The DOL-DFL Nexus

The Relationship between the Degree of Operating Leverage (DOL) and the Degree of Financial Leverage (DFL)

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Abstract

In the present paper, I have modelled the Degree of Operating Leverage (DOL) and the Degree of Financial Leverage (DFL) using the percentage variations of the economic quantities. I devoted a great effort to encompass the investment dynamic and its financing mix to design a robust model implementable in a business context.

The relationship discovered between DOL and DFL is complex and manifold: first, it appears asymmetrical because DOL can influence DFL, but the former is unrelated to the latter. Second, there is an infra-annual relationship measurable through partial derivatives. Eventually, the stress tests shed light on some long-term impacts of one-off shocks even when the steady-state conditions are restored, disclosing an inter-annual relationship.

The DOL-DFL nexus appears to be negatively related, but I also discovered positive relations and unrelated conditions. As argued in the economic literature, they cannot always behave as substitutes. The mathematical DOL-DFL model developed can admit positive, negative, and unrelated relations even though management might intervene to choose the right combination. Also, the Business Case shows positive and negative relationships, both at the infra-annual and inter-annual levels. The DOL-DFL nexus depends on circumstances and management decisions. Empirical evidence should find how management uses such a nexus and how effective such decisions have been over time.

Keywords: debt ratio, DFL, DOL, EBIT, financial leverage, fixed costs, net profit, operating leverage, tax rate, variable costs

1. Introduction

The DOL-DFL nexus involves important corporate decisions in capital budgeting and structure that impact profitability and risk. The economic literature considers the Degree of Operating Leverage (from now on DOL) and the Degree of Financial Leverage (from now on DFL) as substitutes even though the empirical evidence is controversial. Besides, the precise analytical relationship is still debated and to worsen the situation, DOL and DFL definitions are not univocal.

Beyond the debate on Trade-off and Pecking Order hypothesis summarised perfectly by Frank et al. (2007) and Glover et al. (2013), the analytical relationship between DOL and DFL, including their determinants or factors, has been only recently addressed by Sarkar (2020) and Chen et al. (2019) in essential contributions. Since I presented the methodology of using the percentage variations of the economic quantities to examine the DOL in Paganini (2019), in the present paper, I extend such a method to investigate the DFL to reach a persuasive solution.

Before addressing the subject, I have pointed out already that the definitions of DOL, DFL, Operating Leverage, and Financial Leverage in the economic literature are not always univocal. The DOL and DFL definitions used in this paper are the following:

$$DOL_t = \frac{\Delta\%EBIT_t}{\Delta\%S_t} \tag{1}$$

$$DFL_t = \frac{\Delta\%\Pi_t}{\Delta\%EBIT_t} = \frac{\Delta\%\Pi_t}{DOL_t * \Delta\%S_t} \tag{2}$$

where:

$\Delta\%S_t$ = Revenue percentage variation for the period t compared to the previous period;

$\Delta\%EBIT_t$ = EBIT percentage variation for the period t compared to the previous period;

$\Delta\%\Pi_t$ = Net Profit percentage variation for the period t compared to the previous period.

From the previous equations (1) and (2), we can observe that DFL depends on DOL, being the latter in the denominator of equation (2), while the opposite appears unlikely. Furthermore, we have to consider that at the company level, what matters is the impact that the Revenue percentage variation $\Delta\%S_t$, that means Revenue growth, determines on the Net Profit percentage variation $\Delta\%\Pi_t$:

$$\frac{\Delta\%\Pi_t}{\Delta\%S_t} = DOL_t * DFL_t = DTL_t \tag{3}$$

where:

DTL_t = Degree of Total Leverage for the period t.

The reason why I adopted the previous definitions of DOL and DFL depends on the possibility of connecting them to the Net Profit percentage variation $\Delta\%\Pi_t$ and particularly to the Net Profit Π_t for the period t thanks to the following equation:

$$\Pi_t = \Pi_{t-1} * (1 + DOL_t * DFL_t * \Delta\%S_t) = \Pi_{t-1} * (1 + DTL_t * \Delta\%S_t) \tag{3 bis}$$

from which appears the indirect link that unites DOL, DFL, and $\Delta\%S_t$ on one side and systematic risk β on the other, as already presented by Mandelker et al. (1984). Perhaps alternative definitions of DOL and DFL could be practical for specific purposes, but equation (3 bis) is the gateway to the systematic risk analysis.

2. Economic Literature and Development Strategy

We can find alternative definitions of DOL and DFL and different nomenclatures in the economic literature, invariably referring to the same topic even though they refer to various objectives and perspectives. See the papers by Dudycz (2006 and 2020) for an in-depth analysis of the different definitions of DOL and DFL.

