wage income only in the first period, while they consume in both periods, c_{yt} and c_{ot+1} representing consumption by the young at time t and consumption by the old at $t+1$. Also agents enjoy housing services in both periods, by renting a home from the existing housing stocks in the economy. There are three assets in the economy: one is a standard discounted bond and another one is a stock which does not include housing capital. Other than these two assets, there is a third asset called *housing capital* or *housing stock* because this capital is assumed reversible and traded in asset markets. Purchasing s units of stocks at price p_t^s in period t implies a random cum-dividend return $(p_{t+1}^s + d_{t+1}^s)s$ in the following period. The superscript s denotes the usual stock. Purchasing b units of the discount bond at price q_t implies one unit of return in the following period. Similar to the non-housing stocks, purchasing H units of housing capital at price p_t^h in period t implies a gross return of $(p_{t+1}^h + d_{t+1}^h)H$ in the following period, where the superscript h denotes the housing stock and d^h implies dividends to the housing stock or the rental earning from housing capital.

2.2 Consumer's Optimization

Young agents born in period t solve the following maximization problem.

$$U(c,h) = u(c_{yt}, h_{yt}) + \beta E u(c_{ot+1}, h_{ot+1}) \tag{1}$$

where

$$u(c_t,h_t) = \frac{g(c_t,h_t)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \tag{2}$$

for $\sigma > 0, v > 0$.

$$g(c_t,h_t) = \left(c_t^{\frac{v-1}{v}} + \omega h_t^{\frac{v-1}{v}}\right)^{\frac{v}{v-1}} \tag{3}$$

The value of CES aggregator g depends on v, which represents the importance of intratemporal smoothing. The higher the intratemporal elasticity of substitution v, the more willing the agents are to substitute housing for the other consumption within the same period. The two goods are perfect substitutes if v is infinite, while they are perfect complements if v is zero. Taking the limit as $v \rightarrow 1$ yields a Cobb-Douglas aggregator. Next is the intertemporal smoothing. The periodic utility function u depends on how willing agents are to substitute the CES aggregated bundle over time. The higher the intertemporal elasticity of substitution σ, the more willing the agents are to substitute for the future. It is well known that the elasticity is in fact the inverse of the coefficient of relative risk aversion, i.e., $\gamma = 1/\sigma$. Therefore, the representative young agent born in period t solves the following maximization problem:

$$Max\ U(c,h) \tag{4}$$

subject to

$$c_{yt} + d_t^h h_{yt} + p_t^h H_{t+1} + p_t^s s_{t+1} + q_t b_{t+1} = w_t e_t \tag{5}$$

$$c_{ot+1} + d_{t+1}^h h_{ot+1} = \left(p_{t+1}^h + d_{t+1}^h\right) H_{t+1}(1-\delta) + \left(p_{t+1}^s + d_{t+1}^s\right) s_{t+1} + b_{t+1} \tag{6}$$

where $w_t e_t$ represents a stochastic income and $\{b_{t+1}, s_{t+1}, H_{t+1}\}$ is each asset demand for the next period, while h_t is housing service and δ is the depreciation rate of housing stock, the total of which is fixed for simplicity. Assume $h_t = \eta H_t$, implying housing service is a constant stream out of housing stock each time.

2.2.1 Optimality

From the description above, we set up the following Lagrangian for a maximization problem of one objective with two constraints:

$$L = \frac{\left(\left(c_{yt}^{\frac{v-1}{v}} + \omega h_{yt}^{\frac{v-1}{v}}\right)^{\frac{v}{v-1}}\right)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \beta E_t \frac{\left(\left(c_{ot+1}^{\frac{v-1}{v}} + \omega h_{ot+1}^{\frac{v-1}{v}}\right)^{\frac{v}{v-1}}\right)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \tag{7}$$

$$+\lambda\left[w_t e_t - \left(c_{yt} + d_t^h h_{yt} + p_t^h H_{t+1} + p_t^s s_{t+1} + q_t b_{t+1}\right)\right]$$
$$+\mu\left[\left(p_{t+1}^h + d_{t+1}^h\right)H_{t+1}(1-\delta) + \left(p_{t+1}^s + d_{t+1}^s\right)s_{t+1} + b_{t+1} - \left(c_{ot+1} + d_{t+1}^h h_{ot+1}\right)\right]$$

where λ and μ are the Lagrangian multipliers *w.r.t.* the budget constraints of the household when young and old respectively. Solving the Lagrangian with choice variables of $\{c_{yt}, c_{ot+1}, h_{yt}, h_{ot+1}, b_{t+1}, s_{t+1}, H_{t+1}\}$ leads to the following optimality condition:

$$\left(c_{yt}^{\frac{v-1}{v}} + \omega h_{yt}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}} \left(c_{yt}\right)^{\frac{-1}{v}} = \lambda \tag{8}$$

