International Journal of Economics and Finance; Vol. 15, No. 7; 2023
ISSN 1916-971X E-ISSN 1916-9728
Published by Canadian Center of Science and Education

The Detection of Asset Price Bubbles in the Cryptocurrency Markets with an Application to Risk Management and the Measurement of Model Risk

Michael Jacobs, Jr.[1]

[1] PNC Financial Services Group, Balance Sheet Analytics & Modeling / Model Development, New York, N.Y., USA

Correspondence: Michael Jacobs, Jr., Ph.D., CFA, Lead Quantitative Analytics & Modeling Expert, Head – Wholesale 1st Line Model Validation & Quality Assurance Services, PNC Financial Services Group, Balance Sheet Analytics & Modeling / Model Development, 340 Madison Avenue, New York, N.Y., 10022, USA. Tel: 1-917-324-2098. E-mail: michael.jacobsjr@pnc.com

Received: May 14, 2023 Accepted: June 8, 2023 Online Published: June 15, 2023

doi:10.5539/ijef.v15n7p46 URL: https://doi.org/10.5539/ijef.v15n7p46

Abstract

This study presents an analysis of the impact of asset price bubbles on the markets for cryptocurrencies and con-siders the standard risk management measure *Value-at-Risk* ("VaR"). We apply the theory of local martingales, present a styled model of asset price bubbles in continuous time and perform a simulation experiment featuring one- and two-dimensional *Stochastic Differential Equation* ("SDE") systems for asset value through a *Constant Elasticity of Variance* ("CEV") process that can detect bubble behavior. In an empirical analysis across several widely traded cryptocurrencies, we find that the estimated parameters of one-dimensional SDE systems do not show evidence of bubble behavior. However, if we estimate a two-dimensional system jointly with an equity market index, we do detect a bubble, and comparing bubble to non-bubble economies it is shown that asset price bubbles result in materially inflated VaR measures. The implication of this finding for portfolio and risk management is that rather than acting as a diversifying asset class, cryptocurrencies may not only be highly correlated with other assets but have anti-diversification properties that materially inflate the downside risks in portfolios combining these asset types. We also measure the model risk arising from mispecifying the process driving cryptocurrencies by ignoring the relationship to another representative risk asset through applying the *principle of relative entropy*, where we find that across all cryptocurrencies studied that the distributions of a distance measure between the simulated distributions of VaR are almost all highly skewed to the right and very heavy-tailed. We find that in the majority of cases that the model risk "multipliers" range in about two to five across cryptocurrencies, estimates which could be applied to establish a model risk reserve as part of an economic capital calculation for risk management of cryptocurrencies.

Keywords: cryptocurrencies, model risk, asset price bubbles, value-at-risk, stochastic differential equations, constant elasticity of variance

JEL Classification: C15, E58, G12, G17, G18, G21, G28.

1. Introduction and Motivations

The financial crises of the last decades have been the impetus behind a movement to better understand the relative merits of various risk measures, classic examples being *Value-at-Risk* ("VaR") and related quantities (Jorion, 2006; Inanoglu & Jacobs, 2009). The importance of an augmented comprehension of these measures is accentuated in the realm of new asset classes such as cryptocurrencies, as observed in the recent meltdown in these markets. We have subsequently learned from episodes such this that the pricing models have failed in not incorporating the phenomenon of asset price bubbles, which in turn added to the severity of the downturn for investors and risk managers who mis-measured their potential adverse exposure to market risk in this domain. This manifestation of *model risk* (The U.S. Board of the Governors Federal Reserve System, SR 11-7), wherein a modeling framework lacks a key element of an economic reality and therefore fails, was due to some extent to a lack of basic understanding. This failure of the modeling paradigm in cryptocurrencies spans gaps in the measurement, characterization and economics of asset price bubbles.

In this paper we leverage the deep economics literature concerning local martingale theory as applied asset price bubbles using historical time series data in a continuous time and finite horizon setting (see Brunnermeier and Oehmke (2012) for a survey). As discussed in Jarrow et al. (2010), in the most general model structure possible, there are three types of bubbles: Type 1, Type 2 and Type 3. A Type 1 bubble exists only in infinite horizon models, and it captures a bubble in fiat money, a security with zero cash flows but strictly positive value. A type 2 bubble also exists only in infinite horizon models, and it corresponds to an asset whose price process (under the risk neutral probability measure) is a martingale but not a uniformly integrable martingale. Intuitively, the sum of the risk adjusted expected discounted cash flows and liquidation value at time infinity (i.e. the asset's fundamental value) does not equal the market price. Type 2 bubbles are studied within an infinite horizon model where the market price of an asset is compared to its fundamental value and estimated using a model for the asset's dividends and discount rate. However, there are two challenges associated with testing for Type 1 and Type 2 bubbles that have resulted in conflicting results in the literature. First, since in both cases the models assume an infinite horizon, the model estimation requires a very large time series sample, which creates a problem when there are structural breaks or non-stationarities in financial markets (Note 1). Second, in the case of Type 2 bubbles there is no consensus on the model for an asset's fundamental value, which leads to a joint hypothesis issue. Moreover, this setup is not applicable to cryptocurencies, as they have no cash flows.

Finally, a Type 3 bubble exists only in continuous trading models, and it corresponds to an asset whose price process is a local martingale but not a martingale, which is the type of bubble that is the subject of this study. In economic terms, in this case the risk adjusted expected discounted cash flows and liquidation value at some finite time horizon does not equal the market price, implying the asset's fundamental value not being equal to its market price. Type 3 bubbles arise when investors attempt to capture short-term trading profits through trading over a finite horizon where the market price for an asset exceeds its fundamental value, the latter being interpreted as the price paid for the asset to buy and hold until liquidation. Jarrow et al. (2011) show it is possible to test for the existence of Type 3 bubbles without estimating an asset's fundamental value, thereby avoiding the joint hypothesis issue, and discuss how local martingale theory is the basis for this mode of testing for Type 3 bubbles. Cryptocurrencies are naturally suited to this form of testing since as they have no have no cash flows, and the fundamental value corresponds to the currency's liquidation value at the model's horizon, which implies that bubbles exist in cryptocurrencies when speculators buy to resell before the model's horizon. We believe this situation to be rather plausible in the case of novel cryptocurrencies, which are mainly used as a medium of exchange. Theoretically, if purchased to buy and hold and to use as needed, the transaction demand for these as-sets should be constrained by the usage of other more standard currencies to execute transactions. However, this expectation is at odds with historical experience, as seen in the unprecedented expansion of cryptocurrency markets over the last decade.

In view of analyzing the impact of asset price bubbles on market risk measures and economic capital determination, we construct various hypothetical economies, having and also not having asset price bubbles. In a stylized structural asset pricing model framework (Merton, 1974), we simulate a cryptocurrency asset value processes in each of these economies, computing the standard risk measure VaR. We present a model of asset price bubbles in continuous time, and perform a simulation experiment of a one- and two- dimensional *Stochastic Differential Equation* ("SDE") system for asset value. Comparing bubble to non-bubble economies, it is shown that asset price bubbles may cause a traditional market risk measures such as VaR to decline, due to a reduced right skewness of the loss distribution. In an empirical experiment across several widely traded cryptocurrencies, we find that estimated parameters of one-dimensional SDE systems do not show evidence of bubble behavior. However, if we estimate a two-dimensional system jointly with an equity market index (in this case the NASDAQ), we do detect a bubble, and comparing bubble to non-bubble economies, it is shown that asset price bubbles result in materially inflated VaR measures. The implication of this finding for portfolio and risk management is that rather than acting as a diversifying asset class, cryptocurrencies may not only be highly correlated with other assets but have anti-diversification properties that materially reduce diversification benefits in portfolios.

The results of our experiment demonstrate that the existence of an asset price bubble, which occurs for certain parameter settings in the CEV model, results in the cryptocurrency loss distributions having more right-skewness and higher kurtosis. This augmented non-normality of the cryptocurrency's returns due to bubble expansion results in an increase in the VaR risk measures, and an under-statement in the risk of the cryptocurrency, when non-bubble dynamics are inferred from incorrectly modeling the asset in isolation. Based on these measures alone, their more mildly declining asset values imply that in the presence of asset price bubbles, less economic capital is required. However, as shown by the joint modeling with an equity asset class proposed in the present paper, this conclusion is incorrect. This market loss measure increases in bubble economies and is due to bubble bursting, with accompanying market risk losses on the bubble-bursting paths.

As asset price bubbles are inevitably bound to burst, causing significant mark-to-market loss to holders of cryptocurrencies, more market risk capital should be held for these bubble-bursting scenarios. Unfortunately, the severity of these bubble-bursting scenarios is not adequately captured by such a misspecification of modeling cryptocurrencies in isolation. However, if these bubble-bursting scenarios are captured with market risk measures derived from assuming the correct CEV dynamics that admit asset price bubbles, and estimated over a long enough historical time period, we are more likely to anticipate the bursting of a bubble.

We also measure the model risk arising from mispecifying the process driving cryptocurrencies by ignoring the relationship to another representative risk asset through applying the *principle of relative entropy*, where we find that across all cryptocurrencies studied the distribution of a distance measure between the simulated distributions of VaR are almost all highly skewed to the right and exhibit extremely heavy tails. We find that in the majority of cases that the model risk "multipliers" range in about two to five across cryptocurrencies, estimates which could be applied to establish a model risk reserve as part of an economic capital calculation for market risk management of cryptocurrencies.

We conclude this introduction with a discussion of the implications of this research for prudential supervision and public policy. In the wake of the downturn in the cryptocurrency markets that began in earnest in 2022 that coincided with central banks raising interest rates, and the several high-profile blow-ups of several cryptocurrency exchanges and trading firms (e.g., most infamously FTX) that followed, the debate about the proper scope for supervision over this asset class has transitioned into a phase of a fevered pitch. The central question has been not only if but how cryptocurrencies should be brought under the supervisory umbrella, including which asset classes the various cryptocurrencies should be classified as (e.g., securities vs. commodities), but also possible unintended consequences of ill designed regulation (Sauce, 2022). In view of our findings, that there is a powerful interaction between cryptocurrencies and another major risk asset that leads to a self-reinforcing vicious cycle of bubble behavior, any such regulatory regime should account for these linkages. Such a regulatory regime should include an emphasis on coordination between different supervisory bodies with authority and domain knowledge across different asset classes, such as in the U.S. the SEC, CFTC, Federal Reserve, etc.

An outline for this paper is as follows. Section 2 presents a review of the literature. Section 3 presents our market model incorporating the effect of asset price bubbles. Section 4 describes the cryptocurrencies that constitute our modeling data. Section 5 contains the results of the estimation of the models and our VaR simulation experiment. Section 6 describes the mathematics and results of our measurement of model risk. Finally, Section 7 summarizes the implications of our analysis for market and model risk management, and provides directions for future research.

2. Review of the Literature

Modern credit risk modeling (e.g., Merton, 1974) increasingly relies on advanced mathematical, statistical and numerical techniques to measure and manage risk in market portfolios. This gives rise to model risk (The U.S. Board of Governors of the Federal Reserve System, SR 11-7), defined as the potential that a model used to assess financial risks does not accurately capture those risks, and the possibility of understating inherent dangers stemming from very rare yet plausible occurrences perhaps not in reference datasets or historical patterns of data (Note 2), a key example of this being the inability of the market risk modeling paradigm to accommodate the phenomenon of asset price bubbles.