The second issue I have to address concerns the methodology used to mathematically and economically model DOL and DFL functions: if we define both as the relationship between measures related to percentage variations of economic quantities, it becomes essential to use a suitable methodology that allows us to represent and explain the variations occurred in the same way. The considerations exposed by Paganini (2019) on DOL are extensible to the studies on the DFL with the further observation that there has been no attempt to model it by including the financing mix and understanding whether DFL depends primarily on tax burden or financial variables.

Prezas (1987) includes the safe debt but excludes both the financing mix and the taxation, while Sarkar (2020) embeds the debt ratio in his model even though the latter is not the DFL.

The third issue concerns the supposed negative relationship between DOL and DFL, either proposed by the "trade-off" hypothesis or predicted by the theoretical literature, not always observed empirically. The introduction of the paper by Sarkar (2020) is enlightening. Mandelker et al. (1984) "found a significant correlation between the two types of leverage", concluding that "firms engage in trade-offs between DOL and DFL seems to have gained strong empirical evidence" while Dugan et al. (1994) found the same in only one kind of firms out of two. Also, Lord (1996) was unable to find "an interaction between DOL and DFL where risks were concerned". Prezas (1987) demonstrated that "the changes in DOL and DFL, caused by changes in debt, are of the same sign in many cases".

It is arduous to assume such a relationship negative for granted *sic et simpliciter*, lacking a convincing model of how DOL and DFL relate to each other.

The fourth issue concerns the implementation in a business context of a model that can examine and evaluate DOL-DFL nexus starting from the complete set of company's data without which it is impossible to investigate DOL. We can anticipate that DFL depends exclusively on financial statement data, while DOL needs either managerial accounting data or indispensable conjectures. Some models proposed in the literature seem hard to be implemented at the company level, despite being innovative and intriguing.

The present working paper tries to overcome some limitations of the existing models by setting four objectives:

1)   to adopt a definition of DOL and DFL that can be linked simply to systematic risk β and with the least

possible number of variables, as previously stated;

2) to use a methodology that can explain consistently with the definition assumed for DOL and DFL how management can influence the starting conditions through the percentage variation of the economic and financial quantities of the business;

3) to reach a DOL-DFL nexus that makes it possible to understand and verify their mutual relationship;

4) to use the DOL-DFL nexus in the analysis of Business Cases, provided the availability of all the accounting information necessary to determine DOL.

The present working paper has been organised in the following paragraphs bearing in mind the four objectives aforementioned: paragraph 3 resumes the DOL model developed by Paganini (2019), in paragraph 4 develops a brand new DFL model using the percentage variations already used for DOL; in paragraph 5, it is necessary to investigate how the variations inside the Income Statement impact on the Balance Sheet since DFL, unlike DOL, depends on the financing mix of the firm. The ensuing two paragraphs allow us to deepen how DOL and DFL functions are running together in a business model initially set in steady-state (paragraph 6) and later with a selected set of one-off shocks to verify the behaviour of the complete model (paragraph 7).

Finally, the DOL-DFL nexus is tested in a Business Case (paragraph 8) to check what type of relationship establishes at the infra-annual and inter-annual levels. Conclusions and discussions will follow to close the paper with paragraph 9.

3. The Degree of Operating Leverage (DOL)

The definition of DOL for the period t reported in equation (1) is the ratio between the EBIT and the Revenue percentage variations for the period t compared to the previous period.

From the paper by Paganini (2019), we can extract the following equation representing the DOL function:

$$DOL_t = \frac{\Delta\%EBIT_t}{\Delta\%S_t} = PDOL_t * \left\{ \frac{S_{t-1}}{CM_{t-1}} - \frac{VC_{t-1}*[(1+\Delta\%uvc_t)*(1+\Delta\%qmvc_t)-1]+ FC_{t-1}*\Delta\%F_t}{CM_{t-1}*[(1+\Delta\%up_t)*(1+\Delta\%qms_t)-1]} \right\} \qquad (1 \text{ bis})$$

where:

$PDOL_t$ = Potential DOL for the period t;

S_{t-1} = Revenue for the period $t\text{-}1$;

VC_{t-1} = Variable Costs for the period $t\text{-}1$;

CM_{t-1} = Contribution Margin for the period $t\text{-}1$;

FC_{t-1} = Fixed Costs for the period $t\text{-}1$;

$\Delta\%up_t$= unit price percentage variation for the period t compared to the previous period;

$\Delta\%uvc_t$= unit variable cost percentage variation for the period t compared to the previous period;

$\Delta\%qms_t$= quantity/mix percentage variation of Revenue for the period t compared to the previous period;

$\Delta\%qmvc_t$= quantity/mix percentage variation of Variable Costs for the period t compared to the previous period;

$\Delta\%F_t$= Fixed Costs percentage variation for the period t compared to the previous period.

