Portfolio Value at Risk Bounds Using Extreme Value Theory

Skander Slim
E-mail: skander.slim@u-paris10.fr
Imed Gammoudi
E-mail: imed_.gamoudi@yahoo.fr
Lotfi Belkacem
Quantitative Finance Laboratory, Higher Institute of Commercial Studies
University of Sousse, Route de la ceinture - B.P:40, Sousse 4054, Tunisia
Tel: 216-73-368-351 E-mail: Belkacem@isgs.rnu.tn

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Abstract

The aim of this paper is to apply a semi-parametric methodology developed by Mesfioui and Quessy (2005) to derive the Value-at-Risk (VaR) bounds for portfolios of possibly dependent financial assets when the marginal return distribution is in the domain of attraction of the generalized extreme value distribution while the dependence structure between financial assets remains unknown. However, These bounds are very sensitive to location changes and depend heavily on the actual location. Modified VaR bounds are derived through an extension of the Vermaat, Does and Steerneman (2005) contribution on quantile estimation of large order to a multivariate setting which enjoy the interesting property of location invariance. Empirical studies for several market indexes are carried out to illustrate our approach.

Keywords: Copulas, Extreme value theory, Location invariance, Portfolio value at risk, Dependent risks

1. Introduction

Portfolio managers have to face questions related to extremal events for handling problems concerning the probable maximal loss of investment strategies. As highlighted by the past turbulences in financial markets such as the Russian financial crisis in 1998, the burst of the speculative information technology bubble in 2001, or recently the subprime crisis, extreme price fluctuations may expose at times portfolio managers to high levels of market risk. In this specific context, funds managers have to pay more attention on the distribution of « large» stock returns values and implement suitable risk management tools to avoid big losses. Some funds managers and other market participants have been driven to adopt such quantitative risk management techniques not only in reaction to their own experience of market turbulence, but also because of regulatory climate. Hence, pension funds and insurance companies portfolio managers are not allowed to take high risks and are constrained in their management by some statutory regulatory restrictions, with the aim to ensure diversification for limiting investment risks. Therefore, given the non-Gaussian character of financial returns distributions and in consequence the limitation of the variance as an indicator for describing the amount of uncertainty in their fluctuations, various risk measures have been proposed, such as VaR (Jorion, 1997; Duffie and Pan, 1997). VaR techniques aim to quantify the worst expected loss of a portfolio over a specific time interval at a given confidence level α. In other words, VaR is the highest α-quantile of potential loss that can occur within a given portfolio during a specified time period. Although VaR is widely adopted by the financial community, it has undesirable mathematical characteristics such as a lack of sub additivity and convexity. So that, VaR is not a coherent measure of risk for non-elliptical portfolios (Artzner, Delbaen, Eber and Heath., 1999). Hence, the risk associated to a given portfolio may be greater than the sum of the risks of the individual assets, when measured by VaR. Besides dependence seems to be particularly pronounced during sock market crises, which emphasizes that financial assets become more dependent in the lower tail during extreme market movements (Longin and Solnik, 2001; Ang and Chen, 2002; Cappiello, Engle, and Sheppard 2006). The key feature of measuring the risk of an aggregate position is to accurately depict the underlying dependence structure between the marginal business lines. In the context of elliptical distributions the correlation matrix is naturally used to describe the dependence structure of the individual risks. However, outside the family of elliptical distributions, correlation only might provide partial or misleading information on the actual underlying dependencies (Embrechts, McNeil and Straumann, 2002). Copulas seem to be an interesting alternative for dependence characterization. In order to estimate portfolio VaR in a classical way, one have to assume a marginal return distribution and a specific dependence structure between the portfolio components by fitting a parametric copulas to data (Mendes and Souza, 2004; Junker and May, 2005; Miller and Liu, 2006). However, this may lead to a model-dependent view of stochastic dependence. A more flexible and reliable approach consists in assessing portfolio risk when there is no information on the asset's dependence structure but only their marginal distributions are known. It is obvious, in this specific framework, of unknown dependence structure, that an explicit portfolio VaR is unreachable but one is interested in finding lower and upper bounds on VaR**.** Recent contributions include (Denuit, Genest , and Marceau, 1999; Denuit, Dhaene, Goovaerts, and Kaas, 2005; Embrechts, Höing, and Puccetti, 2005; Embrechts and Puccetti, 2006a ; Kaas, Laeven, and Nelsen, 2009; Luciano and Marena, 2002; Mesfioui and Quessy, 2005) and refer to Embrechts and Puccetti (2006b) for an extension to the general case where the marginal distributions of each of the assets are different. In a such case, a numerical procedure is suggested to compute VaR bounds. In this paper, we study VaR bounds for sums of possibly dependent financial assets using the recent contribution by Mesfioui and Quessy (2005) when no information on the dependence between them is available. The underlying dependence involving the portfolio constituents is captured through the empirical copula. The Generalized Extreme Value (GEV) distribution is assumed for marginals, as its ability to replicate the tail behaviour of asset returns (Bali, 2003). Besides, we derive modified bounds for portfolio VaR when a large shift of location is applied to asset returns as a generalization of the quantile estimator suggested by Vermaat et al. (2005) to the multivariate framework. These modified bounds have the advantage of being insensitive with respect to location changes. The outline of the paper is as follows. Section 2 recalls the notion of portfolio VaR and some mathematical background about the concept of copulas and the Fréchet bounds. In section 3, the GEV distribution is introduced for modelling big losses of financial assets. Explicit expressions for the lower and the upper bounds of VaR are then given under the assumption of GEV distribution for marginals when a shift of location is applied to the original data. The relevance of these bounds for describing portfolio risk is discussed through several applications including five international stock market indexes: empirical studies are carried out in section 5 and some concluding remarks are collected in section 6.