The relative merits of various risk measures, classic examples being VaR and related quantities, have been discussed extensively by prior research (Alexander, 2001; Jeanblanc et al., 2009; Jorion, 1997, 2006). Risk management as a discipline in its own right, distinct from either general finance or financial institutions, is a relatively recent phenomenon. A general result of mathematical statistics due to Sklar (1956), allowing the combination of arbitrary marginal risk distributions into a joint distribution while preserving a non-normal correlation structure, readily found an application in finance. Among the early academics to introduce this methodology is Embrechts et al. (1999, 2003). This was applied to credit risk management and credit derivatives by Li (2000). The notion of copulas as a generalization of dependence according to linear correlations is used as a motivation for applying the technique to understanding tail events ss found in Frey and McNeil (2003). This treatment of tail dependence contrasts to Poon et al. (2004), who instead use a data intensive multivariate extension of extreme value theory, which requires observations of joint tail events. Inanoglu and Jacobs (2009) contribute to the modeling effort by providing tools and insights to practitioners and regulators, utilizing data from major banking institutions' loss experience, exploring the impact of business mix and inter-risk correlations on total risk, and comparing alternative established frameworks for risk aggregation on the same datasets across banks.

In this paper we perform analysis of alternative asset pricing models and resultant VaR measures, and we quantify the model risk arising from potential model misspecification, which is a key component of the supervisory framework for managing model risk. Jacobs (2015a) discusses various approaches to measuring and aggregating model risk across an institution, including a sensitivity analysis approach to quantification of model risk. Jacobs (2015a, 2017) applies a model for asset price bubbles in a credit risk context to *stress testing* ("ST"), a key supervisory tool for supplementing VaR measures such as VaR or economic capital. Jacobs (2013) surveys practices and supervisory expectations for *stress testing* ("ST"), in a credit risk framework for trading book exposures; including simple and practical ST examples, a ratings migration based approach and anther a top-down time series modeling approach. Combining these concepts, Jacobs et al. (2015) presents an example of model risk quantification in the realm of ST, comparing alternative models in two different classes, Frequentist and Bayesian approaches to modeling stressed bank losses.

Regarding our choice of VaR as the risk measure of interest, we note that recently in the realm of regulatory capital measurement of market risk there has been a movement toward *expected shortfall* ("ES"), which is defined as the expected or average loss in the tail of the loss distribution in excess of a VaR level at a given confidence level. We offer three justifications for studying VaR and not ES in this paper. First, VaR and not ES is still predominant amongst practitioners in market risk for non-regulatory purposes, as it is numerically more stable and easier to explain to the lines of business (Carver, 2014). Second, in Section 5 of this paper on the measurement of model risk we simulate the entire distribution of VaR, which yields a more robust and comprehensive view of the risk in excess of VaR than only the mean, the latter being less meaningful due to the asymmetry in the VaR distribution. Finally, by measuring the risk in the VaR model relative to an alternative and plausible model, we develop a more holistic view of the risk in the VaR measure than ES would represent, in that ES is more akin to a confidence bound measuring only parameter uncertainty in estimating VaR.

Since the 2007 crisis, the mathematical finance literature has made significant advances in the modeling and testing of asset price bubbles (Jarrow & Protter, 2010; Hong et al., 2006). Protter (2011), Protter et al. (2010) and Jarrow et al. (2007, 2010, 2011, 2014, 2015) apply these new insights to determine the impact that asset price bubbles have on the common risk measures used in practice for the determination of equity capital. Jacobs (2015b) and Jacobs (2016) provide an extension of the latter literature the realms of credit and liquidity risk, respectively. Jacobs (2017) extend Jacobs (2016) with an addition of a sensitivity analysis as well as an empirical implementation with an application to the stress testing of credit risk.

There have been several papers over the past decade that have empirically investigated whether there are asset price bubbles in markets for cryptocurrencies. Cheung et al. (2015) conduct an econometric investigation of the existence of asset price bubbles in the bitcoin market based on the technique of Phillips et al. (2013). Over the period 2010-2014 the authors detect three huge bubbles in the latter part of the period 2011-2013 lasting from 66 to 106 days, with the last and biggest one leading to the demise of the Mt. Gox exchange. Phillips and Gorse (2017) build predictive models to detect asset price bubbles for a number of cryptocurrencies using a hidden Markov model previously utilized to detect influenza epidemic outbreaks, based on the behavior of novel online social media indicators. The authors validate their methodology through implementing a trading strategy that is built and tested on historical data which they find to outperform a buy and hold strategy. Geuder et al. (2019) and Cheah and Fry (2015) investigate the presence of asset price bubbles in Bitcoin. Bouri et al. (2019) argue that the cryptocurrency market is prone to herding behavior, and they find evidence of a high degree of co-movement in the cross-sectional returns across different cryptocurrencies. Agosto and Cafferata (2020) investigate co-explosivity amongst cryptocurrencies in unit root a model that accounts for possible shock propagation channels that can potentially improve the prediction of market collapses. Shahzad et al. (2022) detect episodes of price explosivity and collapse in Bitcoin and its contender Dogecoin using four-hourly data. The results show multiple bubble episodes in both cryptocurrencies, with a more frequent occurrence in Bitcoin that are related to Elon Musk's tweets that are more general cryptocurrency related, whereas his Dogecoin specific tweets have contributed to price explosivity in Dogecoin only. Most recently and in a methodology closest to that employed in this research, Choi and Jarrow (2022) employ an asset price bubble detection algorithm based upon local Martingale theory to test for the existence of price bubbles in eight cryptocurrencies from January 1, 2019 to July 17, 2019. The authors find that five of the eight cryptocurrencies exhibit asset price bubbles, and that the remainder are inconclusive, which they argue provides strong evidence for the prevalence of asset price bubbles in cryptocurrencies.

Finally, we review some of the foundational studies in the quantification of model risk according to the principle of relative entropy. Hansen and Sargent (2007) propose an alternative paradigm to the standard theory of decision making under uncertainty that is based on a statistical model that informs an optimal distribution of outcomes. The authors adapt robust control techniques through developing a theory of model risk measurement that admits acknowledgement of misspecification in economic modeling and applies this framework to a variety of problems in dynamic macroeconomics. Glasserman and Xu (2014) apply this framework to financial risk measurement that relies on models of prices and other market variables that inevitably rely on imperfect assumptions that give rise to model risk. They develop a framework for quantifying the impact of model error through measuring and minimizing risk in a way that is robust to model error. Their robust approach starts from a baseline model and finds the worst-case error in risk measurement that would be incurred through a deviation from a baseline model, given a precise constraint on the plausibility of the deviation. Using relative entropy to constrain model distance leads to an explicit characterization of worst-case model errors that lends itself to Monte Carlo simulation, allowing straightforward calculation of bounds on model error with very little computational effort beyond that required to evaluate performance under the baseline nominal model. The authors apply this technique to a variety of applications in finance such as problems of portfolio risk measurement, credit risk, delta hedging and counterparty risk measured through credit valuation adjustment. Skoglund (2019) applies the principle of relative entropy to quantify the model risk inherent in loss-projection models used in macroeconomic stress testing and impairment estimation in an application to a retail portfolio and a delinquency transition model. He argues that this technique can complement traditional model risk quantification techniques, where a specific direction or range of model misspecification reasons, such as model sensitivity analysis, model parameter uncertainty analysis, competing models and conservative model assumptions, is usually considered. Jacobs (2022) addresses the building of obligor level hazard rate corporate *probability-of-default* ("PD") models for ST, building models based upon varied of financial, credit rating, equity market and macroeconomic factors, using an extensive history of large corporate firms sourced from Moody's. He develops a *distance-to-default* ("DTD") risk factor and designs hybrid structural/Merton-reduced form models as challengers to versions of the models containing only the other variables. Measuring the model risk attributed to various modeling assumptions according to the principle of relative entropy, where the loss metrics are bounds on the stressed PD forecasts, he observes that the omitted variable bias with respect to the DTD risk factor, neglect of interaction effects and incorrect link function specification has the greatest, intermediate and least impacts, respectively.

3. A Model for Asset Price Bubbles

We model the evolution of asset prices, incorporating the phenomenon of price bubbles, using the approach of Jarrow et al. (2007, 2014a, 2014b). The setting is a continuous time trading economy, without loss of generality having a finite horizon $[0, \boldsymbol{\tau}]$, with randomness described by the *filtered probability space* $(\Omega, \Im, \mathrm{F}, \mathrm{P})$, where we define: the *state space* Ω, the σ *-algebra* $\Im$, the *information partition* $F = \{\Im_t\}_{t\in[0,\tau]}$ and the *physical probability measure* P (or *actuarial*, as contrasted to a *risk-neutral probability measure*, commonly denoted by the symbol Q). We assume, again without loss of generality and for the purpose on focusing on the application to market risk, a single asset value process $\{V_t\}_{t\in[0,\tau]}$ that is adapted to the filtration F. Note that this could also represent a share of stock owned by a representative equity investor, which is a claim on the single productive entity or firm in this economy. In the general setting V_t follows an *Ito diffusion process* (Øksendal, 2003) having the following *stochastic differential equation* ("SDE") representation:

$$dV_t = \mu(V_t,t)\,dt + \sigma(V_t,t)\,dW_t^L, \tag{1}$$

where $\mu(V_t,t)$ is the *instantaneous drift process,* $\sigma(V_t,t)$ is the *instantaneous diffusion process,* $W_t^L \sim N(0,t)$ is a *standard Weiner process* (or a *Brownian motion process*) on the filtered probability space $(\Omega, \Im, \mathrm{F}, \mathrm{P})$, and dW_t are its infinitesimal increments. In order to complete this economy, we assume that there exists a traded money market account process M_t, which grows according to a risk-free rate process r_t, the latter also adapted to the filtration of the aforementioned probability space:

$$M_t = \exp\left\{\int_{s=0}^{t} r_s\,ds\right\}. \tag{2}$$

Without a loss of generality, we assume that the asset has no cashflows, which could have been incorporated into the model by assuming a dividend process and studying the dividend-reinvested stock price process (Back, 2010), but as previously argued as appropriate to a non-yielding cryptocurrency setting, we choose to not do so.

We model an economy potentially having an price bubbles through the assumption that the risky asset's prices follows a *constant elasticity of variance* ("CEV:) process (Emanuel & MacBeth, 1982; Schroder, 1989), as in Jarrow et al. (2014b), which is the following restricted version of the Ito diffusion process in equation (1):

$$dV_t = \mu V_t dt + \sigma V_t^{\theta} dW_t^V, \tag{3}$$

where μ is the drift, σ is the volatility and the CEV parameter governs the state of the risky asset price process exhibiting a price bubble or not. An asset price bubble is defined as the situation where the market price for an asset exceeds its *fundamental value* (Jarrow et al., 2007, 2010), the latter being defined conventionally price an investor would pay to hold the asset perpetually without rebalancing. This fundamental value is determined through the imposing some additional structure on the economy, requiring at minimum two additional assumptions. First, we need to assume that the absence of any arbitrage opportunities (Delbaen & Schachermayer, 1998), which guarantees the existence of a risk-neutral probability Q measure equivalent to P, such that the asset value process V_t normalized by the money market account M_t is a *local Martingale process*:

$$E^Q[V_t^{\tau^*} / M_t \mid \Im_{t'}] = [V_{(t,\tau^*)^-} / M_t] = V_t \quad \forall t' < t, \tag{4}$$

where $V_t^{\tau^*} \equiv V_{(t,\tau^*)^-}$ is the *stopped process* of V_t and $\tau^*: \Omega \to [0, +\infty)$ is a sequence of stopping times that satisfy certain technical conditions (Note 3). The mechanism in (4) involving the risk-neutral probability measure affords us a means of computing present values where we shift the mass of the probability distribution (magnitude of the cash-flows) such that we can recover the same prices as under actuarial measure with the original cash-flows – but note that is arbitrary. In order to pin down this risk-neutral distribution, we assume from this point on a *complete market*, which means that that enough derivatives on the risky assets trade in order to replicate its cash flows in a suitably constructed *arbitrage portfolio*. The first condition is satisfied because the CEV process given in expression (3) admits an equivalent local martingale measure, so by construction it satisfies the absence of arbitrage opportunities (Note 4). Under this incremental structure that we impose upon the economy, an asset's fundamental value FV_t given the time t information set $\Im_t$, is defined as the asset's discounted future payoff from liquidation at time at horizon $\tau > t$:

$$FV_t[V_\tau \mid \Im_t] = E^Q[V_\tau / M_t \mid \Im_t] M_\tau. \tag{5}$$

It follows that we may define the asset's price bubble as the difference between the market price and its fundamental value FV_t:

$$\mathrm{B}_t^V[V_\tau \mid \Im_t] = V_\tau - FV_t[V_\tau \mid \Im_t]. \tag{6}$$

Since as a conditional expectation, the fundamental value normalized by the value of the money market account is a martingale under Q, a bubble exists if and only if the asset's normalized price is a strict local martingale and not a martingale under Q. In the case of the CEV process, it can be shown (Jarrow et al., 2011) that the asset's normalized price $(V_t^{\tau^*}/M_t)$ is a martingale under Q when $\theta \le 1$ in (3) (i.e., no asset price bubble), and a strict-local martingale under Q where $\theta > 1$ in that equation (i.e., an asset price bubble). Note that the boundary case of $\theta = 1$ yields the geometric Brownian motion underlying the *Black–Scholes–Merton* ("BSM") option pricing model (Merton, 1974), which is called the *BSM economy*, and can be shown to exhibit no price bubble (Delbaen & Schachermayer, 1995).