I have to remember that $PDOL_t$ is the ex-ante or Potential Degree of Operating Leverage for the period t and is determined as follows:

$$PDOL_t = \frac{CM_{t-1}}{EBIT_{t-1}} \qquad (3)$$

where:

$EBIT_{t-1}$ = Earning Before Interest and Taxes for the period $t\text{-}1$;

and this indicates that DOL_t tends to the ratio between the Contribution Margin and EBIT for the period preceding t whenever the difference of the absolute variation of the Contribution Margin and the Fixed Costs in period t compared to the previous period is equal to $CM_{t-1} * \Delta\%S_t$ that represents the potential Contribution Margin percentage variation. To summarise, multiple solutions lead DOL_t to match $PDOL_t$ since the solution in which the Fixed Costs variation is zero is not the only one.

The Contribution Margin CM_t is the difference between Revenue and Variable Costs for the period t while

$EBIT_t$ is the difference between Contribution Margin and Fixed Costs for the period t.

The equation (1 bis) comes out by the simple methodology of comparing the main economic variables of the Income Statement, i.e. Revenue, Variable Costs, and Fixed Costs, of two financial periods by relating them with each other and replace the increase of the economic quantity for the period t with the value of the previous period multiplied by the percentage variation occurred in the period t. The analytical dynamic thus obtained of the Revenue, Variable Costs, and Fixed Costs percentage variations explain very well the impact that both management and markets have had on the EBIT variation between two periods.

For any further information on the DOL issue, please refer to Paganini (2019).

4. The Degree of Financial Leverage (DFL) and Its Relationship with DOL

To investigate the DFL_t of the period t, I employed the same methodology used to examine DOL by applying it to the variables that determine DFL.

Let it start with the definition of DFL adopted with equation (2).

$$DFL_t = \frac{\Delta\%\Pi_t}{\Delta\%EBIT_t} = \frac{\Delta\%\Pi_t}{DOL_t * \Delta\%S_t} \tag{2}$$

From equation (2), it is evident that only the dynamic of Net Profit $\Delta\%\Pi_t$ needs to be determined. The definition of Net Profit Π_t for the period t is the following:

$$\Pi_t = (EBIT_t - FO_t) * (1 - \alpha_t) = EBT_t * (1 - \alpha_t) \tag{4}$$

where:

Π_t = Net Profit for the period t;

$EBIT_t$ = Earning Before Interest and Taxes for the period t;

FO_t = Financial Charges for the period t;

EBT_t= Earning Before Taxes for the period t;

α_t = Corporate Income Tax Rate for the period t.

Therefore we can move on to define the Net Profit percentage variation for the period t compared to the previous period obtaining the following equation:

$$\Delta\%\Pi_t = \frac{\Pi_t}{\Pi_{t-1}} - 1 = \frac{(EBIT_t - FO_t)*(1-\alpha_t)}{(EBIT_{t-1} - FO_{t-1})*(1-\alpha_{t-1})} - 1 = \frac{EBIT_t*\left(1 - \frac{FO_t}{EBIT_t}\right)*(1-\alpha_t)}{EBIT_{t-1}*\left(1 - \frac{FO_{t-1}}{EBIT_{t-1}}\right)*(1-\alpha_{t-1})} - 1 =$$

$$= (1 + \Delta\%EBIT_t) * \left(1 + \Delta\%\rho_t\right) * (1 + \Delta\%\nu_t) - 1 \tag{5}$$

where:

$$\left(1 + \Delta\%\rho_t\right) = \frac{\rho_t}{\rho_{t-1}} = \frac{1-\phi_t}{1-\phi_{t-1}} = \frac{1 - \frac{FO_t}{EBIT_t}}{1 - \frac{FO_{t-1}}{EBIT_{t-1}}} \tag{6}$$

and

$$(1 + \Delta\%\nu_t) = \frac{\nu_t}{\nu_{t-1}} = \frac{1-\alpha_t}{1-\alpha_{t-1}} \tag{7}$$

where:

ϕ_n = EBIT share allocated to Financial Charges for the period n;

ρ_n = EBIT share allocated to EBT for the period n;

ν_n = EBT share allocated to Net Profit for the period n;

$\Delta\%\rho_t$= percentage variation of EBIT share allocated to EBT for the period t compared to the previous period;

$\Delta\%\nu_t$= percentage variation of EBT share allocated to Net Profit for the period t compared to the previous period.

I have to point out that by definition, the sum of ϕ_n and ρ_n must be 1, as the sum of ν_n and α_t.