$$\omega\left(c_{yt}^{\frac{v-1}{v}} + \omega h_{yt}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}} \left(h_{yt}\right)^{\frac{-1}{v}} = \lambda d_t^h \tag{9}$$

$$\beta E_t\left(c_{ot+1}^{\frac{v-1}{v}} + \omega h_{ot+1}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}} \left(c_{ot+1}\right)^{\frac{-1}{v}} = \mu \tag{10}$$

$$\omega\beta E_t\left(c_{ot+1}^{\frac{v-1}{v}} + \omega h_{ot+1}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}} \left(h_{ot+1}\right)^{\frac{-1}{v}} = \mu d_{t+1}^h \tag{11}$$

$$\mu = \lambda q_t \tag{12}$$

$$\mu(p_{t+1}^s + d_{t+1}^s) = \lambda p_t^s \tag{13}$$

$$\mu(p_{t+1}^h + d_{t+1}^h)(1-\delta) = \lambda p_t^h \tag{14}$$

From Eq. (8) and Eq. (9) or from Eq. (10) and Eq. (11), we get the following intratemporal optimality implication.

$$\omega\left(c_{yt}^{\frac{v-1}{v}} + \omega h_{yt}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}} \left(h_{yt}\right)^{\frac{-1}{v}} = \left(c_{yt}^{\frac{v-1}{v}} + \omega h_{yt}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}} \left(c_{yt}\right)^{\frac{-1}{v}} d_t^h \tag{15}$$

which simply leads to

$$\left(\frac{c_{yt}}{h_{yt}}\right)^{\frac{1}{v}} = \frac{d_t^h}{\omega} \tag{16}$$

implying that each time the young household's consumption of regular goods and housing service should be allocated according to their relative utility weights. The Eq. (16) represents the relative importance of housing to the household subjectively to the one objectively due to depreciation. Likewise, several equations for intertemporal optimality can be derived:

$$\beta E_t\left(c_{ot+1}^{\frac{v-1}{v}} + \omega h_{ot+1}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}} \left(c_{ot+1}\right)^{\frac{-1}{v}} = q_t\left(c_{yt}^{\frac{v-1}{v}} + \omega h_{yt}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}} \left(c_{yt}\right)^{\frac{-1}{v}} \tag{17}$$

$$\beta E_t\left(c_{ot+1}^{\frac{v-1}{v}} + \omega h_{ot+1}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}} \left(c_{ot+1}\right)^{\frac{-1}{v}}(p_{t+1}^s + d_{t+1}^s) = p_t^s\left(c_{yt}^{\frac{v-1}{v}} + \omega h_{yt}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}} \left(c_{yt}\right)^{\frac{-1}{v}} \tag{18}$$

$$(1-\delta)\beta E_t\left(c_{ot+1}^{\frac{v-1}{v}} + \omega h_{ot+1}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}} \left(c_{ot+1}\right)^{\frac{-1}{v}}(p_{t+1}^h + d_{t+1}^h) = p_t^h\left(c_{yt}^{\frac{v-1}{v}} + \omega h_{yt}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}} \left(c_{yt}\right)^{\frac{-1}{v}} \tag{19}$$

Each of these equations directly shows the inter-temporal marginal utility related with each choice variable among assets $\{b_{t+1}, s_{t+1}, H_{t+1}\}$. One may also notice that the other variables of consumption goods and housing services $\{c_{yt}, c_{ot+1}, h_{yt}, h_{ot+1}\}$ appear in all three equations aligned along their periodic states. Because these elements are common in these equations, we may infer that they can serve for asset pricing building blocks. In fact, an asset pricing kernel or stochastic discount factor may be constructed from any of the common variables using an intertemporal marginal rate of substitution (MRS) between two intertemporal goods. In general, MRS over consumption goods is used for this and the usual C-CAPM utilizes this consumption-based pricing kernel. When there is no other consumption element such as housing service, the simplest c-pricing kernel is $\beta u'(c_{t+1})/u'(c_t)$. Intertemporal optimization implies that a household determining how much he would consume today and tomorrow (next period) by allocating an asset should equate marginal utility over time. In other words, the household adjusts the asset allocation until the expected MRS between the two periods is equal to one: $E[|[\beta u'(c_{t+1})/u'(c_t)]x_{t+1}|] = 1$, where x_{t+1} represents expected return of an asset (Note 12). When today's marginal utility is low due to higher asset holding for tomorrow, then it should be compensated by higher marginal utility of the next period through asset payoff. Therefore it is clear that through the asset pricing kernel, asset prices are linked to the household's marginal utility of consumption over time. For the CES aggregator in the model, the pricing kernel with consumption as numeraire is (Note 13):

$$M_{t+1}^{c} = \beta\left(\frac{1+\omega\left(\frac{h_{ot+1}}{c_{ot+1}}\right)^{\frac{v-1}{v}}}{1+\omega\left(\frac{h_{yt}}{c_{yt}}\right)^{\frac{v-1}{v}}}\right)^{\frac{\sigma-v}{\sigma(v-1)}} \tag{20}$$