2. Copulas and VaR Bounds: Some Basics

In this section we give the basic concepts about copulas and the fundamental results about the problem of bounding VaR for functions of dependent risks, while we refer the interested reader to Nelsen (1999) for a more detailed treatment of copulas as well as their relationship to the different concepts of dependence

Definition *Let $f:\mathbb{R}\to\mathbb{R}$ be a non-decreasing function. Its generalized left continuous inverse is the mapping $f^{-1}:\mathbb{R}\to\mathbb{R}$ defined by $f^{-1}(t)=\inf\{s\in\mathbb{R} \mid f(s)\geq t\}$. The VaR at probability level $\alpha\in[0,1]$ for a random variable X with (right-continuous) distribution function F is defined by*

$$VaR_\alpha = F^{-1}(\alpha)$$

In order to evaluate the risk level of a portfolio of possibly dependent financial assets, one is involved in the dependence structure among the risks. This can be carried out through the use of copulas. The concept of copula was introduced by Sklar (1959), but only recently its potential for applications in finance has become clear. A review of applications of copulas in finance can be found in Embrechts, Höing and Juri (2003) and Cherubini, Luciano and Vecchiato (2004). Let the multivariate distribution function of a random vector $X=(X_1,...,X_d)$ be defined as $H(x_1,...,x_d)=\mathrm{P}(X_1\leq x_1,...,X_d\leq x_d)$ and denote by $F_1,...,F_d$ the set of the associated marginal distributions. The sklar's theorem (Sklar, 1959) states that there exists a multidimensional copula C such that $H(x_1,...,x_d)=C(F_1(x_1),...,F_d(x_d))$ for all $(x_1,...,x_d)\in\mathbb{R}^d$. For continuous marginal distributions C is unique. Therefore, The multivariate distribution function H is linked to the marginal distributions via the copula function $C:[0,1]^d\to[0,1]$.

Let us denote the risky asset losses with $X_1,...,X_d$, where $X_i:\Omega\to\mathbb{R}, i=1,2,...,d$, and consider the portfolio strategy consisting in investing a fixed relative amount $w_i\in[0,1]$ of the capital in the i-th asset (short sales being excluded), so that $\sum_{i=1}^d w_i=1$ (the portfolio is fully invested). For given weights, the portfolio loss can be expressed as the sum of the random variable $S_i\overset{d}{=}w_iX_i$ with known continuous marginal distribution functions $F_1,\ldots,F_d$. It is assumed throughout this paper that the underlying dependence structure of $S_1,\cdots,S_d$ described by the copula C is unknown. However, it is supposed that there exist copulas C_L and C_U such that

$C \geq C_L$ and $C^D \geq C_U^D$. Let denote by F_S the distribution function of the portfolio loss, $S = S_1 + \cdots + S_d$. For high dimensional portfolios, $d > 2$, stochastic bounds for F_S, easily obtained from the multivariate version of Makarov (1981) in Cossette, Denuit, and Marceau (2002), are $F_L(s) \leq F_S(s) \leq F_U(s)$, where

$$F_L(s) = \sup_{u_1+\cdots+u_d=s} C_L\{F_1(u_1),\ldots,F_d(u_d)\}$$

and

$$F_U(s) = \inf_{u_1+\cdots+u_d=s} C_U^D\{F_1(u_1),\ldots,F_d(u_d)\}$$

The distribution functions (Note 1), F_L and F_U, known respectively as the lower and the upper Fréchet bounds, provide the best possible bounds of F_S in terms of stochastic dominance (Lehmann, 1966). When no information about the dependence structure of d risks is available and if for each $i \in \{1,\ldots,d\}$ there exist a number s_i^* such that the density $f_i(s)$ is non-increasing for all $s \leq s_i^*$, it is shown in Mesfioui and Quessy (2005) that portfolio VaR for a given confidence level α, denoted by $VaR_\alpha(S)$, is bounded as follows

$$\underline{VaR}(\alpha) \leq VaR_\alpha(S) \leq \overline{VaR}(\alpha)$$

where

$$\overline{VaR}(\alpha) = \inf_{u_1+\cdots+u_d=\alpha+d-1} \sum_{i=1}^{d} F_i^{-1}(\alpha_i) \tag{1}$$

and

$$\underline{VaR}(\alpha) = \max_{1\leq i\leq d}\left\{F_i^{-1}(\alpha) + \sum_{1\leq j\neq i\leq d} F_j^{-1}(0)\right\}, \quad \alpha \leq \min\left\{F_1(s_1^*),\ldots,F_n(s_n^*)\right\} \tag{2}$$

The above result is crucial since it allows to easily compute explicit VaR bounds for possibly dependent risks. One only needs to assume an appropriate distribution function for the marginal asset losses. Explicit, analytical bounds exist when all the S_i r.v.'s belong to the same distribution class.