Equation (5) indicates that the Net Profit percentage variation $\Delta\%\Pi_t$ will be equal to the EBIT percentage variation $\Delta\%EBIT_t$ if there are no changes in the percentage allocation of:

- EBIT between Financial Charges and EBT and

- EBT between Taxes and Net Profit

corresponding to the condition that $\Delta\%\rho_t$ and $\Delta\%v_t$ are both zero; another solution is they can perfectly offset, which means $(1 + \Delta\%\rho_t) * (1 + \Delta\%v_t)$ be equal to 1.

In conclusion, the Net Profit percentage variation $\Delta\%\Pi_t$ will be the higher,

a. the higher the EBIT variation is, influenced in turn by the factors determining DOL;

b. the higher the EBIT variation allocated to EBT is by lowering the share apportioned to Financial Charges;

c. the higher the EBT variation assigned to Net Profit is by decreasing the part apportioned to Taxes.

We have examined point a. already; point c. it is relatively simple to deal analytically while the allocation of EBIT between Financial Charges and EBT depends on many variables such as the Debt level, its cost, the debt ratio, and their percentage variations that occurred in period t.

Now through a few algebraic steps, it is possible to prove that:

$$\left(1 + \Delta\%\rho_t\right) = 1 - \frac{\phi_{t-1}}{1-\phi_{t-1}} * \Delta\%\phi_t \qquad \text{(6 bis)}$$

where:

$\Delta\%\phi_t$ = percentage variation of EBIT share allocated to Financial Charges for the period t compared to the previous period;

and similarly:

$$(1 + \Delta\%v_t) = 1 - \frac{\alpha_{t-1}}{1-\alpha_{t-1}} * \Delta\%\alpha_t \qquad \text{(7 bis)}$$

where:

$\Delta\%\alpha_t$ = percentage variation of EBT share allocated to Taxes (or more simply the Tax Rate α_t percentage variation) for the period t compared to the previous period;

Equation (7 bis) measures the percentage variation of the EBT share allocated to Net Profit starting from the Tax Rate for the period t-1 and its percentage variation for the period t compared to the previous one. Equation (6 bis) measures the percentage variation of the EBIT share allocated to EBT starting from the impact that the Financial Charges had for the period t-1 and their percentage variation for the period t compared to the previous one. Equation (6 bis) allows to link $\Delta\%\phi_t$ to the debt exposure and its related cost that originate the Financial Charges of the two periods:

$$\Delta\%\phi_t = \frac{\phi_t}{\phi_{t-1}} - 1 = \frac{FO_t}{EBIT_t} * \frac{EBIT_{t-1}}{FO_{t-1}} - 1 = \frac{i_t*D_t}{i_{t-1}*D_{t-1}} * \frac{EBIT_{t-1}}{EBIT_t} - 1 = \frac{(1+\Delta\%i_t)*(1+\Delta\%D_t)}{(1+\Delta\%EBIT_t)} - 1 \qquad \text{(8)}$$

where:

i_n = Cost of Debt for the period n;

D_n = debt exposure at the end of the period n;

$\Delta\%i_t$ = percentage variation of the Cost of Debt for the period t compared to the previous period;

$\Delta\%D_t$ = percentage variation of Debt Exposure for the period t compared to the former period.

We must point out that $\Delta\%i_t$ is obtained as the ratio, not as difference, between i_t and i_{t-1} decreased by one unit. Using Income Statement and Balance Sheet data, i_n is the ratio between FO_n e D_n:

$$i_n = \frac{FO_n}{D_n} \qquad \text{(9)}$$

FO_n are the Financial Charges for the period n shown in the Income Statement, and D_n is the debt exposure at the end of the period n shown in the Balance Sheet. The approach adopted is questionable since the Cost of Debt i_n thus obtained may not represent its cost. I could have used a weighted average Cost of Debt, but in this case, the value of the exposure D_n is not the value observed in the Balance Sheet, even though it would have a more significant economic meaning. Such an option allows us to reach a more convincing economic explanation, but the definition of invested capital needs to be changed.

I decided to remain tied to the figures of the Balance Sheet.

We introduce the following definition of Debt-to-Equity ratio that corresponds to the financial leverage

definition used in several financial papers and handbooks, Brigham (2011) and Damodaran (2001) for instance:

$$t_{d_t} = \frac{D_t}{E_t} \tag{10}$$

where:

t_{d_t} = Debt-to-Equity ratio at the end of the period t;

E_t = Equity at the end of the period t.

From the Balance Sheet at the end of the period t, we can observe the value of E_t.

Now comparing the Debt-to-Equity ratios of two financial periods, we can obtain the following equation:

$$\frac{t_{d_t}}{t_{d_{t-1}}} = (1 + \Delta\%t_{d_t}) = \frac{D_t}{E_t} * \frac{E_{t-1}}{D_{t-1}} = \frac{1+\Delta\%D_t}{1+\Delta\%E_t} \tag{11}$$

where:

$\Delta\%t_{d_t}$= percentage variation of Debt-to-Equity ratio for the period t compared to the previous period;

$\Delta\%E_t$= percentage variation of the Equity for the period t compared to the previous period.