As Piazzesi et al. (2007) argue, the CES aggregator gives a pricing kernel that represents the composition risk. In equation (20), the pricing kernel is not only determined by the usual consumption growth but also by the consumption ratio growth: ratio between the two arguments of goods and housing services and the growth between the two periods of young and old. However, there is another way to construct the pricing kernel because we have one more element (housing service) that gives utility to the household in the model. It is the pricing kernel that is obtained when we set housing service as the numeraire. The kernel is:

$$M_{t+1}^{h} = \beta\left(\frac{1+\omega\left(\frac{h_{ot+1}}{c_{ot+1}}\right)^{\frac{v-1}{v}}}{1+\omega\left(\frac{h_{yt}}{c_{yt}}\right)^{\frac{v-1}{v}}}\right)^{\frac{\sigma-v}{\sigma(v-1)}}\left(\frac{h_{ot+1}}{h_{yt}}\right)^{\frac{-1}{v}} \tag{21}$$

which implies that the pricing kernel is not only determined by the housing service growth rate but also by the relative ratio growth rate. It is easy to see how one of the kernels is related to the other by looking at the following equation from the intratemporal optimality.

$$M_{t+1}^{c} = M_{t+1}^{h}\frac{d_t^h}{d_{t+1}^h} \tag{22}$$

2.2.2 Asset Prices and Returns

Asset prices can be constructed using one of the kernels. Because the value of an asset is equal to the asset's expected payoff discounted by a pricing kernel, information regarding each asset's expected payoff is necessary. In the case of a discounted bond, the payoff is simply one. Payoffs to usual stocks and housing stocks are $p_{t+1}^s + d_{t+1}^s$ and $p_{t+1}^h + d_{t+1}^h$, respectively. Thus if we use the consumption-numeraired pricing kernel, then the bond price is obtained by

$$q_t = E_t\frac{\beta\left(c_{ot+1}^{\frac{v-1}{v}}+\omega h_{ot+1}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}}(c_{ot+1})^{\frac{-1}{v}}}{\left(c_{yt}^{\frac{v-1}{v}}+\omega h_{yt}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}}(c_{yt})^{\frac{-1}{v}}} = E_t M_{t+1}^{c} \tag{23}$$

or equivalently, by occupying the housing-numeraired pricing kernel, the bond price is written by

$$q_t = E_t\frac{\omega\beta\left(c_{ot+1}^{\frac{v-1}{v}}+\omega h_{ot+1}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}}(h_{ot+1})^{\frac{-1}{v}}\frac{1}{d_{t+1}^h}}{\omega\left(c_{yt}^{\frac{v-1}{v}}+\omega h_{yt}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}}(h_{yt})^{\frac{-1}{v}}\frac{1}{d_t^h}} = E_t M_{t+1}^{h}\frac{d_t^h}{d_{t+1}^h} \tag{24}$$

Similarly, the stock price is obtained by

$$p_t^s = E_t\frac{\beta\left(c_{ot+1}^{\frac{v-1}{v}}+\omega h_{ot+1}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}}(c_{ot+1})^{\frac{-1}{v}}}{\left(c_{yt}^{\frac{v-1}{v}}+\omega h_{yt}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}}(c_{yt})^{\frac{-1}{v}}}(p_{t+1}^s + d_{t+1}^s) \tag{25}$$

which is equal to $E_t[M_{t+1}^c(p_{t+1}^s + d_{t+1}^s)]$ or by housing numeraire, $p_t^s = E_t[M_{t+1}^h[d_t^h/d_{t+1}^h](p_{t+1}^s + d_{t+1}^s)]$. Likewise, the housing price is:

$$p_t^h = E_t \frac{\beta(1-\delta)\left(c_{ot+1}^{\frac{v-1}{v}}+\omega h_{ot+1}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}}(c_{ot+1})^{\frac{-1}{v}}}{\left(c_{yt}^{\frac{v-1}{v}}+\omega h_{yt}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}}(c_{yt})^{\frac{-1}{v}}}(p_{t+1}^h + d_{t+1}^h) \tag{26}$$

which returns to $E_t[(1-\delta)M_{t+1}^c(p_{t+1}^h + d_{t+1}^h)]$ or $p_t^h = E_t[(1-\delta)M_{t+1}^h[d_t^h/d_{t+1}^h](p_{t+1}^h + d_{t+1}^h)]$. If we turn to gross returns of these assets, we have: because the discount bond at price q_t implies one unit of return in the following period, the bond returns in terms of each pricing kernel is (Note 14):

$$x_{t+1}^q = \frac{1}{q_t} = \frac{1}{E_t M_{t+1}^c} = \frac{1}{E_t M_{t+1}^h \frac{d_t^h}{d_{t+1}^h}} \tag{27}$$