3. On Portfolio Extreme Loss Distribution and Quantile Estimation

In this section, we focus on the maximal relative loss (Note 2) of weighted financial assets over a large period of time n, that is to say on $M_n = \max_{t=1,\ldots,n} S(t)$, of which fluctuations may be characterized by an asymptotic extreme value distribution. The asymptotic theory for extremes is well developed and univariate modelling basically falls in one of the two categories: modelling standardized maxima through the GEV distribution, or modelling excesses over high thresholds via the generalized Pareto distribution (refer to Embrechts, Klüppelberg, and Mikosch, 1997, for further details on extreme value theory). We use GEV to approximate the distribution of suitably normalized extreme asset losses $(M_n - \mu)/\sigma$, where σ and μ are scale and location parameters, respectively. The GEV distribution function is given by

$$F_\gamma(s) = \exp\left\{-(1+\gamma s)^{-1/\gamma}\right\}, 1+\gamma s \geq 0, \gamma \in \mathbb{R}$$

which is interpreted as $\exp\left\{e^{-s}\right\}$ for $\gamma = 0$. The tail index γ characterizes the extreme upper behaviour of an i.i.d. sequence $(S(t))_{1\leq t\leq n}$ drawn from its unknown distribution function F regarding to its maximum value. It is a well-known result in extreme value theory that there are only three types of possible limit distributions for the maximum of i.i.d. r.v.'s under positive affine transformations, depending on the tail behaviour of their common density. The Fisher-Tippet Theorem (Fischer and Tippett, 1928; Gnedenko, 1943) implies that, The limit distribution of the normalized maxima, when $n \to \infty$, must be one of the Gumbel, the Weibull and the Fréchet distribution which are respectively related to $\gamma = 0$, to $\gamma < 0$, and to $\gamma > 0$. The Gumbel distribution is reached for thin-tailed return distributions. The Fréchet distribution is obtained for fat-tailed distributions of returns. Finally, the Weibull distribution is obtained when the distribution of returns has no tail. Therefore, the class of GEV distribution is very flexible with the tail index parameter controlling the shape of the tails of the three different families of distributions subsumed under it. Let $S_{(1)} < S_{(2)} < \cdots < S_{(n)}$ denote the order statistics of the weighted asset returns sequence $S_1, \ldots, S_n$ and m the number of the largest order statistics. When no assumption being made on the tail index γ, one may use the moment estimator proposed by Dekkers, Einmahl, and Haan (1989) for quantiles of large order defined by

$$\hat{\gamma}_n = M_n^{(1)} + 1 - \frac{1}{2}\left\{1 - \frac{(M_n^{(1)})^2}{M_n^{(2)}}\right\}^{-1}, \tag{3}$$

where

$$M_n^{(r)} = \frac{1}{m}\sum_{j=1}^{m} \log\left(\frac{S_{(n-j+1)}}{S_{(n-m)}}\right)^r, \text{ for } r = 1, 2.$$

Then, $M_n^{(1)}$ is the well known Hill estimator (Hill, 1975). The Hill estimator and the moment estimator have the best performance especially in the case $\gamma > 0$. However, even these estimators are scale invariant, they are not location invariant. The $(1-\alpha)$ – quantile of the distribution function F_γ for $0 < \alpha < 1/2$ is given by

$$F_\gamma^{-1}(1-\alpha) = S_{(n-m)} + \frac{(m/n\alpha)^{\hat{\gamma}_n}}{\hat{\gamma}_n}\left(1 - \min(\hat{\gamma}_n, 0)\right) S_{(n-m)} M_n^{(1)}.$$

4. Convergence of portfolio VaR bounds

In order to establish explicit bounds for VaR of a portfolio composed by possibly dependent assets, we combine the Dekkers et al. (1989) quantile estimator of large order with VaR bounds obtained by Mesfioui and Quessy (2005) and discussed in section 2. Moreover, following the methodology of Vermaat et al. (2005), we provide location change invariant bounds for quantile estimator of large order. The upper bound of VaR can be interpreted as the "worst possible outcome" at a given confidence level, in the absence of any information on the actual dependence structure between the different sources of risk and only the marginal distributions are known. A straightforward application of VaR bounds defined in (1) and (2) to the special case of a GEV distribution for marginal extreme losses gives

$$\overline{VaR}(\alpha) = \sum_{i=1}^{d} S_{i(n-m_i)} + \frac{\left(m_i/nq_i(\alpha)\right)^{\hat{\gamma}_{in}} - 1}{\hat{\gamma}_{in}}\left(1 - \min(0, \hat{\gamma}_{in})\right) S_{i(n-m_i)} M_{in}^{(1)}, \tag{4}$$

and

$$\underline{VaR}(\alpha) = \sum_{i=1}^{d} S_{i(n-m_i)} + \max_{1\le i\le d}\left\{\frac{\left(m_i/n(1-\alpha)\right)^{\hat{\gamma}_{in}} - 1}{\hat{\gamma}_{in}}\left(1 - \min(0, \hat{\gamma}_{in})\right) S_{i(n-m_i)} M_{in}^{(1)} \right.$$
$$\left. + \sum_{1\le j\ne i\le d} \frac{\left(m_j/n\right)^{\hat{\gamma}_{jn}} - 1}{\hat{\gamma}_{jn}}\left(1 - \min(0, \hat{\gamma}_{jn})\right) S_{j(n-m_i)} M_{jn}^{(1)}\right\}. \tag{5}$$