From equation (11), we can obtain:

$$(1 + \Delta\%D_t) = (1 + \Delta\%td_t) * (1 + \Delta\%E_t) \tag{11 bis}$$

which can be replaced in the equation (8) obtaining:

$$\Delta\%\phi_t = \frac{(1+\Delta\%i_t)*(1+\Delta\%td_t)*(1+\Delta\%E_t)}{(1+\Delta\%EBIT_t)} - 1 \tag{8 bis}$$

which, in turn, it will go into equation (6 bis) and then into equation (5), obtaining:

$$\Delta\%\Pi_t = (1 + \Delta\%EBIT_t) * \left(1 + \Delta\%\rho_t\right) * (1 + \Delta\%v_t) - 1 =$$

$$= (1 + \Delta\%EBIT_t) * \left\{1 - \frac{\phi_{t-1}}{1-\phi_{t-1}} * \left[\frac{(1+\Delta\%i_t)*(1+\Delta\%td_t)*(1+\Delta\%E_t)}{(1+\Delta\%EBIT_t)} - 1\right]\right\} * (1 + \Delta\%v_t) - 1 \tag{5 bis}$$

Proceeding now with a double replacement:

$$A_{t-1} = \frac{\phi_{t-1}}{1-\phi_{t-1}} \tag{12}$$

and

$$T_t = (1 + \Delta\%v_t) \tag{13}$$

we can reach the following simplified algebraic notation:

$$\Delta\%\Pi_t = \left[\left(1 + A_{t-1}\right) * (1 + \Delta\%EBIT_t) - A_{t-1} * (1 + \Delta\%i_t) * (1 + \Delta\%t_{d_t}) * (1 + \Delta\%E_t)\right] * T_t - 1 \tag{5 ter}$$

Equation (5 ter) presents an interesting dynamic of the Net Profit percentage variation $\Delta\%\Pi_t$ linked to both income and equity dynamics but lacking the dynamic generated by the Invested Capital variation even though it contains clues about its funding through Equity and Debt.

We need to find a relationship explaining the dynamic of the Invested Capital.

From the ratio between Invested Capital at the end of period t compared to the previous one, we can achieve the following relation:

$$\frac{CI_t}{CI_{t-1}} = (1 + \Delta\%CI_t) = \frac{E_t+D_t}{E_{t-1}+D_{t-1}} = \frac{E_t*(1+\frac{D_t}{E_t})}{E_{t-1}*(1+\frac{D_{t-1}}{E_{t-1}})} = \frac{E_t*(1+t_{dt})}{E_{t-1}*(1+t_{d_{t-1}})} = (1 + \Delta\%E_t) * \frac{1+t_{dt}}{1+t_{d_{t-1}}} \tag{14}$$

where:

CI_n= Invested Capital at the end of period n;

$\Delta\%CI_t$= percentage variation of the Invested Capital for the period t compared to the previous period.

By definition, we can introduce the function ψ_t that represents the ratio between Invested Capital and Revenue for the period t compared to the same ratio in the previous period:

$$\psi_t = \frac{CI_t}{S_t} * \frac{S_{t-1}}{CI_{t-1}} = \frac{1+\Delta\%CI_t}{1+\Delta\%S_t} \tag{15}$$

Using equation (15) to modify equation (14), we can achieve the following:

$$(1 + \Delta\%CI_t) = (1 + \Delta\%E_t) * \frac{1+t_{dt}}{1+t_{d_{t-1}}} = \psi_t * (1 + \Delta\%S_t) \tag{14 bis}$$

from which we can get:

$$(1 + \Delta\%E_t) = \psi_t * (1 + \Delta\%S_t) * \frac{1+t_{d_{t-1}}}{1+t_{d_t}} \tag{14 ter}$$

that can be useful to replace $(1 + \Delta\%E_t)$ in equation (5 ter) to obtain:

$$\Delta\%\Pi_t = \left[\left(1 + A_{t-1}\right) * (1 + \Delta\%EBIT_t) - A_{t-1} * (1 + \Delta\%i_t) * (1 + \Delta\%td_t) * \psi_t * (1 + \Delta\%S_t) * \frac{1+t_{d_{t-1}}}{1+t_{d_t}}\right] * T_t - 1 \tag{5 quater}$$