Finally, we extend the CEV model of equation (3) by estimating a two-equation joint SDE process:

$$dV_t^C = \mu_c V_t^C dt + \sigma_c V_t^{\theta_c} dW_t^{V_C}, \tag{7}$$

$$dV_t^E = \mu_E V_t^E dt + \sigma_E V_t^{\theta_E} dW_t^{V_E}, \tag{8}$$

$$\rho_t^{C,E} = \mathrm{Corr}_t[dW_t^{V_C}, dW_t^{V_E}] = (1/dt) \times \mathrm{E}_t[dW_t^{V_C} \times dW_t^{V_E}], \tag{9}$$

where V_t^C represents the asset value process of a cryptocurrency, V_t^E represents the same with respect to an equity price and $\rho_t^{C,E}$ is the instantaneous correlation between the Brownian motions governing the respective processes.

Given these dynamics, the statistics to be estimated are the market loss distribution's moments under the physical probability measure, characterizing the changes in the market value of instruments that includes both positive and negative mark-to-market values. In a market risk management application, we are actually only interested in losses, to which end we seek to understand the right tail of the market loss distribution, and compute the *high quantile risk measure* VaR to measure market risk. Apart from asset price bubbles, even though its limitations are widely known (Alexander, 2001; Jorion, 1997), such measures are widely used in the industry. An estimator for the VaR at a given confidence level c is given by:

$$VaR_\tau(c) = \mathrm{Quantile}_{i \le k \le N_p(c)}, \tag{10}$$

where

$$\text{Quantile}_{i \le k \le N_p(c)}(c) = \inf_x \{(1/N_p) \times [\sum_{k=1}^{N_p} I_{CL_\tau^k \ge x}] \ge c\}, \tag{11}$$

where $I_{CL_\tau^K \ge x}$ is an indicator function that takes the value 1 if $CL_\tau^K \ge x$ and 0 otherwise. As the cryptocurrencies and the NASDAQ are on different scales, in the analysis of VaR we report the *normalized VaR* measures, which is minus the difference of VaR and the starting value of the series expressed as a percentage of the starting value, which is denoted by:

$$VaR_\tau^N(c) = -(VaR_\tau^N(c) - V_0)/V_0, \tag{12}$$

Where V_0 is the starting value of the series.

4. Modeling Data and Summary Statistics

In our empirical experiment we consider the top 6 most widely traded cryptocurrencies (Bitcoin, Etherium, Stellar, Bancor Cash, Cardano and Dogecoin). Summary statistics for the cryptocurrencies are show below in Table 1 and they are described below in Table 2. All data run through the 2nd quarter of 2022 and have start dates ranging from the year 2013 for Bitcoin to as recent as 2018 for Dogecoin, as can be seen in the time series plots of the series levels and percent changes below in Figure 1 (Figures 2 through 7) for the NASDAQ (cryptocurrencies), including the respective data histograms for each series. The prices are all measured at a daily frequency, where in the case of the NASDAQ we take the closing prices, and in the case of the cryptocurrencies we take the price that is closest on an hourly basis to when trading in the NASDAQ ends for the day. While the scales differ markedly amongst the cryptocurrency series, what is striking is the commonality that all of them are extremely volatile, excessively skewed to the right and have fat tails far in excess of normality, with Dogecoin departing the furthest from normality. The NASDAQ also exhibits high volatility relative to the mean and fat tails, but unlike the cryptocurrencies has negative excess skewness.

Table 1. Summary statistics of cryptocurrencies in the empirical experiment

		Count	Min.	1st Quartile	Median	Mean	3rd Quartile	Max.	Stdev.	Skew.	Kurt.
NASDAQ	Level	3,336	3,166	4,813	6,606	7,499	8,721	16,057	3,336	0.97	2.77
	% Δ		-12.00%	0.00%	0.00%	0.05%	1.00%	9.00%	1.27%	-0.33	10.59
Bitcoin	Level	3,336	65.16	447.48	3,859.1	10,378.0	9,997.7	67,108.2	15,851	1.87	-1.75
	% Δ		-25.00%	0.00%	0.00%	-0.01%	0.00%	23.00%	0.95%	5.31	260.21
Etherium	Level	2,193	7.06	135.74	244.29	763.62	691.63	4,777.8	1,123	1.86	5.30
	% Δ		-30.42%	-2.3365%	0.20%	0.41%	2.92%	33.13	5.60%	0.24	6.54
Stellar	Level	1,481	0.032	0.0788	0.1593	0.19	0.28	0.73	0.13	0.99	3.65
	% Δ		-35.82%	-3.03%	0.0000%	0.16%	2.69%	78.88%	6.34%	1.89	23.67
Bancor	Level	1,690	0.14	0.60	1.63	2.23	3.35	10.44	1.99	1.20	4.025
	% Δ		-39.75%	-2.9t%	-0.05%	0.29%	3.00%	64.56%	7.65%	1.29	13.57
Cardano	Level	1,450	0.022	0.052	0.10	0.49	0.99	2.972	0.69	1.49	4.10
	% Δ		-36.48%	-2.9412%	0.00%	0.25%	3.03%	28.49%	5.78%	0.31	5.89
Dogecoin	Level	1,755	0.01	0.02	0.04	0.01	0.02	0.12	0.01	24.43	639.98
	% Δ		-98.79%	0.00%	0.00%	1.66%	0.00%	2214.3%	57.4%	34.87	1295.1

Table 2. Description of cryptocurrencies in the empirical experiment

Bitcoin (BTC)	Bitcoin is the original blockchain-based cryptocurrency. Created in 2009 by the pseudonymous Satoshi Nakomoto, Bitcoin has since attracted millions of investors, becoming the largest cryptocurrency by market capitalization. Bitcoin is inherently scarce: only 21 million Bitcoin will ever be minted. Bitcoin's proof-of-work blockchain has become a template for other cryptocurrencies in building decentralized consensus mechanisms.
Etherium (ETH)	Ethereum was created in 2014 by Vitalik Buterin, a Russian-Canadian programmer, and Gavin Wood, an English computer scientist who later contributed to other cryptocurrency projects. The Ether currency is built on top of the Ethereum blockchain, which operates smart contracts. Unlike Bitcoin, which investors primarily view as a store of value, Ether's value derives from its enablement of smart contracts in decentralized applications. Most "DeFi" (decentralized finance) projects are built on Ethereum. Ether's supply is unconstrained, meaning the total number of Ether minted is still undecided, but will be determined by Ethereum's community members. Recently the network has been split into the existing proof-of-work mechanism and a new proof-of-stake mechanism.
Stellar (XLM)	Stellar is an open source blockchain whose native currency is Lumen. The network was founded in 2014 by Jed McCaleb, a cryptocurrency evangelist who previously co-founded Ripple Labs and the infamous Mt. Gox Exchange. Stellar's goal is to enable inexpensive transactions in underdeveloped markets. The blockchain eschewed a standard mining network for transaction validations, relying instead on what's known as a "federated byzantine agreement" algorithm.
Bancor (BNT)	Bancor Network Token (BNT) is an Ethereum token that powers the Bancor protocol. The protocol describes itself as "a fully on-chain liquidity protocol that can be implemented on any smart contract-enabled blockchain model.
Cardano (ADA)	Cardano was founded in 2015 by Charles Hoskinson, a computer scientist and cofounder of Ethereum, who left the project over disagreements with its other founders. Cardano's cryptocurrency, ADA, is secured by a proof-of-stake protocol named Ouroboros, which runs both permissioned and permissionless blockchains. The Cardano Foundation, a Switzerland based not-for-profit group, supervises the development of the project. The group has carried out extensive research and experimentation, writing over 90 papers on blockchain technology. Much of this academic work underlies Cardano's technology.
Dogecoin (DOGE)	Dogecoin began in 2013 as a joke. The token's mascot appropriates the doge internet meme, and was intended as an ironic take on the growth of so-called "altcoins" (cryptocurrencies that aren't Bitcoin). Dogecoin has a large, unconstrained supply, which means the coin could inflate infinitely. The cryptocurrency attracted millions of new investors in 2021, when Tesla CEO Elon Musk, NBA owner Mark Cuban and other celebrities began tweeting about the erstwhile little-known cryptocurrency.