These bounds will be referred as standards bounds of VaR. Since the quantile estimator based on the Dekkers et al. (1989) work depends heavily on the actual location because of the steepness of the log function, one can easily verify that the standard VaR bounds do not react adequately to a shift of the data. Therefore, we introduce a large shift of location, denoted by K, to the original data and we calculate the quantile bounds for the transformed data. Then, to restore the quantile bounds for the original data, an inverse transformation of the calculated bounds is performed. In our setting, we investigate the limiting values for VaR bounds when the shift goes to infinity. Then, we get the modified upper and lower bounds estimation for the quantile of the sum $S = S_1 + \cdots + S_d$, which are invariant with respect to location change. The results are given in the following proposition. The proof of the proposition as well as standard VaR bounds defined in (4) and (5) are given in the appendix.

Proposition *Define* $(S_i^*)_{1\le i\le d} = (S_i)_{1\le i\le d} + K$ *where* S_i *is a random variable with an absolutely continuous probability distribution function* F_γ^i *and K is a constant. Suppose that* $m_i/n(1-\alpha) \ge 1$. *When no information is* *available about the dependence structure of* $(S_1,\ldots,S_d)$ *and if* $K \to \infty$ *, then the modified upper and lower VaR bounds denoted respectively by* $\overline{VaR}^*(\alpha)$ *and* $\underline{VaR}^*(\alpha)$ *with* $1/2 < \alpha < 1$ *are given by*

$$\overline{VaR}^*(\alpha) = \sum_{i=1}^{d}\left\{ S_{i(n-m_i)} + D_i\left(\frac{m_i}{nq_i(\alpha)}\right)\frac{1}{m_i}\sum_{j=1}^{m_i}\left(S_{i(n-j+1)} - S_{i(n-m_i)}\right)\right\},$$

and

$$\underline{VaR}^*(\alpha) = \sum_{i=1}^{d} S_{i(n-m_i)} + \max_{1\le i\le d}\left\{ D_i\left(\frac{m_i}{n(1-\alpha)}\right)\frac{1}{m_i}\sum_{l=1}^{m_i}\left(S_{i(n-l+1)} - S_{i(n-m_i)}\right)\right.$$
$$\left. + \sum_{1\le j\ne i\le d} D_j\left(\frac{m_j}{n}\right)\frac{1}{m_j}\sum_{l=1}^{m_j}\left(S_{j(n-l+1)} - S_{j(n-m_j)}\right)\right\}, \tag{6}$$

where

$$q_i(\alpha) = \frac{(L_i)^{1/G_{in}+1}}{\sum_{i=1}^{d}(L_i)^{1/G_{in}+1}}(1-\alpha),$$

and

$$D_i\left(\frac{m_i}{nq_i(\alpha)}\right) = \frac{(m/nq_i(\alpha))^{G_{in}} - 1}{G_{in}}\left(1 - \min(0, G_{in})\right),$$

with

$$G_{in} = 1 - \frac{1}{2(1-Q_{in})}, \tag{7}$$

and

$$Q_{in} = \frac{\left[\frac{1}{m_i}\sum_{j=1}^{m_i}\left(S_{i(n-j+1)} - S_{i(n-m_i)}\right)\right]^2}{\frac{1}{m_i}\sum_{j=1}^{m_i}\left(S_{i(n-j+1)} - S_{i(n-m_i)}\right)^2}.$$

The tail index estimator defined in (7) will be referred as the modified moment estimator since it stands for the limiting value of the moment estimator as the shift goes to infinity. From our experience with several data sets, as a rule of thumb, the modified estimator provides a rather better approximation of the actual tail index than the classical moment estimator. As shown by Figure 1, the modified tail index estimator converges to the true value of the shape parameter of a sample of size T = 5000 generated from a GEV distribution with parameters $\mu = 0.5$, $\sigma = 0.1$ and $\gamma = 0.3$, while the original moment estimator is significantly above the actual value, thus overestimating γ.