Through the following simplifications:

$$M_t = (1 + \Delta\%i_t) \tag{16}$$

and

$$P_t = \psi_t * (1 + \Delta\%td_t) * \frac{1+t_{d_{t-1}}}{1+t_{d_t}} = \psi_t * \frac{DFL^*_t}{DFL^*_{t-1}} \tag{17}$$

where:

$$DFL^*_n = \frac{D_n}{D_n + E_n} \tag{18}$$

which represents the debt ratio (Brigham, 2011) for the period *n* and that many authors define improperly as "degree of financial leverage" (Sarkar, 2020), we obtain the following compact equation for $\Delta\%\Pi_t$:

$$\Delta\%\Pi_t = \left[\left(1 + A_{t-1}\right) * (1 + DOL_t * \Delta\%S_t) - A_{t-1} * M_t * P_t * (1 + \Delta\%S_t)\right] * T_t - 1 \tag{5 quinquies}$$

Through the following simplifications:

$$N_t = (1 + A_{t-1}) * T_t \tag{19}$$

and

$$Q_t = A_{t-1} * M_t * P_t * T_t \tag{20}$$

we can finally reach the following equation:

$$\Delta\%\Pi_t = N_t * (1 + DOL_t * \Delta\%S_t) - Q_t * (1 + \Delta\%S_t) - 1 \tag{5 sexies}$$

The equation (5 sexies) needs some explanations; the Net Profit percentage variation $\Delta\%\Pi_t$ has the following determinants:

1) N_t which depends on the product of:

   a. an initial parameter A_{t-1} related to the period *t-1*, increased by one unit, which represents the ratio between the EBIT share allocated to Financial Charges and the share allocated to EBT; $1 + A_{t-1}$ is the inverse of EBIT share apportioned to EBT, i.e. $1/\rho_{t-1}$;

   b. a variable T_t related to the corporate Tax Planning or the tax burden variation for the period *t* compared to the previous one.

2) Q_t that depends on the product of:

   a. A_{t-1} and T_t which already influence N_t;

   b. the variable M_t, which represents the percentage variation of the Cost of Debt; such a variation, in turn, depends on:

      i. the macroeconomic framework and

      ii. the different risk perceived by lenders caused by:

         1. market changes in risk premium or

         2. Corporate risk changes *stricto sensu*.

   c. the variable P_t, which depends on corporate policy:

      i. to fund the Invested Capital and

      ii. to adapt the Invested Capital according to the Revenue trend;

3) by the variable DOL_t that affects $\Delta\%EBIT$;

4) the Revenue percentage variation $\Delta\%S_t$, positively related to the EBIT percentage variation $\Delta\%EBIT$ and negatively to the Invested Capital percentage variation $\Delta\%CI_t$.

In summary, the Net Profit percentage variation depends on two factors that measure:

1. the first one, based on N_t, is the after-tax impact in the absence of changes in the EBIT allocation between Financial Charges and EBT for the period *t* compared to *t-1*;

2. the second one, based on Q_t, is the after-tax impact resulting from changes in the financing mix and related costs for the period *t* compared to the previous one, which originates a different EBIT allocation among Financial Charges, Taxes, and Net Profit.

Furthermore, N_t is linked to the EBIT percentage variation $\Delta\%EBIT$ while Q_t relates to the Revenue percentage variation $\Delta\%S_t$.

Thanks to the equation (5 sexies), we can finally reach the definition of DFL_t:

$$DFL_t = \frac{\Delta\%\Pi_t}{\Delta\%EBIT_t} = \frac{N_t*(1+DOL_t*\Delta\%S_t)-Q_t*(1+\Delta\%S_t)-1}{DOL_t*\Delta\%S_t} = N_t + \frac{N_t-1}{DOL_t*\Delta\%S_t} - Q_t*\frac{1+\Delta\%S_t}{DOL_t*\Delta\%S_t} \qquad (2\ bis)$$

Equation (2 bis) highlights how DFL_t depends on four variables:

1) the degree of operating leverage DOL_t;

2) the Revenue growth $\Delta\%S_t$;

3) the variable N_t measures the distribution of EBIT between EBT and Financial Charges in the period *t-1* to which is applied the variation of the EBT share allocated to Net Profit through the taxation of the period *t* compared to the previous one;

4) the variable Q_t measures the after-tax impact of the Invested Capital variation, which, in turn, induces changes in the financing mix, its cost, and the EBIT allocation between EBT and Financial Charges.