or $E_t\left[M_{t+1}x_{t+1}^q\right] = 1$ for $M_{t+1} = M_{t+1}^c = M_{t+1}^h[d_t^h/d_{t+1}^h]$. Also because both non-housing stocks and housing stocks purchased at p_t imply random cum-dividend return $\{p_{t+1} + d_{t+1}\}$ in the following period, the return in terms of a unified notation is:

$$x_{t+1} = E_t \frac{p_{t+1}+d_{t+1}}{p_t} \tag{28}$$

From $E_t[M_{t+1}x_{t+1}] = 1$, we can construct the risk premium. The equation gives $E_t[M_{t+1}]E_t[x_{t+1}] + Cov_t[M_{t+1}\ ,\ x_{t+1}] = 1$. For riskless bonds, the gross return (Note 15) satisfies $x_{t+1}^q E_t[M_{t+1}] = 1$, and thus the housing market risk premium k^h is defined by

$$k_{t+1}^h = E_t[x_{t+1}^h] - x_{t+1}^f = -\frac{Cov_t\left[M_{t+1}^h \frac{d_t^h}{d_{t+1}^h}\ ,\ x_{t+1}^h\right]}{E_t\left[M_{t+1}^h \frac{d_t^h}{d_{t+1}^h}\right]} \tag{29}$$

or $k_{t+1}^h = -\frac{Cov_t\left[M_{t+1}^c,\ x_{t+1}^h\right]}{E_t\left[M_{t+1}^c\right]}$ using consumption as the numeraire.

3. Equilibrium

The Model OLG economy is summarized by a set of preference parameters $\{\beta, \sigma, \omega, v\}$ and stochastic processes for consumption and housing service for both young and old. The following definition describes the asset prices under the optimal behavior in a competitive system.

Definition *A competitive (partial) equilibrium of the economy is a collection of asset price processes* $\{q_t, p_t^s, p_t^h\}$, *allocation processes of consumption and housing service of young and old,* $\{(c_{yt}, h_{yt}), (c_{ot+1}, h_{ot+1})\}$, *and a series of asset holdings by the young,* $\{b_{t+1}, s_{t+1}, H_{t+1}\}$ *such that given the price processes, the following is satisfied:*

i) Given the price system, the allocation {young, old} solves the consumer's maximization problem each time.

ii) For the given pair of allocations, all the asset markets clear, i.e. net supply of bond is zero, net supply of equity is unity and net supply of housing is fixed by H *for each time.*

3.1 Optimal Allocation

To explore the meaning of the market clearing condition, consider the optimal allocation at time *t* under the equilibrium prices:

$$c_{yt} = w_t e_t - p_t^s - p_t^h H_t - d_t^h h_{yt} \tag{30}$$

$$c_{ot} = p_t^s + d_t^s + (p_t^h + d_t^h)H_t(1-\delta) - d_t^h h_{ot} \tag{31}$$

This follows from the market clearing condition that net supply of bond is zero and net supply of equity is unity. Thus

$$c_{yt} + c_{ot} = w_t e_t + d_t^s - p_t^h H_t - d_t^h\left(h_{yt} + h_{ot}\right) + (p_t^h + d_t^h)H_t(1-\delta) \tag{32}$$

Assume that the housing capital market clears at $H_{t+1} = H_t = H$. This implies that there is construction by the depreciation amount but zero growth in net construction and $h_t = \eta H_t$. Then

$$c_t = c_{yt} + c_{ot} = w_t e_t + d_t^s + [d_t^h(1-\eta) - \delta(p_t^h + d_t^h)]H \tag{33}$$

or $c_t = w_t e_t + d_t^s + [(1-\eta-\delta)d_t^h - \delta d_t^h]H$. This is because it is true that the housing stocks satisfy

$$H_{yt} + H_{ot} = H_t \tag{34}$$

while the housing services

$$h_{yt} + h_{ot} = \eta H_{yt} + \eta H_{ot} = \eta H_t \tag{35}$$

Let $e_t = 1$, then the steady state consumption is obtained by

$$c = w + d^s + [(1-\eta-\delta)d^h - \delta p^h]H \tag{36}$$

For each generation of young and old,

$$c_y = w - p^s - p^h H - d^h h_y \tag{37}$$

and

$$c_o = p^s + d^s + (p^h + d^h)H(1-\delta) - d^h h_o \tag{38}$$

Finally notice that the steady state housing stock and service satisfy $H_y + H_o = H$ and $h_y + h_o = \eta H.$