5. Applications

In this section we apply our approach to estimate VaR bounds for international equity portfolios. We use daily return series of international stock market indexes over the period running from July-09-1987 to December-17-2007 (the length of the financial time series studied is thus $n = 5333$) listed in Table 1. For comparison purpose, inferential and descriptive statistics concerning the loss distribution of single equity indexes are also given in Table 1. The estimation of the Dekkers et al. (1989) tail index $\hat{\gamma}_n$ and the modified tail index $\hat{G}_n$ requires to choose the number m_i, $1 < i < d$, of upper order statistics for each portfolio component $S_1,\cdots,S_d$. It is well-known that tail index estimators are very sensitive to the choice of the sample fraction m. This problem has been the subject of intense research in theoretical statistics, still developing. The optimal sample fraction m is selected through the minimization of the asymptotic mean squared error as in Danielsson, De Haan, Peng and De Vries (2001) or Drees and Kaufmann (1998). In a general fashion, by looking at these indicators one can see that the single market indexes loss distributions are skewed and heavy-tailed.

As a first illustration of our approach for measuring risk in the case of $d = 2$, we consider two equally weighted portfolios. The first portfolio (PF1) is composed of the French and the German stock market indexes, the second portfolio (PF2) contains the Japanese and the Chilean stock market indexes. The dependence for the French-German stock market returns in PF1 is persistent for both positive and negative returns, as illustrated by Figure 2a, extreme variations of CAC40 and DAX30 returns occur simultaneously in most cases suggesting strong positive dependence. In contrast, the dependence between the Japanese-Chilean stock market returns is much weaker, which is indicative of a presumably independence between returns in PF2 (see Figure 2b).

The idea behind the construction of portfolios PF1 and PF2 is to illustrate the behaviour of VaR bounds in different (supposed unknown) dependence situations since stock market indexes returns pairs in these two portfolios are characterized by completely different dependence structures. In Figure 3, the upper and the lower VaR bounds for PF1 are displayed under the assumption of GEV distribution for extreme marginal loss of stock market indexes. Given the positive dependence between CAC40 and DAX30 returns, one is interested in the upper bounds of VaR and compare these bounds to the exact $VaR_\alpha(S)$ under the assumption of comonotonicity (the so-called perfect positive dependence). It is important to notice that the worst possible scenario may occur when the portfolio assets have a very special dependence beyond the perfect positive dependence. Indeed,

because of the non-coherence of VaR, $VaR_\alpha(S) = VaR_\alpha(\sum_{i=1}^{d} S_i)$ may be greater than $\sum_{i=1}^{d} VaR_\alpha(S_i)$ when

the marginal distributions of the portfolio components are very skewed or heavy-tailed as illustrated in Table 1. Therefore, the upper bound in particular is interesting, from the point of view of risk management, since it represents the worst possible portfolio loss at a given confidence level, when there is no knowledge of the joint distribution nor of their dependence structure and only the marginal distributions are known. As expected, the upper bounds are closer to $VaR_\alpha(S)$ than the lower bounds. Moreover the modified upper bound provides a rather good approximation to $VaR_\alpha(S)$ than the standard upper bound. One may observe from Figure 3a that VaR estimates under the GEV assumption for marginal losses are close to the empirical quantiles for the different confidence levels between 0.95 and 0.99 suggesting a good fit of the GEV model to data. Besides, the modified upper bound is very close to the empirical quantile under the assumption of comonotonicity at high levels of probability. Given the quasi-independence between the NIKKEI225 and the IGPA returns, Figure 3b shows that the lower bounds are closer to the exact $VaR_\alpha(S)$ under the assumption of independence than the upper bounds. Even if the standard lower bound provides a better approximation of $VaR_\alpha(S)$ than the modified lower bound for the probability level interval $[0.95, 0.993]$, the modified lower bound is identical to the empirical quantile for extremely high confidence levels. As a second illustration, we also applied our method to the case of a five dimensional portfolio (PF3) including all the stock market indexes listed in Table 1. Unlike the case of portfolios PF1 and PF2 where the unknown dependence between returns pairs is characterized in Figure 2 and hence one can either look at the upper bound as an approximation of VaR when asset returns are comonotonic or come across the lower bound when portfolio components are independent or countermonotonic, the actual dependency structure of the asset returns in the high dimensional portfolio PF3 could not be illustrated any more for comparison purpose. Therefore, the risk manager have to consider lower and upper bounds as an approximation of the exact portfolio VaR in any scenario of dependence between asset returns. As a matter of fact, the modified upper bound is closer to the comonotonic VaR than the standard upper bound. However, the standard lower bound yields a better approximation of the exact VaR under the assumption of independence (see Figure 4).

6. Conclusion

In this paper explicit VaR bounds for portfolios of possibly dependent financial assets are evaluated. The upper bound in particular is interesting, from the point of view of risk management, since it comes into sight as the worst possible portfolio loss at a given probability level. By exploiting recent contribution of Mesfioui and Quessy (2005) on bounding VaR of the sum of several risks, we provide explicit bounds for portfolio risk when the asymptotic marginal distribution of extreme negative returns belongs to the domain of attraction of GEV distribution. These standard bounds are obtained without imposing a specific assumption on the dependence structure between asset returns. The underlying dependence between the portfolio components is captured throughout the empirical copula. In order to consider possibly location change, the standard bounds are evaluated when a large shift of location is applied to the original data. The reverse translated bounds applied to the shifted data converge to the modified bounds which enjoy the desirable property of location invariance, as well as scale invariance, a property also common to the classical Dekkers et al. (1989) estimator. Empirical applications are carried out on three international equity portfolios under various dependence scenarios and show that the modified upper bound is closer than the standard one to the exact VaR under the assumption of comonotonicity. However, the modified lower bound outperforms the standard lower bound, when asset risks are independent or countermonotonic, only for extremely high levels of probability.