Suppose that DOL_t be exogenously specified and equal to 100%, a condition of equilibrium that occurs in case of an even percentage variation of Revenue, Variable Costs and Fixed Costs for the period *t* compared the previous one, then equation (2 bis) becomes:

$$DFL_t = \frac{\Delta\%\Pi_t}{\Delta\%EBIT_t} = (N_t - Q_t)*\left(1+\frac{1}{\Delta\%S_t}\right) - \frac{1}{\Delta\%S_t} \qquad (2\ ter)$$

Also, DFL_t has its equilibrium value to which tends in case of steady-state conditions identifiable as follows:

$$DOL_t = M_t = T_t = \frac{DFL^*_t}{DFL^*_{t-1}} = 1$$

DFL_t tends to its potential value, namely $PDFL_t$ when the above conditions are satisfied:

$$PDFL_t = 1 + A_{t-1}*(1-\psi_t)*\left(1+\frac{1}{\Delta\%S_t}\right) \qquad (21)$$

We can point out that whenever the Invested Capital percentage variation $\Delta\%CI_t$ matches the Revenue percentage variation $\Delta\%S_t$, ψ_t would be equal to 1 makes $PDFL_t$ too equal to 100% for any value of $\Delta\%S_t$.

The presence of DOL_t at the denominator of the DFL_t function suggests that the lower the former is, the higher is its impact on the latter, without being able to define the direction of the effect. *A priori,* we cannot assess whether the relationship between DOL_t and DFL_t be positive or negative: for sure, there is an influence of the former on the latter and not vice versa. The impact that DOL_t exerts on DFL_t depends on the circumstances. It seems sound not to consider both as exogenously specified, for the time being. We must investigate how DFL_t change in different financial periods and what impact DOL_t has on DFL_t in each of them.

A more realistic approach is to consider the determinants of DOL_t and DFL_t depending on the macroeconomic, market, and business conditions as exogenously specified or chosen optimally; consequently, we must evaluate the degrees of operating and financial leverage in light of the contingent situation.

5. The Relationship between Invested Capital and Revenue

In paragraph 4, we introduced one of the determinants of DFL, the function ψ_t, given by the ratio between the Invested Capital and the Revenue for the period *t* compared to the same measure for the previous period by using the equation (15):

$$\psi_t = \frac{CI_t}{S_t}*\frac{S_{t-1}}{CI_{t-1}} = \frac{1+\Delta\%CI_t}{1+\Delta\%S_t} \qquad (15)$$

From an ex-post perspective, the function ψ_t is an identity; consequently, it becomes a parameter. On the other hand, when we consider ψ_t from an ex-ante perspective, it needs further investigation that will be preparatory to develop the model presented in paragraph 6.

The Revenue percentage variation $\Delta\%S_t$ that appears in the denominator of the equation (15) has been treated already in detail by Paganini (2019), while the Invested Capital percentage variation $\Delta\%CI_t$ needs further analysis; we can consider it depending on:

1) the sensitivity of liquidity and Net Working Capital (NWC) to vary according to the Revenue absolute level: the higher the Revenue, the greater the need for cash and NWC is;

2) the Net Fixed Investment at the end of the period *t-1* and its depreciation and amortisation (D&A from now on) for the subsequent period; the Net Fixed Investment must be net of D&A;

3) the New Fixed Investment and its related D&A for the period *t*, necessary to maintain or strengthen the firm's operational capabilities and competitiveness, depending on the Revenue growth.

From these simple assessments, it follows that the Invested Capital at the end of period *t* will depend on the following function:

$$CI_t = S_t * (\lambda + \omega) + (FI_{t-1} - OA_t) + NFI_t = S_t * (\lambda + \omega) + (FI_{t-1} - OA_t) + \Delta S_t * \gamma * (1 - \theta) \quad (22)$$

where:

λ = sensitivity of liquidity to the Revenue absolute level S_t;

ω = sensitivity of NWC to the Revenue absolute level S_t;

FI_{t-1} = Net Fixed Investment at the end of period *t-1*;

OA_t = D&A of FI_{t-1} for the period *t*;

NFI_t = New Fixed Investment for the period *t*, net of its D&A;

ΔS_t = Revenue variation for period *t*;

γ = sensitivity of NFI_t to the Revenue variation ΔS_t, γ will be zero for negative ΔS_t.

θ = D&A rate of NFI_t.

Through some simple algebraic steps, we obtain ψ_t:

$$\psi_t = \frac{1 + \Delta\% CI_t}{1 + \Delta\% S_t} = \frac{S_{t-1}}{CI_{t-1}} * (\lambda + \omega) + \frac{1}{1 + \Delta\% S_t} * \left[\frac{S_{t-1}}{CI_{t-1}} * \gamma * (1 - \theta) * \Delta\% S_t + \frac{FI_{t-1} - OA_t}{CI_{t-1}} \right] =$$

$$= K1_{t-1} * K2_t + \frac{1}{1 + \Delta\% S_t} * \left[K1_{t-1} * K3_t * \Delta\% S_t + K4_t \right] \quad (15 \text{ bis})$$

where:

$$K1_{t-1} = \frac{S_{t-1}}{CI_{t-1}} \quad (23)$$

$$K2_t = \lambda + \omega \quad (24)$$

$$K3_t = \gamma * (1 - \theta) \quad (25)$$

$$K4_t = \frac{FI_{t-1} - OA_t}{CI_{t-1}} \quad (26)$$

In summary, ψ_t is a function that links the Invested Capital variation to the Revenue one, to some parameters $K2_t$ and $K3_t$ depending on the industry or company and contingent situations, not necessarily constant over time, and the initial conditions $K1_{t-1}$ and $K4_t$, since this last parameter depends on the D&A for the period *t* that we can calculate or predetermine, except in cases of divestment.