3.2 Steady State Asset Prices

From Equations (23), (25), (26), the non-stochastic steady state asset price equations in terms of consumption numeraire satisfy:

$$q = M^c \tag{39}$$

$$p^s = M^c(p^s + d^s) \tag{40}$$

$$p^h = (1-\delta)M^c(p^h + d^h) \tag{41}$$

where

$$M^c = \beta \left(\frac{1+\omega\left(\frac{h_o}{c_o}\right)^{\frac{v-1}{v}}}{1+\omega\left(\frac{h_y}{c_y}\right)^{\frac{v-1}{v}}} \right)^{\frac{\sigma-v}{\sigma(v-1)}} \left(\frac{c_o}{c_y}\right)^{\frac{-1}{v}} \tag{42}$$

Because at steady states, both pricing kernels are identical (Note 16), it follows that $M^c = M^h \equiv M$. Thus the steady state asset prices are:

$$q = M \tag{43}$$

$$p^s = \frac{M}{1-M} d^s \tag{44}$$

$$p^h = \frac{M(1-\delta)}{1-M(1-\delta)} d^h \tag{45}$$

If the housing depreciation rate is equal to zero, then p^s and p^h are not different from each other but dividends. The following figures demonstrate a set of simulation results for the steady state asset prices when the steady state consumption and housing are set to certain normalized values following a 2013 consumer expenditure survey (Note 17). According to this survey, the housing expenditure is about one half of the nonhousing consumption expenditure. Furthermore, we assume that consumption of old age is lower than the consumption of young age, while the housing expenditure remains about the same. Therefore we set $\{c_y, c_o, h_y, h_o\}$={1, 0.75, 0.5, 0.5}. Because one period in a two-period OLG model is about 30~35 years, we may set the parameter values at $\{\beta, \delta\}$ ={0.7, 0.5}. Also following Piazzesi et al. (2007), we can set the intratemporal and intertemporal parameters at $\{v, \sigma\}$ ={1.5, 2}. Finally regarding the normalized dividend value: because the depreciation appears in the valuation of housing stock, but not in nonhousing stock, the dividend stream should be higher at equilibrium with housing stocks than with nonhousing stocks (Note 18). Thus, to incorporate this relationship, we may set $\{d^s, d^h\}$ ={0.5, 2.5}. Figure 1 and Figure 2 show the steady state asset prices of stock, housing, and bond, from the baseline model with both $\sigma = 0.5$ and $\sigma = 2$, over different values of ω. Because the change in stock value is visible from these figures, we want to see the stock price graph in isolation along with different sigma values. Figure 3 shows this result. When we set ω= 0.5, we further obtain the graphs with varying σ (Figure 4) or varying v (Figure 5).

4. Conclusion

In this paper we develop an asset pricing model with housing and derive a new type of pricing kernel using housing service as the numeraire. Like Piazzesi et al. (2007), this paper occupies a consumption-based asset pricing model, in which there is an explicit role of housing as both a consumption good and an asset. In Piazzesi et al. (2007), a representative agent consumes housing service and nonhousing consumption goods, which is the numeraire for their ILA model in frictionless asset markets. Instead of occupying the standard consumption good-based asset pricing kernel, this paper develops a new type of housing-numeraired asset pricing kernel in an OLG model and explores its property. It is shown that housing based numeraire is closely related with consumption based numeraire and in non-stochastic steady states, the two are not different from each other. A simulation result for steady states shows that stock price responds more than the other prices to variation of consumer parameters. By the interrelationship among many consumer parameters in asset pricing model with housing, we may better understand the movement of housing asset prices.