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Notes

Note 1. For $d = 2$ the upper and lower bounds are themselves copula. However, for higher dimension $d > 2$, the lower bound F_L is not any more a distribution function (Embrechts et al., 2003)

Note 2. Following convention, a loss is treated as a positive number and extreme events take place when losses come from the upper tail of the loss distribution.

Table 1. Extreme value statistics based on daily losses from 5 international equity markets, for the period July 1987 to December 2007. For each market index, m is the number of order statistics used for computing tail index estimates defined in (3) and (7), respectively.

Country (Index)	**Skewness**	**Kurtosis**	m	γ_n	G_n
Germany (DAX30)	0.33	9.09	318	0.18	0.12
France (CAC40)	0.15	7.55	240	0.18	0.13
Japan (Nikkei225)	0.06	9.74	266	0.14	0.19
U.K. (FTSE100)	0.58	12.39	355	0.23	0.21
Chile (IGPA)	0.31	14.28	305	0.15	0.18

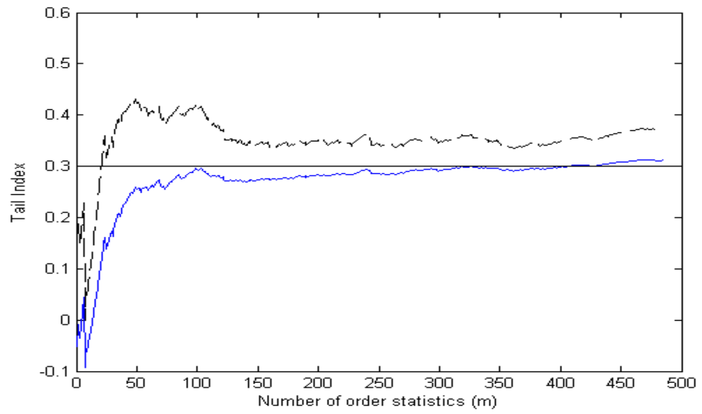

Figure 1. Tail index estimates from simulated data: moment estimator (dashed line) and modified moment estimator (solid line).

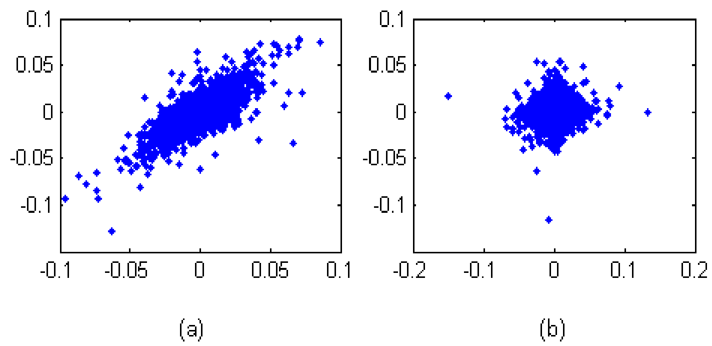

Figure 2. Dependence structure of returns pairs. (a) France (vertical axis) and Germany (horizontal axis) (b) Japan (vertical axis) and Chile (horizontal axis)

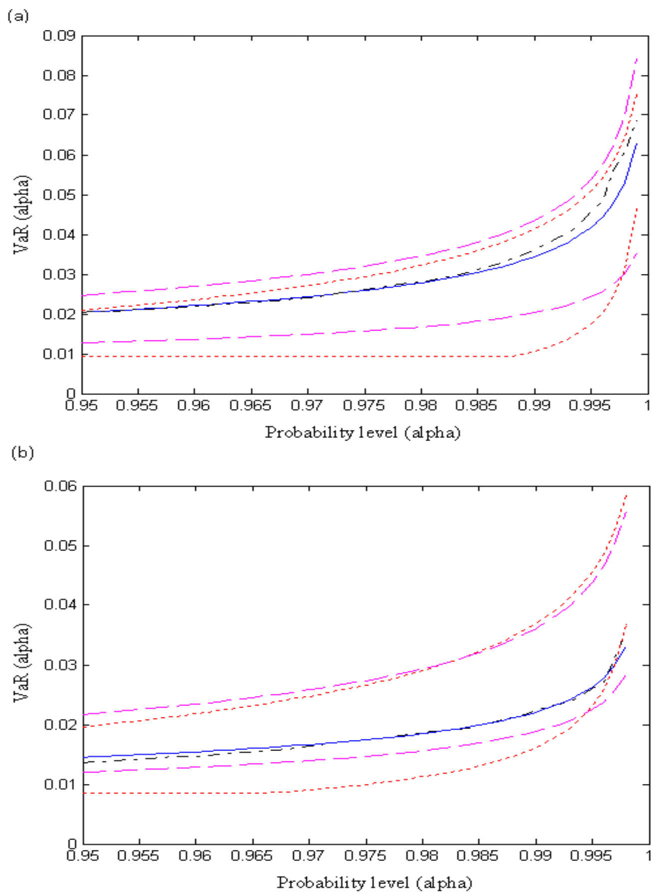

Figure 3. (a) Bounds on VaR of PF1 (dashed lines) and modified bounds (dotted lines) compared to the exact $VaR_\alpha (S)$ when the portfolio components are comonotonic under the assumption of GEV for loss distribution (solid line) and empirical distribution (dash dotted line). (b) Bounds on VaR of PF2 compared to the exact $VaR_\alpha (S)$ when the portfolio components are independent.