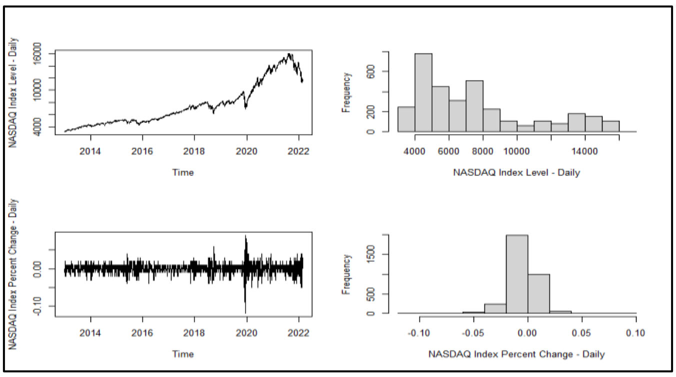

Figure 1. Time series plots and data histograms – NASDAQ

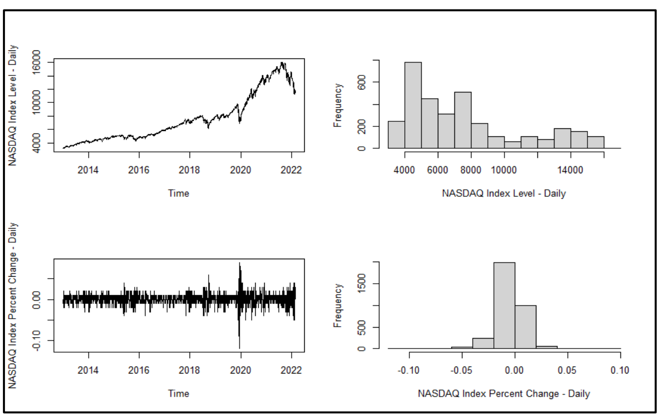

Figure 2. Time series plots and data histograms – Bitcoin

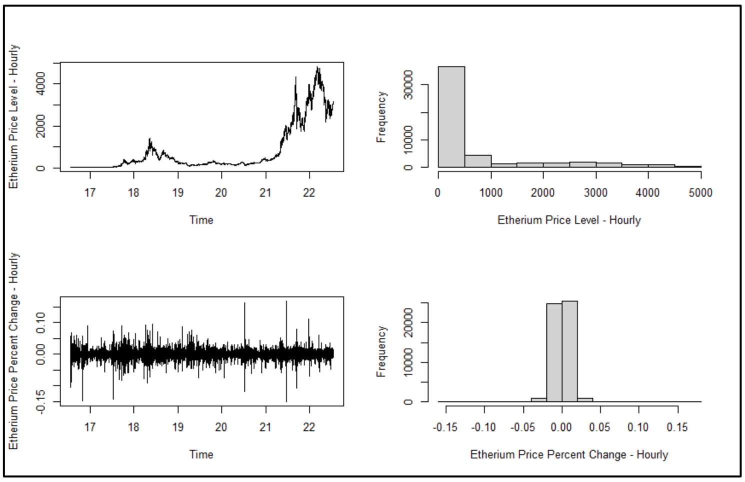

Figure 3. Time series plots and data histograms – Etherium

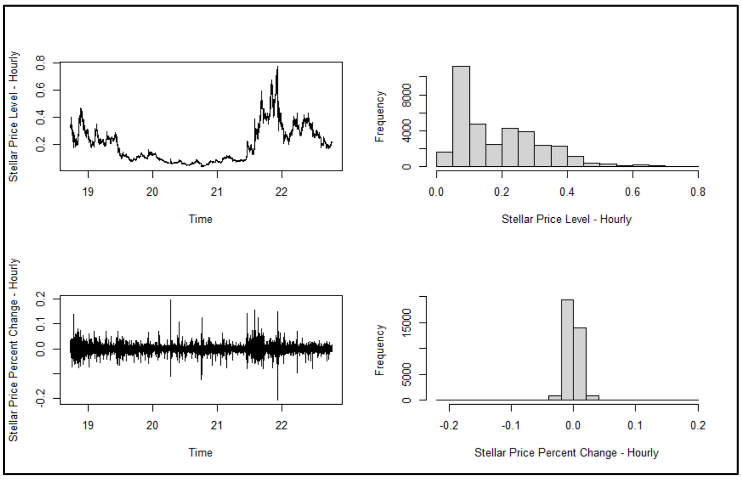

Figure 4. Time series plots and data histograms – Stellar

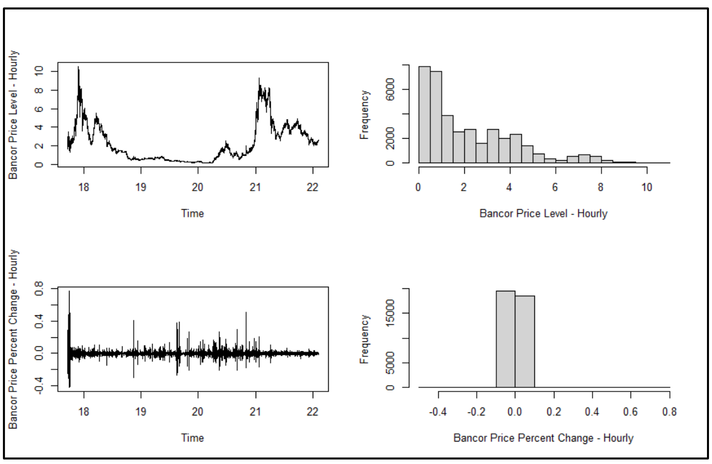

Figure 5. Time series plots and data histograms – Bancor

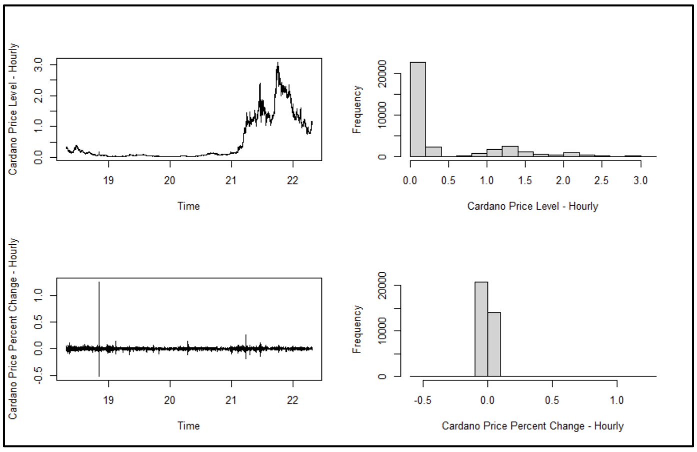

Figure 6. Time series plots and data histograms – Cardano

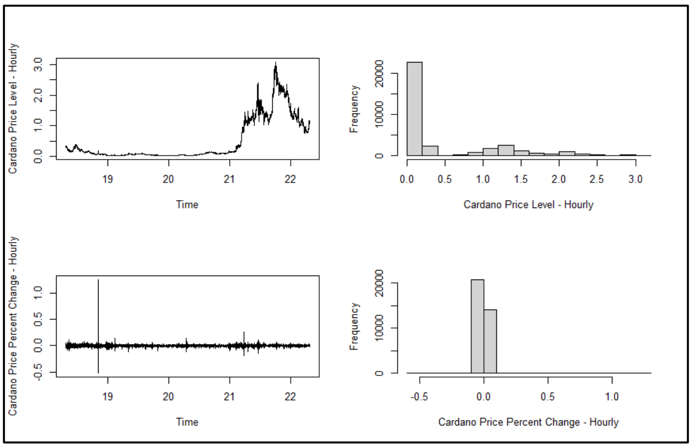

Figure 7. Time series plots and data histograms – Dogecoin

5. Estimation Results

In this section we discuss the estimation results for the one- and two-dimensional SDE models for the six cryptocurrencies under consideration and the NASDAQ (Note 5), as well as the simulation of daily VaR for each of these models. We normalize the VaR measures by calculating the proportion of the starting value of the process, which would be the price of the cryptocurrency or level of the NASDAQ on the last available trading day in the historical time series, lost at the VaR level in the simulated loss distribution. In Tables 3 through 8 below we show the estimation results. Each table corresponds to a cryptocurrency and the NASDAQ, and for each price process we show the results for the one- and the two-dimensional SDE system. The statistics tabulated include the parameter estimates of the CEV process, corresponding standard errors and p-values, log-likelihoods, *Aikaike information criteria* (”AIC”) and the normalized VaR measures.

Consistently across all six cryptocurrencies, we observe that when we estimate the SDE separately for each of them and the NASDAQ, the parameter estimate for the CEV parameter is either statistically indistinguishable from unity, else is less than one and we would reject the null hypothesis that it exceeds one, which is indicative of no bubble. However, when we estimate the two-dimensional systems, considering the correlation between the cryptocurrency and equity index process, all CEV parameter estimates are greater than one and enough so that we would reject the null hypothesis that it is less than or equal to one, which is evidence of a bubble in the joint price processes. For example, in the case of Bitcoin and NASDAQ, for the one- (two) dimensional system we estimate a CEV parameter of 0.9035 (1.0390) in the case of the cryptocurrency with a standard error of 0.0012 (0.0064), and the corresponding estimate for the equity index is 1.1012 (1.5196) with a standard error of 0.0851 (0.0014). While the conclusions regarding detection of a bubble are consistent across all cases, we note that there is wide variation in the CEV estimates between cryptocurrencies and for the NASDAQ in isolation or considered jointly, which reflects the particular market dynamics and structures across cryptocurrencies and how these interact with other risky assets such as equities. For example, we have just described that the NASDAQ appears to have extreme bubble behavior when modeled with Bitcoin, whereas considering Dogecoin the bubble behavior in the NASDAQ appears to be milder (a CEV estimate of 1.1254), and for Dogecoin itself in the two-dimensional system the CEV estimate is showing much more extreme bubble behavior (a CEV estimate of 1.4079), whereas the bubble behavior for Bitcoin is milder in the two-dimensional model.

The second major observation is that the normalized VaR measures are materially elevated in the cases of the two-dimensional SDE models where we detect bubble behavior as compared to the one-dimensional cases, which holds across all cryptocurrencies as well as the NASDAQ. As an example, in the case of Bitcoin, for the cryptocurrency itself the normalized VaR estimate is 0. 8568 (0.9903) in the one- (two-) dimensional SDE model, and for the NASDAQ the corresponding measure is 0.2718 (0.4930). As with the CEV exponent and other parameter estimates, there is a wide variation across cryptocurrencies and the equity index, although we note a consistent pattern that the VaR estimates are much higher in the case of the former as compared to the latter, which reflects the extremely fat tails and asymmetry of the distribution to the right (i.e., positive skewness) in cryptocurrencies as compared to equities as we saw in the analysis of distributional statistics. Another feature worthy of note is that we see across all cryptocurrencies and the equity index that in the two-dimensional models the drift and volatility estimates are lower versus the one-dimensional models.

Table 3. One- and two-dimensional SDE system estimation results – NASDAQ and Bitcoin

			Drift	Vol.	CEV Exp.	Corr.	Log-L	AIC	99^{th} Prcntl. VaR
		Estimate	0.1012	1.6586	0.9035				
	1 Dim. SDE	Std. Err.	0.0249	0.0167	0.0012	N/A	71,643.73	71,644.33	0.8568
Bitcoin		Estimate	0.9095	1.1167	1.0390	0.9128			
	2 Dim. SDE	Std. Err.	0.2784	0.0638	0.0064	0.0282	41,048.81	41,054.81	0.9902
		Estimate	0.0635	0.0565	1.1012				
NASDAQ	1 Dim. SDE	Std. Err.	0.0242	0.0022	0.0851	N/A	60,432.10	60,372.10	0.2718
Equity		Estimate	0.0752	0.0802	1.5196	0.9128			
Index	2 Dim. SDE	Std. Err.	0.0064	0.0000	0.0014	0.0282	41,048.81	41,054.81	0.4930

Table 4. One- and two-dimensional SDE system estimation results – NASDAQ and Etherium

			Drift	Vol.	CEV Exp.	Corr.	Log-L	AIC	99^{th} Prcntl. VaR
		Estimate	0.0644	0.0866	0.9962				
Etherium	1 Dim. SDE	Std. Err.	0.0035	0.0091	0.0018	N/A	23,287.20	23,287.80	0.7296
		Estimate	0.0739	0.0546	1.0462	0.8482			
	2 Dim. SDE	Std. Err.	0.0044	0.0055	0.0088	0.0359	17,664.41	17,670.41	0.9898
NASDAQ		Estimate	0.03992	0.10117	1.00523				
Equity	1 Dim. SDE	Std. Err.	-0.00114	0.00880	0.00157	N/A	27,881.12	27,851.12	0.3188
Index	2 Dim. SDE	Estimate	0.2032	0.0149	1.0684	0.8482	17,664.41	17,670.41	0.4309

Table 5. One- and two-dimensional SDE system estimation results – NASDAQ and Stellar

			Drift	Vol.	CEV Exp.	Corr.	Log-L	AIC	99^{th} Prcntl. VaR
		Estimate	0.1005	0.1483	1.0121				
	1 Dim. SDE	Std. Err.	0.0048	0.0153	0.0102	N/A	35,656.17	35,655.57	0.8175
Stellar		Estimate	0.0785	0.1643	1.1687	0.5421			
	2 Dim. SDE	Std. Err.	0.0059	0.0085	0.0256	0.0248	19,207.75	19,195.75	0.8707
NASDAQ		Estimate	0.03993	0.15566	1.01530				
Equity	1 Dim. SDE	Std. Err.	0.00133	0.00330	0.01986	N/A	3,571.63	3,565.63	0.3589
Index	2 Dim. SDE	Estimate	0.0412	0.0994	1.1202	0.5421	2,310.41	2,311.16	0.5481