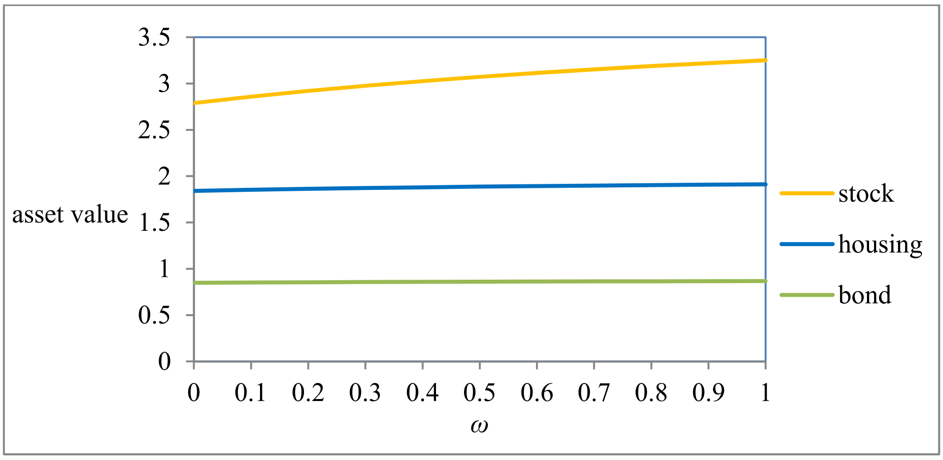

Figure 1. Steady state asset prices over *ω*, when *σ*=2, *ν*=1.5

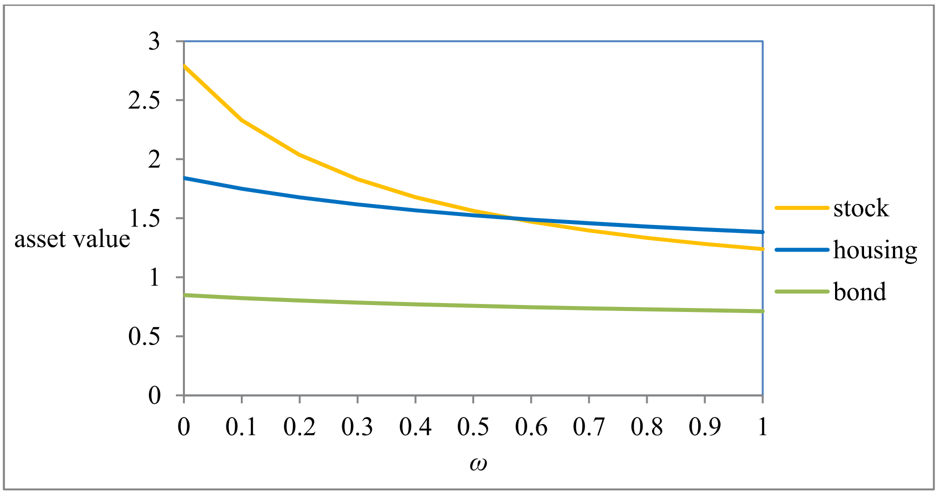

Figure 2. Steady state asset prices over *ω*, when *σ*=0.5, *ν*=1.5

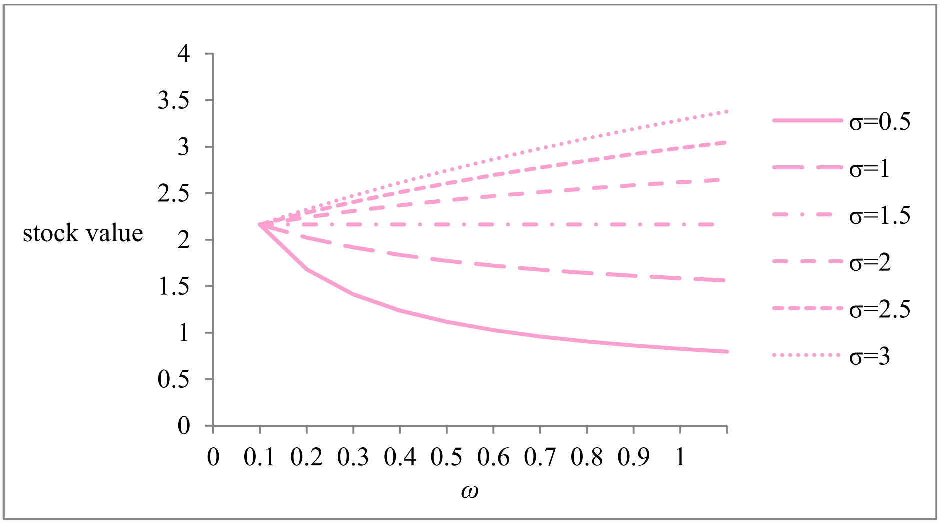

Figure 3. Steady state asset prices over *ω* with various *σ*, when *ν*=1.5

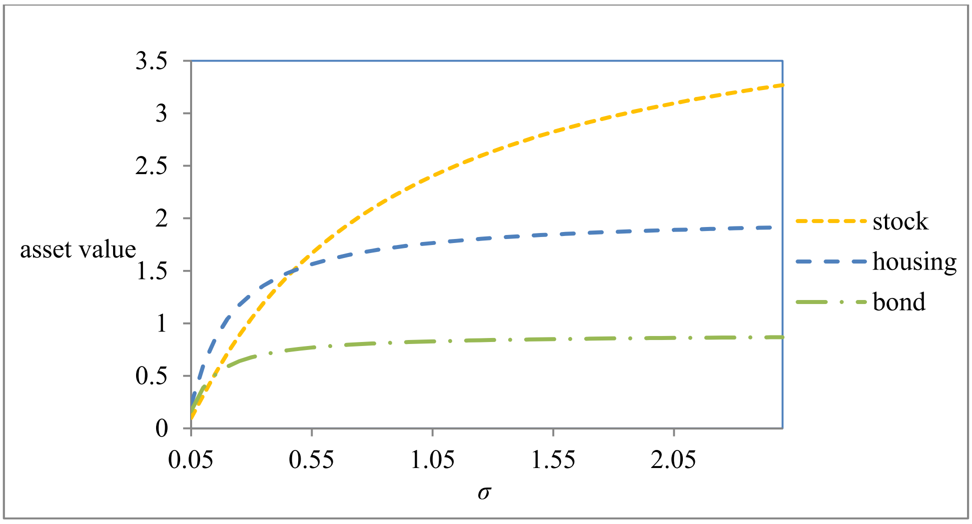

Figure 4. Steady state asset prices over σ, when ω=0.5, v=1.5

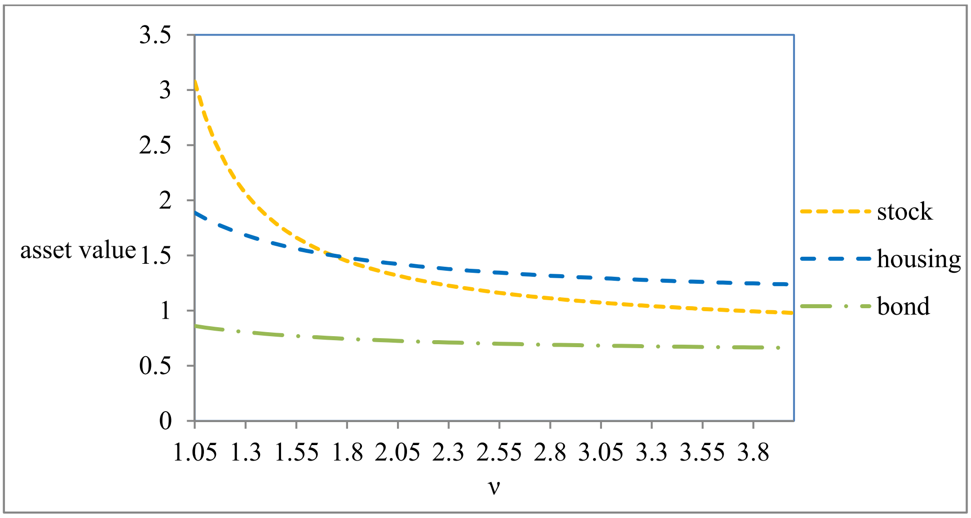

Figure 5. Steady state asset prices over v, when ω=0.5, σ=1

References

Bansal, R., & Yaron, A. (2004). Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles. *Journal of Finance, 59*, 1481-1509. http://dx.doi.org/10.1111/j.1540-6261.2004.00670.x

Bansal, R., Tallarini, T., & Yaron, A. (2008). *The return to wealth, asset pricing and the intertemporal elasticity of substitution*. Mimeo, Duke University.

Bohm, V., Kikuchi, T., & Vachadze, G. (2008). Asset Pricing and Productivity Growth: The Role of Consumption Scenarios. *Computational Economics, 32*, 163-181. http://dx.doi.org/10.1007/s10614-008-9137-3

Brunnermeier, M. K., & Julliard, C. (2008). Money Illusion And Housing Frenzies. *The Review of Financial Studies, 21*, 135-180. http://dx.doi.org/10.1093/rfs/hhm043

Campbell, J., & Cochrane, J. (1999). Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior. *Journal of Political Economy, 107*, 205-251. http://dx.doi.org/10.1086/250059

Cochrane, J., & Hansen, L. (1992). Asset Pricing Explorations for Macroeconomics. *NBER Macroeconomics Annual*, 115-182. http://dx.doi.org/10.2307/3585001