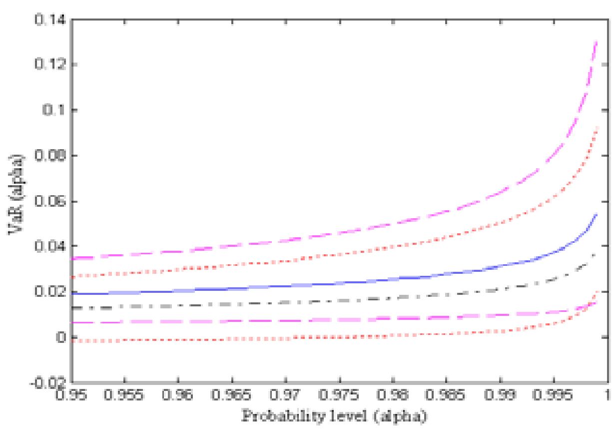

Figure 4. Bounds on VaR of PF3 (dashed lines) and modified bounds (dotted lines) compared to the exact $VaR_\alpha (S)$ when the portfolio components are comonotonic (solid line) and when the portfolio components are independent (dash dotted line).

Appendix. Proof of Proposition

By replacing $(S_i)_{1\le i\le d}$ with $(S_i^*)_{1\le i\le d}$ the definitions of $\hat{\gamma}_{in}$ and $M_{in}^{(r)}$ are modified, respectively, into $\hat{\gamma}_{in}^*$ and $M_{in}^{*(r)}$. The $(1-q_i)$ -quantile of the distribution function of $(S_i^*)_{1\le i\le d}$ can be easily obtained from

$$\hat{F}_i^{-1}(q_i,\hat{\gamma}_{in}^\star) = S_{i(n-m_i)}^\star + \frac{\left(\frac{m_i}{n(1-q_i)}\right)^{\hat{\gamma}_{in}^\star}-1}{\hat{\gamma}_{in}^\star}(1-\min(0,\hat{\gamma}_{in}^\star))S_{i(n-m_i)}^\star M_{in}^{\star(1)} - K,$$

then

$$VaR_\alpha \le \overline{VaR}(\alpha) = \inf_{\sum_{i=1}^d q_i=\alpha+d-1} \sum_{i=1}^d \frac{\left(m_i/n(1-q_i)\right)^{\hat{\gamma}_{in}^*}-1}{\hat{\gamma}_{in}^*}\left(1-\min\left(0,\hat{\gamma}_{in}^*\right)\right)S_{i(n-m_i)}^* M_{in}^{*(1)} + \sum_{i=1}^d S_{i(n-m_i)}. \tag{8}$$

Since the function, defined in (8), to be minimized is convex, the problem can be solved using the Lagrange multiplier method, which gives

$$\left(\frac{m_i}{n}\right)^{\hat{\gamma}_{in}^*}\left(\frac{1}{1-q_i}\right)^{\hat{\gamma}_{in}^*+1}\left(\min\left(0,\hat{\gamma}_{in}^*\right)-1\right)S_{i(n-m_i)}^* M_{in}^{*(1)} = \lambda, \quad \text{and}$$

$$\sum_{i=1}^d \left(1-q_i\right) = 1-\alpha. \tag{9}$$

Now we write

$$C_i\left(\frac{1}{1-q_i}\right)^{\hat{\gamma}_{in}^*+1} = \lambda, \quad \text{where} \quad C_i = \left(\frac{m_i}{n}\right)^{\hat{\gamma}_{in}^*}\left(\min\left(0,\hat{\gamma}_{in}^*\right)-1\right)S_{i(n-m_i)}^* M_{in}^{*(1)},$$

and $$\lambda = \left[\frac{\sum_{i=1}^d (C_i)^{1/\hat{\gamma}_{in}^*+1}}{1-\alpha}\right]^{\hat{\gamma}_{in}^*+1}$$

The solution to (9) is given by

$$\left(1-q_i\right) = \frac{\left(C_i\right)^{1/\hat{\gamma}_{in}^*+1}}{\sum_{i=1}^d \left(C_i\right)^{1/\hat{\gamma}_{in}^*+1}}\left(1-\alpha\right), = q_i^*(\alpha). \tag{10}$$