Table 6. One- and two-dimensional SDE system estimation results – NASDAQ and Bancor

			Drift	Vol.	CEV Exp.	Corr.	Log-L	AIC	99th Prcntl. VaR
		Estimate	0.3173	0.4326	0.9968				
	1 Dim. SDE	Std. Err.	0.0105	0.0017	0.0091	N/A	151,887.80	151,881.80	0.8678
		Estimate	0.2988	0.3549	1.0750	0.3873			
Bancor	2 Dim. SDE	Std. Err.	0.0168	0.0065	0.0185	0.0237	2,774.60	2,768.60	0.9835
		Estimate	0.06210	0.09299	0.99873				
NASDAQ	1 Dim. SDE	Std. Err.	0.01078	0.01326	0.00623	N/A	20,829.39	20,835.39	0.2608
Equity		Estimate	0.0661	0.0783	1.0856	0.3873			
Index	2 Dim. SDE	Std. Err.	0.0021	0.0077	0.0182	0.0237	2,073.47	2,070.47	0.3316

Table 7. One- and two-dimensional SDE system estimation results – NASDAQ and Cardano

			Drift	Vol.	CEV Exp.	Corr.	Log-L	AIC	99th Prcntl. VaR
		Estimate	0.9633	1.1268	0.9485				
	1 Dim. SDE	Std. Err.	0.0061	0.0072	0.0030	N/A	21,672.17	21,671.77	0.7704
Cardano		Estimate	0.9212	1.1737	1.0365	0.8313			
	2 Dim. SDE	Std. Err.	0.0046	0.0354	0.0069	0.0140	9,184.79	9,178.79	0.9493
NASDAQ		Estimate	0.05692	0.10261	1.01097				
Equity	1 Dim. SDE	Std. Err.	0.00266	0.00663	0.02028	N/A	3,495.30	3,489.30	0.2202
Index	2 Dim. SDE	Estimate	0.0933	0.0733	1.1545	0.8313	3,022.92	3,023.92	0.5284

Table 8. One- and two-dimensional SDE system estimation results – NASDAQ and Dogecoin

			Drift	Vol.	CEV Exp.	Corr.	Log-L	AIC	99th Prcntl. VaR
		Estimate	0.1002	0.4542	1.0023				
Dogecoin	1 Dim. SDE	Std. Err.	0.0438	0.0045	0.0019	N/A	31,612.49	31,612.09	0.7008
		Estimate	0.0574	0.1231	1.4079	0.7104			
	2 Dim. SDE	Std. Err.	0.0047	0.0217	0.0290	0.0206	18,368.45	18,362.45	0.8789
NASDAQ		Estimate	0.0397	0.0943	0.9883				
Equity	1 Dim. SDE	Std. Err.	0.0012	0.0030	0.0179	N/A	4,320.87	4,314.87	0.1873
Index	2 Dim. SDE	Estimate	0.0376	0.0568	1.1254	0.7104	4,307.07	4,308.27	0.3668

Lastly for describing the estimation results and the implication thereof, we comment upon the estimates of the correlation parameters in the two-dimensional SDE models, and the performance of the latter in comparison to the one-dimensional SDE models. First, the estimates of the instantaneous correlations between the Brownian processes driving the cryptocurrency and equity market processes are very high, generally in the range of around 70% to 90% (with Bitcoin having the highest estimate of 0.9218), which holds in four out of five cases (exceptions being the 40-50% range for Stellar and Bancor, where the time series lengths are shorter than for the other cases). While not definitive, as is the analysis of the CEV exponent or normalized VaR estimates, this is consistent with limited, non-existent or even anti-diversification benefit from holding cryptocurrencies and equities in a portfolio. Second, the AIC measures are all materially lower in the two- versus the one-dimensional models, indicative of superior model fit in the latter as compared to the former where we account for this correlation, which is an intuitive result that speaks to model misspecification when not modeling cryptocurrencies and equities jointly.

Finally, in Figures 9 through 12 below we show graphical depictions of the VaR simulations for Bitcoin and the NASDAQ, where for the sake of brevity we do not show these for the five other cryptocurrencies. The extreme non-normality, i.e., extreme excess kurtosis and skewness, of the simulated loss distribution in the case of the two- versus the one-dimensional model, are evident from these plots, and the appearance of these graphics are qualitatively similar for the other five cryptocurrencies not shown herein (the plots are available upon request.)

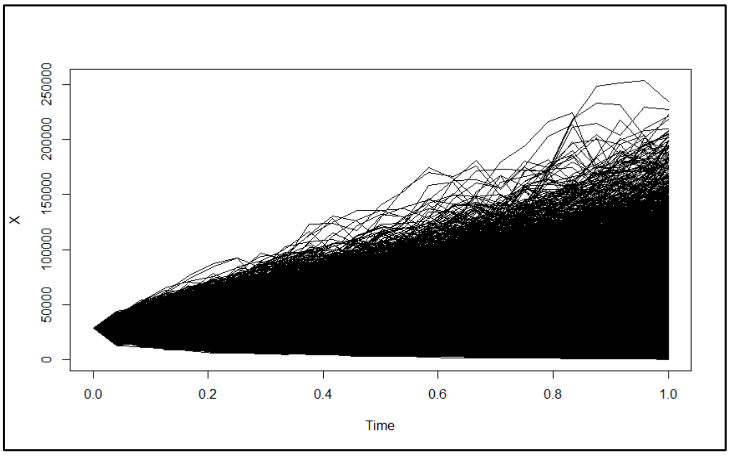

Figure 9. Simulation of one-day VaR from the estimation of a one-dimensional SDE System CEV model – Bitcoin

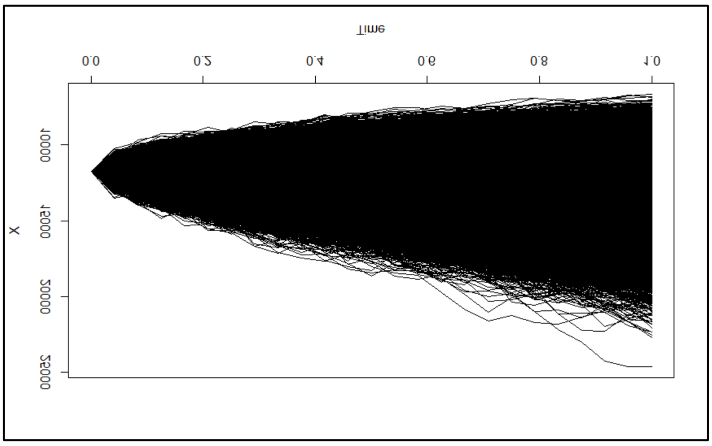

Figure 10. Simulation of one-day VaR from the estimation of a one-dimensional SDE System CEV model – NASDAQ

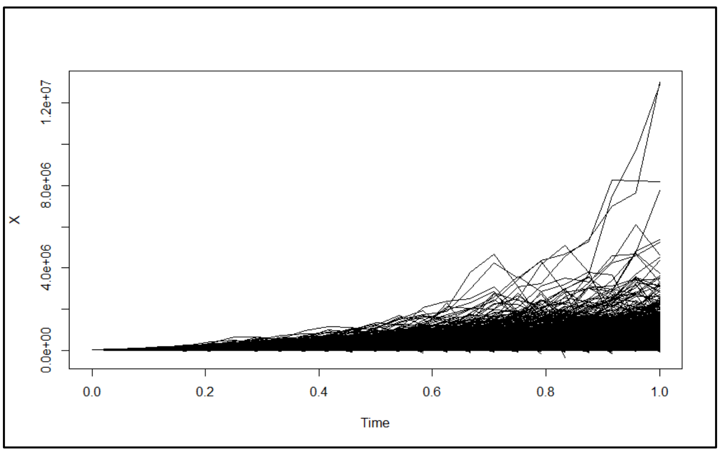

Figure 11. Simulation of one-day VaR from the estimation of a one-dimensional SDE System CEV model – Bitcoin

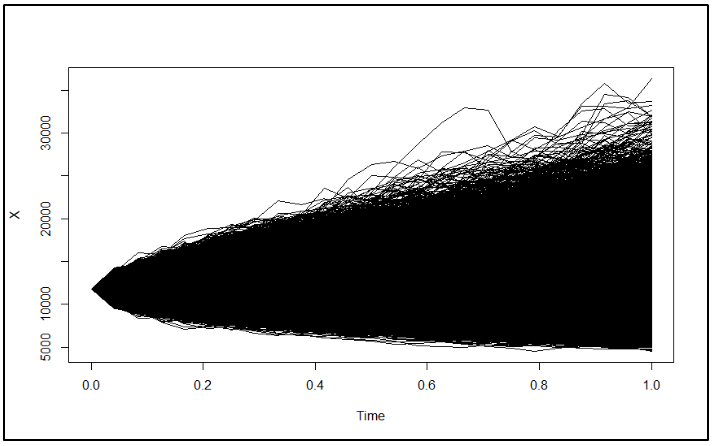

Figure 12. Simulation of one-day VaR from the estimation of a one-dimensional SDE System CEV model – NASDAQ

6. Measurement of Model Risk

In the building of risk models we are subject to errors from model risk, one source being the violation of modeling assumptions. In this section we apply a methodology for the quantification of model risk that is a tool in building models robust to such errors. A key objective of model risk management is to assess the likelihood, exposure and severity of model error in that all models rely upon simplifying assumptions. It follows that a critical component of an effective model risk framework is the development of bounds upon a model error resulting from the violation of modeling assumptions. This measurement is based upon a reference nominal risk model and is capable of rank ordering the various model risks as well as indicating which perturbation of the model has maximal effect upon some risk measure.

In line with the objective of managing model risk in the context of measuring VaR for cryptocurrencies, we calculate confidence bounds around forecasted VaR spanning model errors in a vicinity of a nominal or reference model defined by a set of alternative models. These bounds can be likened to confidence intervals that quantify sampling error in parameter estimation. However, these bounds are a measure of model robustness that instead measures model error due to the violation of modeling assumptions. In contrast, a standard error estimate conventionally employed in managing market risk portfolios does not achieve this objective, as this construct relies in this context upon an assumed joint distribution of the asset returns amongst cryptocurrencies and an equity market index. Note that in applying relative entropy to model risk measurement we need not make this assumption but rather we are able to test whether this assumption is valid.

We meet our previously stated objective in the context of VaR modeling through bounding a measure of loss, in this case the VaR forecasts, which can within reason reflect a level of model error. We have observed that while amongst practitioners one alternative means of measuring model risk is to consider challenger models, an assessment of estimation error or sensitivity in perturbing parameters is in fact a more prevalent means of accomplishing this objective, which captures only a very narrow dimension of model risk. In contrast, our methodology transcends the latter aspect to quantify potential model errors such as incorrect specification of the probability law governing the model without assuming which of these is correct, or herein the specification of the SDE dynamics governing the cryptocurrency and equity index processes, namely whether a one- or two-dimensional SDE between each cryptocurrency and the equity market index is the best specification in terms of model risk.

As these types of model errors under consideration all relate to the likelihood of such error, which in turn is connected to the perturbation of probability laws governing the entire modeling construct, we apply the *principle of relative entropy* (Hansen & Sargent, 2007; Glasserman & Xu, 2013). In Bayesian statistical inference, relative entropy between a posterior and a prior distribution is a measure of information gain when incorporating incremental data. In the context of quantifying model error, relative entropy has the interpretation of a measure of the additional information requisite for a perturbed model to be considered superior to a champion or null model. Said differently, relative entropy may be interpreted as measuring the credibility of a challenger model. Another useful feature of this construct is that within a relative entropy constraint the so-called *worst-case alternative* (e.g., in our case a high quantile of a distribution of VaR estimate differences between the models due to ignoring some feature of the alternative model) can be expressed as an *exponential change of measure*.