Constantinides, G. M., & Duffie, D. (1996). Asset Pricing with Heterogeneous Consumers. *Journal of Political Economy, 104*. http://dx.doi.org/10.1086/262023

Constantinides, G. M., Donaldson, J., & Mehra, R. (2002). Junior can't borrow: A new Perspective on the equity premium puzzle. *Quarterly Journal of Economics, 117*, 269-296. http://dx.doi.org/10.1162/003355302753399508

Davis, M. A., & Martin, R. F. (2009). Housing, home production, and the equity- and value-premium puzzles. *Journal of Housing Economics, 18*, 81-91. http://dx.doi.org/10.1016/j.jhe.2009.04.006

Dimson, E., Marsh, P., & Staunton, M. (2002). *Triumph of the Optimists: 101 Years of Global Investment Returns*. Princeton University Press.

Esptein, L., & Zin, S. (1989). Substitution, Risk Aversion and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica, 57*, 937-969. http://dx.doi.org/10.2307/1913778

Fillat, J. L. (2009). *Housing as a Measure for Long-Run Risk in Asset Pricing*. AFA 2009 San Francisco Meetings Paper.

Hansen, L., & Sargent, T. (2010). Fragile Beliefs and the Price of Uncertainty. *Quantitative Economics, 1*, 129-162. http://dx.doi.org/10.3982/QE9

Jaccard, I. (2012). *Asset Pricing and Housing Supply in a Production Economy*. Working Paper 1454, European Central Bank.

Kwan, Y., Leung, C., & Dong, J. (2014). *Comparing Consumption-based Asset Pricing Models: The Case of an Asian City*. Munich Personal RePEc Archive (MPRA) 60513.