Using equation (10) in the objective function (8) we get

$$\overline{VaR}(\alpha) = \sum_{i=1}^d S_{i(n-m_i)} + \frac{\left(m_i/nq_i^*(\alpha)\right)^{\hat{\gamma}_{in}^*}-1}{\hat{\gamma}_{in}^*}\left(1-\min\left(0,\hat{\gamma}_{in}^*\right)\right)S_{i(n-m_i)}^* M_{in}^{*(1)}.$$

Now we calculate the following limits, if $K\to\infty$

$$\lim_{K\to\infty}\hat{\gamma}_{in}^* = 1-\frac{1}{2\left(1-Q_{in}\right)} = G_{in},$$

where

$$\lim_{K\to\infty}\frac{\left(M_{in}^{(1)*}\right)^2}{M_{in}^{(2)*}}=\frac{\left(\frac{1}{m_i}\sum_{j=1}^{m_i}\left(S_{i(n-j+1)}-S_{i(n-m_i)}\right)\right)^2}{\frac{1}{m_i}\sum_{j=1}^{m_i}\left(S_{i(n-j+1)}-S_{i(n-m_i)}\right)^2}=Q_{in}$$, with $0<Q_{in}<1$ and $G_{in}\in(-\infty,\frac{1}{2})$ with probability one.

We also have

$$\lim_{K\to\infty}q_i^*(\alpha)=\frac{\left(L_i\right)^{1/G_{in}+1}}{\sum_{i=1}^{d}\left(L_i\right)^{1/G_{in}+1}}(1-\alpha)=q_i(\alpha) \tag{11}$$

where

$$L_i=\lim_{K\to\infty}C_i=\left(\frac{m_i}{n}\right)^{G_{in}}\left(\min\left(0,G_{in}\right)-1\right)\frac{1}{m_i}\sum_{j=1}^{m_i}\left(S_{i(n-j+1)}-S_{i(n-m_i)}\right),$$

and

$$\lim_{K\to\infty}\frac{\left(m_i/n\right)^{\hat{\gamma}_{in}^*}-1}{\hat{\gamma}_{in}^*}\left(1-\min\left(0,\hat{\gamma}_{in}^*\right)\right)=\frac{\left(m_i/nq_i\left(\alpha\right)\right)^{G_{in}}-1}{G_{in}}\left(1-\min\left(0,G_{in}\right)\right)=D_i\left(m_i/nq_i\left(\alpha\right)\right). \tag{12}$$

Using (11) and (12), we get if $K\to\infty$

$$\overline{VaR}^*\left(\alpha\right)=\sum_{i=1}^{d}\left\{S_{i(n-m_i)}+D_i\left(m_i/nq_i\left(\alpha\right)\right)\frac{1}{m_i}\sum_{j=1}^{m_i}\left(S_{i(n-j+1)}-S_{i(n-m_i)}\right)\right\},$$

and we have $VaR_\alpha(S)\geq\underline{VaR}(\alpha)$ with

$$\underline{VaR}(\alpha)=\max_{1\leq i\leq d}\left\{S_{i(n-m_i)}+\frac{\left(m_i/n\left(1-\alpha\right)\right)^{\hat{\gamma}_{in}^*}-1}{\hat{\gamma}_{in}^*}\left(1-\min\left(0,\hat{\gamma}_{in}^*\right)\right)S_{i(n-m_i)}^*M_{in}^{*(1)}\right.$$
$$\left.+\sum_{1\leq j\neq i\leq d}S_{j(n-m_j)}+\frac{\left(m_j/n\right)^{\hat{\gamma}_{jn}^*}-1}{\hat{\gamma}_{jn}^*}\left(1-\min\left(0,\hat{\gamma}_{jn}^*\right)\right)S_{j(n-m_j)}^*M_{jn}^{*(1)}\right\}$$

$$=\sum_{i=1}^{d}S_{i(n-m_i)}+\max_{1\leq i\leq d}\left\{\frac{\left(m_i/n\left(1-\alpha\right)\right)^{\hat{\gamma}_{in}^*}-1}{\hat{\gamma}_{in}^*}\left(1-\min\left(0,\hat{\gamma}_{in}^*\right)\right)S_{i(n-m_i)}^*M_{in}^{*(1)}\right.$$
$$\left.+\sum_{1\leq j\neq i\leq d}\frac{\left(m_j/n\right)^{\hat{\gamma}_{jn}^*}-1}{\hat{\gamma}_{jn}^*}\left(1-\min\left(0,\hat{\gamma}_{jn}^*\right)\right)S_{j(n-m_j)}^*M_{jn}^{*(1)}\right\}.$$

If $K\to\infty$, then $\underline{VaR}\left(\alpha\right)$ converges to

$$\underline{VaR}^*\left(\alpha\right)=\sum_{i=1}^{d}S_{i(n-m_i)}+\max_{1\leq i\leq d}\left\{D_i\left(m_i/n\left(1-\alpha\right)\right)\frac{1}{m_i}\sum_{l=1}^{m_i}\left(S_{i(n-l+1)}-S_{i(n-m_i)}\right)+\sum_{1\leq j\neq i\leq d}D_j\left(m_j/n\right)\frac{1}{m_j}\sum_{l=1}^{m_j}\left(S_{j(n-l+1)}-S_{j(n-m_j)}\right)\right\}.$$