Model risk with respect to a champion model $y = f(x)$ is quantified by the *Kullback–Leibler relative entropy* divergence measure to a challenger model $y = g(x)$ and is expressed as follows:

$$D(f,g) = \int (g(x)/f(x)) \times \log(g(x)/f(x)) f(x) dx. \quad (13)$$

In this construct, the mapping $g(x)$ is an alternative model and the mapping $f(x)$ is some kind of base model, the latter being the base one-dimensional SDE VaR model which we have estimated in this paper that may be violating the model assumption that the correct model is a two-dimensional SDE. In a model validation context this is a critical construct as the implication of these relations is a robustness to model misspecification with respect to the alternative model – i.e., we do not have to assume that either the reference or alternative models is correct, and we need only quantify its distance of the alternative from the reference model according to a loss metric to assess the impact of the modeling assumption at play.

Define the *likelihood ratio* $m(f,g)$, characterizing our modeling choice, which is expressed as follows:

$$m(f,g) = g(x)/f(x). \quad (14)$$

As is the standard in the literature, Equation (14) may be expressed as an equivalent expectation of a *relative deviation in likelihood*:

$$\mathrm{E}_f[m \log(m)] = D(f,g) < \delta, \quad (15)$$

where δ is an upper bound to deviations in model risk (which should be small on a relative basis), which may be determined by the model risk tolerance of an institution for a certain model type, interpretable as a threshold for model performance.

A property of relative entropy dictates that $D(f,g) \geq 0$ and $D(f,g) = 0$ only if $f(x) = g(x)$. Given a relative distance measure $D(f,g) < \delta$ and a set of alternative models $g(x)$, model error can be quantified by the following change of numeraire:

$$m_\theta(f,g) = \exp(\theta f(x)) / \mathrm{E}_f[\exp(\theta f(x))], \quad (16)$$

where the solution (or inner supremum) to Equation (16) is formulated in the following optimization:

$$m_\theta(f,g) = \inf{}_{\theta>0} \sup{}_{m(x)} \mathrm{E}_f[m(x) f(x) - (1/\theta)(m(x)\log(m(x)) - \delta)]. \quad (17)$$

Equation (16) features the parameterization of model risk by $\theta \in [0,1]$, where $\theta = 0$ is the best case of no model risk and $\theta = 1$ the worst case of model risk in extremis. The change in measure of Equation (16) has important property of being *model-free*, or not dependent upon the specification of the challenger model $g(x)$. As mentioned previously, this reflects the robustness to misspecification of the alternative model that is a key feature of this construct, and from a model validation perspective is a desirable property. In other words, we do not have to assume that either the champion or the alternative models is correct and only have to quantify the distance of the alternative from the base model according to a loss metric in order to assess the impact of violating the modeling assumptions.

We study the quantification of model risk with respect to the modeling assumptions that the correct VaR model is a single-dimensional SDE through implementing the principle of relative entropy in a bootstrap simulation exercise. In each iteration we resample the data with replacement and re-estimate the models considered in the paper, either a one- or two-dimensional SDE for each cryptocurrency and the equity market index, where our measure of model risk or loss is the difference in the normalized VaR estimates between these models, which we denote by

$$dVaR^b_{\tau;f,g}(c) = VaR^b_{\tau,g}(c) - VaR^b_{\tau,f}(c), \quad (18)$$

where $dVaR^b_{\tau;f,g}(c)$ is the deviation in VaR estimates between of the challenger model $g(x)$ (the two-dimensional SDE model) and of the reference model *f(x)* (the one-dimensional SDE model) in the b^{th} bootstrap, at horizon τ (one day) and confidence level c (the 99th percentile). We then study the distribution of this quantity, as well as the differences between high 99th and low 1st percentiles of these distributions and the mean of the distributions as upper and lower bounds on model risk, respectively.

The results of this exercise are shown below, summary statistics and plots of the normalized VaR deviation model risk measures, in Tables 9 above and 10 below for the former, and in Figures 10 through 15 for the latter. The first major observation is that distributions all have positive support, so that in each case across 100,000 simulations, the VaR in the two-dimensional models always exceeds that in the one-dimensional model. Second, in all cases for the cryptocurrencies with the exception of Stellar, the distributions are extremely skewed to the right, which holds as well in all cases for the NASDAQ. Third, focusing on the cryptocurrencies with the exceptions of Stellar and Dogecoin (the latter being a special case as the right skewness is extreme to an order of magnitude greater than the other right-skewed cryptocurrencies), we observe that the upper bound model risk add-ons range in about 30%-37%, whereas the means of the distributions range in about 8%-26%, which implies that the model risk "multipliers" range in about two to five, where these are shown in the 2nd to last rows of the tables and are defined as:

$$M_{VaR_{\tau}(c)} = (\text{Quantile}_c(\tilde{F}^{B}_{dVaR_{\tau\,f,g}(c)}) - (1/B)\sum_{b=1}^{B} dVaR^{b}_{\tau;f,g}(c)) / ((1/B)\sum_{b=1}^{B} dVaR^{b}_{\tau f,g}(C)), \tag{19}$$

Table 9. Summary statistics – distribution of bootstrapped deviations in normalized VaR estimates between the one- and two-dimensional SDE models for cryptocurrencies

Statistic	Bitcoin	Etherium	Stellar	Bancor	Cardano	Dogecoin
Minimum	0.01%	0.13%	10.18%	2.00E-07	1.00E-04	8.00E-09
1st Quartile	6.62%	15.22%	64.57%	3.86%	1.76%	1.00E-07
Median	12.96%	24.15%	75.34%	8.82%	5.22%	0.01%
Mean	15.52%	25.95%	73.54%	11.65%	8.12%	1.11%
3rd Quartile	21.87%	34.71%	84.31%	16.68%	11.60%	0.42%
Maximum	83.30%	88.56%	99.90%	82.96%	72.82%	65.61%
Standard Deviation	11.48%	13.88%	13.95%	10.17%	8.65%	3.29%
Skewness	1.1013	0.6234	-0.6073	1.3745	1.7721	5.3106
Kurtosis	4.1778	3.0315	3.0107	5.0797	6.8235	41.1076
99th Percentile Upper Bound	35.08%	37.25%	23.21%	33.09%	30.35%	15.91%
1st Percentile Lower Bound	14.97%	22.82%	37.10%	11.49%	8.10%	1.11E-02
VaR Model Risk Multiplier	3.26	2.44	1.32	3.84	4.74	15.39

Table 10. Summary statistics – distribution of bootstrapped deviations in normalized VaR estimates between the one- and two-dimensional SDE models for the NASDAQ

Statistic	Bitcoin	Etherium	Stellar	Bancor	Cardano	Dogecoin
Minimum	0.01%	0.00%	0.04%	0.01%	0.21%	2.11E-05
1st Quartile	7.01%	3.24%	9.91%	10.20%	19.76%	8.58%
Median	13.44%	7.85%	17.37%	17.82%	29.38%	15.51%
Mean	15.94%	10.69%	19.67%	20.00%	30.84%	17.94%
3rd Quartile	22.36%	15.31%	27.06%	27.50%	40.42%	24.96%
Maximum	80.67%	76.35%	85.06%	84.81%	88.14%	80.34%
Standard Deviation	11.54%	9.80%	12.60%	12.59%	14.56%	12.13%
Skewness	1.0553	1.4707	0.8783	0.8464	0.4746	0.9666
Kurtosis	4.0107	5.4559	3.5443	3.4801	2.8051	3.7852
99th Percentile Upper Bound	34.76%	32.55%	36.17%	35.84%	37.22%	35.89%
1st Percentile Lower Bound	15.32%	10.59%	18.36%	18.59%	25.84%	16.98%
VaR Model Risk Multiplier	3.18	4.05	2.84	2.79	2.21	3.00

Where $M_{VaR_{\tau}(c)}$ is the VaR multiplier, $\text{Quantile}_c(\tilde{F}^{B}_{dVaR_{\tau;f,g}(c)})$ is the c^{th} (i.e., 99th) percentile of the bootstrapped distribution of the VaR deviations (in bootstraps) and $\frac{1}{B}\sum_{b=1}^{B}\left(dVaR^{b}_{\tau;f,g}(c)\right)$ is the mean of the bootstrapped distribution. Only in the case of the left skewed Stellar do we get a same order of magnitude as the mean value of 1.32, and in the extremely right-skewed case of Dogecoin do we get an order of magnitude larger than the mean value of 15.39. In the case of NASDAQ, the multipliers all range narrowly in a range of about two to three. Such quantities could be applied to establish a model risk reserve as part of an economic capital calculation for traders or risk managers in cryptocurrencies.

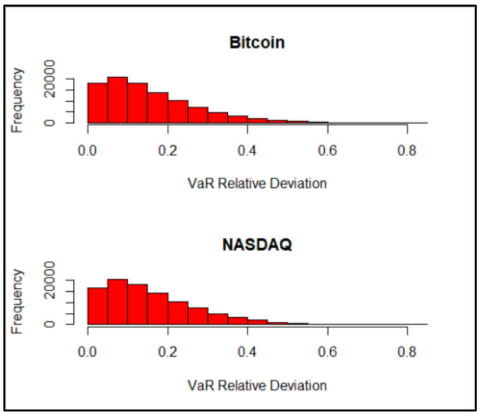

Figure 13. Distribution of bootstrapped deviations in normalized VaR estimates between the one- and two-dimensional SDE models for Bitcoin and the NASDAQ

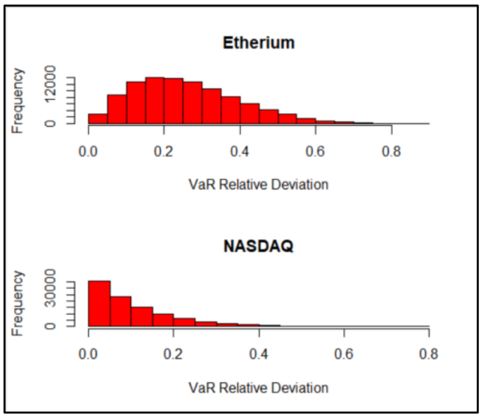

Figure 14. Distribution of bootstrapped deviations in normalized VaR estimates between the one- and two-dimensional SDE models for Etherium and the NASDAQ

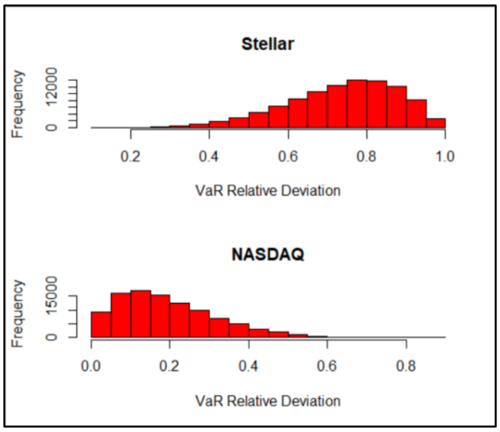

Figure 15. Distribution of bootstrapped deviations in normalized VaR estimates between the one- and two-dimensional SDE models for Stellar and the NASDAQ

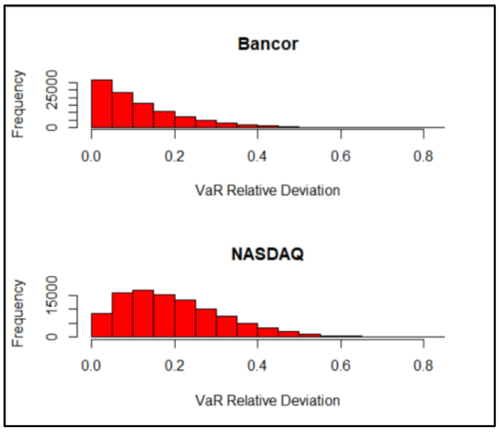

Figure 16. Distribution of bootstrapped deviations in normalized VaR estimates between the one- and two-dimensional SDE models for Bancor and the NASDAQ