Lettau, & Ludvigson. (2001). Consumption, Aggregate Wealth, and Expected Stock Returns. *The Journal of Finance, 56*, 815-849. http://dx.doi.org/10.1111/0022-1082.00347

Leung, C. (2004). Macroeconomics and housing: A review of the literature. *Journal of Housing Economics, 13*, 249-267. http://dx.doi.org/10.1016/j.jhe.2004.09.002

Lustig, H., & Van Nieuwerburgh, S. (2010). How Much Does Household Collateral Constraint Regional Risk Sharing? *Review of Economic Dynamics, 13*, 265-294. http://dx.doi.org/10.1016/j.red.2009.09.005

Mehra, R., & Prescott, E. C. (1985). The Equity Premium: A Puzzle. *Journal of Monetary Economy, 15*, 145-161. http://dx.doi.org/10.1016/0304-3932(85)90061-3

Parker, J., & Julliard, C. (2005). Consumption Risk and the Cross Section of Expected Returns. *Journal of Political Economy, 113*, 185-222. http://dx.doi.org/10.1086/426042

Piazzesi, M., Schneider, M., & Tuzel, S. (2007). Housing, consumption and asset pricing. *Journal of Financial Economics, 83*, 531-569. http://dx.doi.org/10.1016/j.jfineco.2006.01.006

Storesletten, K., Telmer, C., & Yaron, A. (2007). Asset Pricing With Idiosyncratic Risk and Overlapping Generations. *Review of Economic Dynamics, 10*, 519-548. http://dx.doi.org/10.1016/j.red.2007.02.004

Tibaijuka, A. (2011). Building Prosperity: Housing and Economic Development. *Geographical Research, 49*(4), 450-451. http://dx.doi.org/10.1111/j.1745-5871.2010.00680.x

Uhlig, M. (2011). *Observable Long-Run Ambiguity and Long-Run Risk*. Mimeo. Columbia University.

Weil, P. (1989). The equity premium puzzle and the risk-free rate puzzle. *Journal of Monetary Economics, 24*, 401-421. http://dx.doi.org/10.1016/0304-3932(89)90028-7

Notes

Note 1. Regarding the role as collateral, see Lustig and Nieuwerburgh (2010).

Note 2. As shown later in detail, from the intertemporal optimization, we get the kernel: $M_{t+1} = \beta(c_{t+1}/c_t)^{-\gamma}$, where γ is inverse of inter-temporal substitution.

Note 3. For example, average return from equities is 6.7%, while one from T-bills is 0.9%. (Data from Dimson, Marsh, and Staunton, 2002.) Because the standard deviation of equity is 20.2% and the standard deviation of $\Delta\ln c$ is 1.1%, it follows that (6.7%-0.9%)/20.2% = 0.29 $\leq 0.011\gamma$.

Note 4. From the above note 3, the risk aversion (γ) should be greater than 26, which is not supported in most economics literature.

Note 5. The standard model posits an ILA representative agent from homogeneous population with time-separable power utility in complete markets.

Note 6. But this approach engenders volatile interest rates.

Note 7. Composition risk with housing implies the risk associated with the relative share of housing in the consumption bundle.

Note 8. They allow persistent income shock with counter-cyclical volatility.

Note 9. In their finding: (i) risk premium: OLG > ILA; (ii) equity return: ILA (over-predicted) > OLG (reasonable); (iii) bond return: ILA > OLG > Data.

Note 10. In their work, the estimated value is to be ES >1.

Note 11. Piazzesi et al. (2007) assume non-separable utility of consumption and housing with ILA agents and derive composition risks based on consumption-numeraired pricing kernel.

Note 12. With CRRA utility, because $M_{t+1} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$, this implies $E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} x_{t+1}\right]$=1.

Note 13. This is because $M_{t+1}^c = \frac{\beta\left(c_{ot+1}^{\frac{v-1}{v}}+\omega h_{ot+1}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}}(c_{ot+1})^{\frac{-1}{v}}}{\left(c_{yt}^{\frac{v-1}{v}}+\omega h_{yt}^{\frac{v-1}{v}}\right)^{\frac{\sigma-v}{\sigma(v-1)}}(c_{yt})^{\frac{-1}{v}}}$

Note 14. Therefore $E_t\left[M_{t+1}x_{t+1}^q\right] = 1.$ Again, for the CRRA utility specification, the pricing kernel returns to $M_{t+1} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$. Thus, this implies $E_t\left[x_{t+1}^q\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\right] = 1.$

Note 15. $x_{t+1}^q = 1 + r_{t+1}^f$ for risk free assets.

Note 16. This is from $M_{t+1}^c = M_{t+1}^h[d_t^h/d_{t+1}^h]$.

Note 17. The baseline values for this exercise are obtained from the Consumer Expenditures-2013 (BLS, 2014): according to this data, the annual housing expenditure is $17,148 and the annual nonhousing consumption expenditure is about $34,000.

Note 18. Or the housing stock price should be lower than that of nonhousing stock, given the same dividend value.