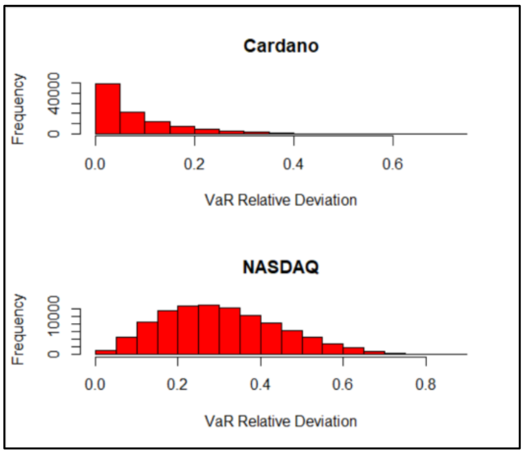

Figure 17. Distribution of bootstrapped deviations in normalized VaR estimates between the one- and two-dimensional SDE models for Cardano and the NASDAQ

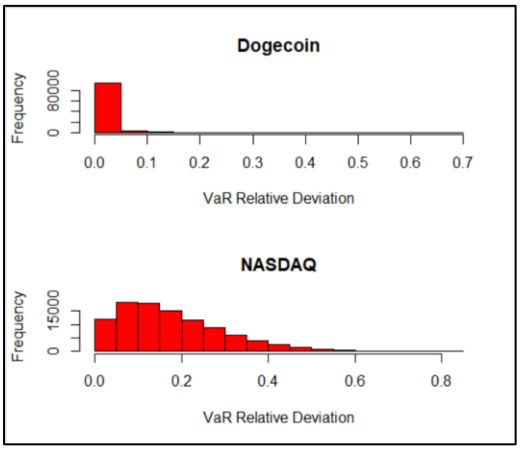

Figure 18. Distribution of bootstrapped deviations in normalized VaR estimates between the one- and two-dimensional SDE models for Dogecoin and the NASDAQ

7. Conclusions and Directions for Future Research

In this study we have leveraged the deep economics literature of local martingale theory as applied asset price bubbles in the markets for cryptocurrencies using historical time series data in a continuous time and finite horizon trading model setting. We have noted that in the case of cryptocurrencies a Type 1 asset price bubble is an inappropriate hypothesis, as such bubbles exists that only in infinite horizon models and a fiat money asset, a security with zero cash flows but strictly positive value. We have also deemed to be an inappropriate hypothesis the case of Type 2 bubbles, as there is no consensus on the model for an asset's fundamental value, which holds especially in the case of cryptocurrencies as they have no cash flows, and which leads to an egregious joint hypothesis problem. In light of these considerations, we have argued that a Type 3 asset price bubble is the most appropriate hypothesis, as they exist only in continuous trading models with finite holding periods, and correspond to an asset whose price process is a local martingale but not a martingale. It has been further noted that in economic terms, for the case where the risk adjusted expected discounted cash flows and liquidation value at some finite time horizon does not equal the market price, this implies that the asset's fundamental value is not equal to its market price. We have proceeded to point out that in this setting, such bubbles arise when investors attempt to capture short-term trading profits through trading over a finite horizon where the market price for an asset exceeds its fundamental value, the latter being interpreted as the price paid for the asset to buy and hold until liquidation. As such, we have concluded that it is possible to test for the existence of Type 3 asset price bubbles without estimating an asset's fundamental value, thereby avoiding the joint hypothesis issue, and we went on to discuss how local martingale theory may be applied as the basis for this mode of testing for Type 3 asset price bubbles. We have then argued that cryptocurrencies are naturally suited to this form of testing as they have no have no cash flows, and the fundamental value corresponds to the cryptocurrency's liquidation value at the model's horizon, which implies that bubbles exist in cryptocurrencies when speculators buy to resell before the model's horizon. Furthermore, we have asserted that this situation is rather plausible especially in the case of novel cryptocurrencies, which are mainly used as a medium of exchange, despite that theoretically if purchased to buy and hold the transaction demand for these assets should be constrained by the usage of other more standard currencies to execute transactions. as the latter expectation is at odds with historical experience with the unprecedented expansion of cryptocurrency markets.

In view of analyzing the impact of asset price bubbles on market risk measures and economic capital determination, we constructed various hypothetical economies, having and also not having asset price bubbles. In a stylized structural asset pricing model framework, we simulated a cryptocurrency asset value processes in each of these economies, computing the standard risk measure VaR. We presented a model of asset price bubbles in continuous time, and performed a simulation experiment of one- and two- dimensional SDE systems for asset values. In an empirical experiment across several widely traded cryptocurrencies, we have found that estimated parameters of one-dimensional SDE systems do not show evidence of bubble behavior. However, if estimating a two-dimensional system jointly with an equity market index we have detected asset price bubbles, and in comparing bubble to non-bubble economies it has been shown that asset price bubbles result in materially inflated VaR measures. We concluded from these findings that the implication for portfolio and risk management is that rather than acting as a diversifying asset class, cryptocurrencies may not only be highly correlated with other assets, but in fact have anti-diversification properties.

The results of our experiment demonstrated that the existence of an asset price bubble, which occurs for certain parameter settings in the CEV model, results in the cryptocurrency loss distributions having more right-skewness and higher kurtosis. It has been shown that this augmented non-normality of the cryptocurrency's returns due to bubble expansion results in an increase in the VaR risk measures, and an understatement in the risk of the cryptocurrencies when non-bubble dynamics are inferred from an incorrect specification that fails to model cryptocurrencies jointly with equity prices. Based on these measures alone, their declining values imply that in the presence of asset price bubbles, a mispecified model results in a lower economic capital requirement. This market loss measure increases in bubble economies and is due to bubble bursting, with accompanying magnified market risk losses on the bubble-bursting paths.

As asset price bubbles are inevitably bound to burst, causing significant mark-to-market losses to holders of cryptocurrencies, we conclude that more market risk capital should be held for these bubble-bursting scenarios. Unfortunately, we have observed that the severity of these bubble-bursting scenarios is not adequately captured by a standard approach to market risk measures in novel asset classes such as cryptocurrencies, whose computations are based on modeling in isolation from other asset classes. Also, modeling time horizons are typically not long enough over which bubble bursting is likely, which coupled with inferring non-bubble dynamics in mispecified models creates further model risk. Furthermore, it has been shown that when these bubble-bursting scenarios are captured in correctly specified CEV model dynamics, derived from calibration of two- versus one- dimensional SDE models where cryptocurrencies are modeled jointly with an equity price index, then market risk measures thus derived admit asset price bubbles and are not understated.

We also measured the model risk arising from mispecifying the process driving cryptocurrencies by ignoring the relationship to another representative risk asset through applying the principle of relative entropy. In a bootstrap simulation implementation of this principle, we studied the distribution of a distance measure between the simulated distributions of VaR, in each iteration measuring the difference in VaR derived from a two-dimensional and a one-dimensional SDE model. We found that across all cryptocurrencies and the equity prices index these distributions to have positive support, excess kurtosis and (with the exception of a single cryptocurrency) extreme right-skewness. We found that in the majority of cases that the model risk "multipliers" range in about two to five across cryptocurrencies, estimates which could be applied to establish a model risk reserve as part of an economic capital calculation for traders in cryptocurrencies.

We have also illustrated implications of this research for prudential supervision and public policy. We have argued that a central question in this domain has been not only if but how cryptocurrencies should be brought under the supervisory umbrella, including which asset classes the various cryptocurrencies should be classified as (e.g., securities vs. commodities), and possible unintended consequences of ill-designed regulation. In view of our findings, we have asserted that there is a powerful interaction between cryptocurrencies and another major risk asset that leads to a self-reinforcing vicious cycle of bubble behavior, and that the regulatory regime should account for these linkages. Therefore, we concluded that any such regulatory regime should include an emphasis on coordination between different supervisory bodies, such as in the U.S. the SEC, CFTC, Federal Reserve, etc.

There are several fruitful avenues of future direction that we can take this line of research on cryptocurrencies and asset price bubbles, including but not limited to:

- Analyzing the interaction of cryptocurrencies and other risk assets such as fixed income, commodities, volatility, etc.;
- modeling several cryptocurrencies jointly in a portfolio management application;
- alternative econometric methodologies, such as non-parametric or machine learning techniques, and,
- alternative methodologies for applying the theory of martingales, such as in Choi et al. (2020).

Acknowledgments

The author acknowledges colleagues in the Professional Risk Management International Association (PRMIA), in particular Steve Lindo and Carl Densem, who provided valuable technical and conceptual input into the conduct of this study.

Disclaimer

The views expressed herein are those of the author do not necessarily represent an official position of PNC Financial Services Group.

References

Acharya, V., Philippon, T., Richardson, M., & Roubini, N. (2009). The financial crisis of 2007-2009: Causes and remedies. *Financial Markets, Institutions & Instruments, 18*(2), 89-137. https://doi.org/10.1111/j.1468-0416.2009.00147_2.x

Agosto, A., & Cafferata A. (2020). Financial bubbles: A study of co-explosivity in the cryptocurrency market. *Risks*, *8*(2), 34. Retrieved from https://doi.org/10.3390/risks8020034

Alexander, C. (2001). *Market Models: A Guide to Financial Data Analysis*. Chichester, UK: John Wiley & Sons. Retrieved from http://sro.sussex.ac.uk/id/eprint/40646/

Back, K. (2010). *Asset Pricing and Portfolio Choice Theory*. Oxford, UK: Oxford University Press. Retrieved from https://global.oup.com/academic/product/asset-pricing-and-portfolio-choice-theory-9780190241148?cc =us&lang=en&

Basel Committee on Banking Supervision. (2005). *An Explanatory Note on the Basel II IRB Risk Weight Functions*. Bank for International Settlements, July. Basel, Switzerland. Retrieved from https://www.bis.org/bcbs/irbriskweight.pdf

Basel Committee on Banking Supervision. (2006). *International Convergence of Capital Measurement and Capital Standards: A Revised Framework*. Bank for International Settlements, June. Basel, Switzerland.

Retrieved from https://www.bis.org/publ/bcbs128.htm

Basel Committee on Banking Supervision. (2009a). *Principles for Sound Stress Testing Practices and Supervision - Consultative Document*. Bank for International Settlements, May, No. 155. Basel, Switzerland. Retrieved from https://www.bis.org/publ/bcbs147.pdf

Basel Committee on Banking Supervision. (2009b). *Strengthening the Resilience of the Banking Sector - Consultative Document*. Bank for International Settlements, December. Basel, Switzerland. Retrieved from https://www.bis.org/publ/bcbs164.pdf

Basel Committee on Banking Supervision. (2010). *Basel III: A Global Regulatory Framework for More Resilient Banks and Banking Systems*. Bank for International Settlements, December. Basel, Switzerland. Retrieved from https://www.bis.org/publ/bcbs189.htm

Bouri, E., Gupta, R., & Roubaud, D. (2019). Herding behaviour in cryptocurrencies. *Finance Research Letters 29*, 216-221. https://doi.org/10.1016/j.frl.2018.07.008

Brunnermeier, M. K., & Oehmke, M. (2012). *Bubbles, financial crises, and systemic risk.* Working paper 19398, National Bureau of Economic Research, Washington, DC. https://doi.org/10.3386/w18398

Carver, L. (2014). Expected shortfall a tall order. *Risk* (June 1), p. 32. Retrieved from https://search.proquest.com/openview/ba9e86142a3e41f0d28c89ced7ff7bcb/1?pq-origsite=gscholar&cbl=32048

Cheah, E. T., & Fry, J. (2015). Speculative bubbles in bitcoin markets? an empirical investigation into the fundamental value of bitcoin. *Economics Letters 130*, 32-36. https://doi.org/10.1016/j.econlet.2015.02.029

Cheung, W. K., Roca, E., & Su, J. J. (2015). Cryptocurrency bubbles: an application of the Phillips–Shi–Yu (2013) methodology on Mt. Gox bitcoin prices. *Applied Economics 47*(23), 2348-2358. https://doi.org/10.1080/00036846.2015.1005827

Choi, S. H., & Jarrow, R. (2020). *Testing the local Martingale theory of bubbles using cryptocurrencies.* Working paper. https://doi.org/10.2139/ssrn.3701960

Delbaen, F., & Schachermayer, W. (1995). Arbitrage possibilities in Bessel processes and their relations to local martingales. *Probability Theory and Related Fields, 102*(3), 357-366. https://doi.org/10.1007/BF01192466

Delbaen, F., & Schachermayer, W. (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen, 312*(2), 215-250. https://doi.org/10.1007/s002080050220

Demirguc-Kunt, A, & Serven, L. (2010). Are all the sacred cows dead? Implications of the financial crisis for macro- and financial policies. *World Bank Research Observer, 25*(1), 91-124. https://doi.org/10.1093/wbro/lkp027

Emanuel, D. C., & MacBeth, J. D. (1982). Further results on the constant elasticity of variance call option pricing model. *Journal of Financial and Quantitative Analysis, 17*(4), 533-554. https://doi.org/10.2307/2330906

Embrechts, P., Lindskog, F., & McNeil, A. J. (2003). Modeling dependence with copulas and applications to risk management. In S. Rachev (Ed.), *Handbook of Heavy Tailed Distributions in Finance* (pp. 329-384). https://doi.org/10.1016/B978-044450896-6.50010-8

Embrechts, P., McNeil, A. J., & Straumann, D. (1999). Correlation: pitfalls and alternatives. *Risk 12*, 69-71. Retrieved from https://www.academia.edu/download/34470776/correlation_and_dependency_in_risk_management_properties_and_pitfalls.pdf

Frey, R., & McNeil, A. J. (2003). *Modeling dependent defaults in models of portfolio credit risk.* Working paper, ETH Zurich. https://doi.org/10.21314/JOR.2003.089

Geuder, J., Kinateder, H., & Wagner, N. F. (2019). Cryptocurrencies as financial bubbles: The case of Bitcoin. *Finance Research Letters, 31*, 179-184. https://doi.org/10.1016/j.frl.2018.11.011

Glasserman, P., & Xu, X. (2013). Robust risk measurement and model risk. *Quantitative Finance, 83*(14), 29-58. https://doi.org/10.1080/14697688.2013.822989

Hansen, L. P., & Sargent, T. (2007). *Robustness*. Princeton, NJ: Princeton University Press. https://doi.org/10.1515/9781400829385

Hong, H., Scheinkman, J., & Xiong, W. (2006). Asset float and speculative bubbles. *Journal of Finance, 61*(3), 1073-1117. https://doi.org/10.1111/j.1540-6261.2006.00867.x

Inanoglu, H., & Jacobs, Jr., M. (2009). Models for risk aggregation and sensitivity analysis: An application to bank economic capital. *The Journal of Risk and Financial Management, 2*(1), 118-189. https://doi.org/10.3390/jrfm2010118

Jacobs, Jr., M. (2013). Stress testing credit risk portfolios. *Journal of Financial Transformation, 37*(April), 53-75. Retrieved from https://www.michaeljacobsjr.com/wp-content/uploads/2022/04/Jacobs_StressTesting_Apr13_JFT_no37_pp53-75.pdf

Jacobs, Jr., M. (2015a). The quantification and aggregation of model risk: perspectives on potential approaches. *The Journal of Financial Engineering and Risk Management, 2*(2), 124-154. https://doi.org/10.1504/IJFERM.2015.074045

Jacobs, Jr., M. (2015b). The impact of asset price bubbles on credit risk measures. *The Journal of Financial Risk Management, 4*(4), 251-266. https://doi.org/10.4236/jfrm.2015.44019

Jacobs, Jr., M. (2016). The impact of asset price bubbles on liquidity risk measures from a financial institutions perspective. *The International Journal of Bonds and Derivatives, 2*(2), 152-182. https://doi.org/10.1504/IJBD.2016.077192

Jacobs, Jr., M. (2017). Asset price bubbles and the quantification of credit risk capital with sensitivity analysis, empirical implementation and an application to stress testing. *The Journal of Risk Model Validation, 11*(4), 1-35. https://doi.org/10.21314/JRMV.2017.178

Jacobs, Jr., M. (2022). Quantification of model risk with an application to probability of default estimation and stress testing for a large corporate portfolio. *The Journal of Risk Model Validation, 15*(3), 1-39. https://doi.org/10.21314/JRMV.2022.023

Jacobs, Jr., M., Karagozoglu, A. K., & Sensenbrenner, F. J. (2015). Stress testing and model validation: Application of the Bayesian approach to a credit risk portfolio. *The Journal of Risk Model Validation, 9*(3), 41-70. https://doi.org/10.21314/JRMV.2015.140

Jarrow, R. A., & Protter, P. (2010). The martingale theory of bubbles: implications for the valuation of derivatives and detecting bubbles. In A. Berd (Ed.), *The Financial Crisis: Debating the Origins, Outcomes, and Lessons of the Greatest Economic Event of Our Lifetime* (pp. 429-448). https://doi.org/10.2139/ssrn.1604175

Jarrow, R. A., & Silva, F. B. G. (2014). Asset price bubbles in incomplete markets. *Mathematical Finance, 20*(2), 145-185. https://doi.org/10.1111/j.1467-9965.2010.00394.x

Jarrow, R. A., & Silva, F. B. G. (2015). Risk measures and the impact of asset price bubbles. *The Journal of Risk 17*(3), 145-185. https://doi.org/10.21314/JOR.2015.321

Jarrow, R. A., Kchia, Y., & Protter, P. (2011). How to detect an asset bubble. *SIAM Journal on Financial Mathematics, 2*(1), 839-865. https://doi.org/10.1137/10079673X

Jarrow, R. A., Protter, P., & Shimbo, K. (2007). Asset price bubbles in complete markets. In M. C. Fu, R. A. Jarrow, J. Y. Yen, & R. J. Elliott (Eds.), *Advances in Mathematical Finance* (pp. 97-121). https://doi.org/10.1007/978-0-8176-4545-8_7

Jarrow, R. A., Protter, P., & Shimbo, K. (2010). Asset price bubbles in incomplete markets. *Mathematical Finance, 20*(2), 145-185. https://doi.org/10.1111/j.1467-9965.2010.00394.x

Jeanblanc, M., Yor, M., & Chesney, M. (2009). *Mathematical Methods for Financial Markets.* Springer. https://doi.org/10.1007/978-1-84628-737-4

Jorion, P. (1997). *Value at Risk: The New Benchmark for Controlling Market Risk* (2nd ed.). Retrieved from https://www.worldcat.org/title/value-at-risk-the-new-benchmark-for-controlling-market-risk/oclc/34658858

Jorion, P. (2006). *Value at Risk: The Benchmark for Managing Financial Risk* (3rd ed.). McGraw Hill Retrieved from https://www.amazon.com/Value-Risk-Benchmark-Managing-Financial/dp/0071464956

Li, D. X. (2000). On default correlation: A copula function approach. *Journal of Fixed Income, 9*(4), 43-54. https://doi.org/10.3905/jfi.2000.319253

Merton, R. (1974). On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance, 29*(2), 449-470. https://doi.org/10.1111/j.1540-6261.1974.tb03058.x

Øksendal, B. K. (2003). *Stochastic Differential Equations: An Introduction with Applications* (6th ed). Springer. https://doi.org/10.1007/978-3-642-14394-6_1

Phillips, P. C. B., Shi, S., & Yu, J. (2013). *Testing for multiple bubbles: Historical episodes of exuberance and collapse in the S&P 500*. Working paper. https://doi.org/10.2139/ssrn.2327609

Phillips, R. C., & Gorse, D. (2017). Predicting cryptocurrency price bubbles using social media data and epidemic modelling. In *IEEE Symposium Series on Computational Intelligence (SSCI)* (pp. 1-7). https://doi.org/10.1109/SSCI.2017.8280809

Poon, S. H., Rockinger, M., & Tawn, J. (2004). Extreme value dependence in financial markets: Diagnostics, models and financial implications. *Review of Financial Studies, 17*(2), 581-610. https://doi.org/10.1093/rfs/hhg058

Protter, P. (2011). How to detect an asset bubble. *SIAM Journal on Financial Mathematics, 2*(1), 839-865. https://doi.org/10.1137/10079673X

Protter, P., & Shimbo, K. (2010). Risk measures and the impact of asset price bubbles. *Journal of Risk, 17*(3), 35-55. https://doi.org/10.21314/JOR.2015.321

R Development Core Team. (2022). *R: A Language and Environment for Statistical Computing.* R Foundation for Statistical Computing, Vienna, Austria. Retrieved from https://www.r-project.org/

Sauce, L. (2022). The unintended consequences of the regulation of cryptocurrencies. *Cambridge Journal of Economics, 46*(1), 57-71. https://doi.org/10.1093/cje/beab053

Schroder, M. (1989). Computing the constant elasticity of variance option pricing formula. *Journal of Finance 44*(1), 211-219. https://doi.org/10.1111/j.1540-6261.1989.tb02414.x

Shahzad, S. J. H., Anas, M., & Bouri, E. (2022). Price explosiveness in cryptocurrencies and Elon Musk's tweets. *Finance Research Letters, 47*(Part B), 102695. https://doi.org/10.1016/j.frl.2022.102695

Sklar, A. (1956). Fonctions de répartition à n dimensions et leurs marges. *Publications de l'Institut de Statistique de L'Université de Paris, 8*, 229-231. Retrieved from https://cir.nii.ac.jp/crid/1573387449735953792

Skoglund, J. (2019). Quantification of model risk in stress testing and scenario analysis. *Journal of Risk Model Validation, 13*(1), 1-23. https://doi.org/10.21314/JRMV.2019.201

U.S. Board of Governors of the Federal Reserve System. (2011). *Supervisory Gguidance on Model Risk management (SR 2011-7).* Washington D.C. Retrieved from https://www.federalreserve.gov/supervisionreg/srletters/sr1107.htm

Notes

Note 1. The data used in this paper ranges in length from around 9 years for Bitcoin and 4-6 years for the other cryptocurrencies under consideration, which is arguably a short enough period that allows us to credibly rule out any drastic regime change.

Note 2. In the wake of the financial crisis (Demirguc-Kunt et al., 2010; Acharya et al., 2009), international supervisors have recognized the importance of ST, especially in the realm of credit risk, as can be seen in the re-vised Basel framework (BCBS 2005, 2006; 2009 a,b; 2010) and the *Federal Reserve's Comprehensive Capital Analysis and Review* ("CCAR") program (Jacobs, 2013; Jacobs et al., 2015).

Note 3. The conditions are that τ^* is *almost surely increasing* $P^Q[\tau_k^* < \tau_{k+1}^*] = 1$ and is almost surely divergent $P^Q[\tau_k^* \to \infty \; as \; k \to \infty] = 1$ (Oksendal, 2003).

Note 4. This condition is sometimes termed "no free-lunch with vanishing risk" or NFLVR (Jeanblanc et al., 2009).

Note 5. We use a proprietary modified code based upon the source code for the R package Sim.DiffProc to both estimate and simulate the one- and two-dimensional system of SDEs for the cryptocurrencies and the NADAQ equity index (R Development Core Team, 2022). The estimation algorithm that we develop is robust to misspecification the distribution of the random noise Weiner processes relative to the assumption that they obey a Gaussian law.