We can transfer the equation (15 bis) to the variable P_t that, in turn, influences Q_t and the degree of financial leverage DFL_t. The transmission mechanism starts from the absolute level of Revenue and its growth which generates an increase in Invested Capital through a greater need for liquidity and NWC, together with an increase in Fixed Investments, namely the Growth Assets, offset by higher D&A.

A firm can fund higher Invested Capital with more Debt on one side, which increases Financial Charges, and more Equity on the other side, in a combination that can modify or maintain untouched the Debt-to-Equity ratio t_{dt}. The final effect on DFL_t depends on so many variables that it requires a considerable ability to plan and forecast environmental and corporate events combined with an uncommon computing power capacity: I believe that many companies, particularly SMEs, consider DOL_t and DFL_t as exogenously specified.

6. Model Development

This paragraph will show the use of the DOL and DFL models in a Business Case planned over seven financial periods.

6.1 Model Build-Up

Before examining a Business Case, it worthwhile to create a model in which some parameters and variables can be left unchanged and then freely varied to evaluate how they impact DOL and DFL over a sufficiently long time horizon and above all within each financial period. It is interesting to wonder what would have happened to the Income Statement and Balance Sheet whether, other things being equal, the value of the Revenue quantity/mix $\Delta\%qms_t$ had been different in a specific financial period.

In particular, how would DOL and DFL vary within a specific financial period when $\Delta\%qms_t$ changes in its domain, from -100% to +∞? Since it is impossible to modify DOL directly, being a function of other variables, we can induce a variation on DOL through an indirect change of $\Delta\%qms_t$ and measure the impact produced on DFL. This exercise makes it possible to verify and measure the exact mathematical relationship between DOL and DFL; after having well understood such a relationship, it will be well-timed to examine the Business Case.

To obtain the analyses that will follow, we set a starting financial period 0; we can apply the percentage variations that determine DOL and DFL from periods 1 to 7 to the Income Statement and the Balance Sheet. We can see the data in Table 1.

Table 1. Parameters and Variables used in the simulation with floating debt. Data in green (or light grey) allow modification of the Income Statement and Balance Sheet figures

Data Set	Variable or Parameter	0	1	2	3	4	5	6	7	Note
Statements Income	Sales	€ 500,000,000	€ 525,000,000	€ 551,250,000	€ 578,812,500	€ 607,753,125	€ 638,140,781	€ 670,047,820	€ 703,550,211	Computed
	Varibale Costs	-€ 300,000,000	-€ 315,000,000	-€ 330,750,000	-€ 347,287,500	-€ 364,651,875	-€ 382,884,469	-€ 402,028,692	-€ 422,130,127	Computed
	Contribution Margin	**€ 200,000,000**	**€ 210,000,000**	**€ 220,500,000**	**€ 231,525,000**	**€ 243,101,250**	**€ 255,256,313**	**€ 268,019,128**	**€ 281,420,085**	Computed
	Fixed Costs	-€ 154,000,000	-€ 158,425,000	-€ 163,003,250	-€ 167,739,693	-€ 172,639,408	-€ 177,707,619	-€ 182,949,690	-€ 188,371,132	Computed
	Depreciation & Amortization	-€ 16,000,000	-€ 18,375,000	-€ 20,868,750	-€ 23,487,188	-€ 26,236,547	-€ 29,123,374	-€ 32,154,543	-€ 35,337,270	Computed
	Total Fixed Costs	-€ 170,000,000	-€ 176,800,000	-€ 183,872,000	-€ 191,226,880	-€ 198,875,955	-€ 206,830,993	-€ 215,104,233	-€ 223,708,402	Computed
	EBIT	**€ 30,000,000**	**€ 33,200,000**	**€ 36,628,000**	**€ 40,298,120**	**€ 44,225,295**	**€ 48,425,319**	**€ 52,914,895**	**€ 57,711,682**	Computed
	Financial Charges	-€ 1,780,000	-€ 1,987,614	-€ 2,172,594	-€ 2,333,766	-€ 2,469,897	-€ 2,579,690	-€ 2,661,781	-€ 2,714,738	Computed
	EBT	€ 28,220,000	€ 31,212,386	€ 34,455,406	€ 37,964,354	€ 41,755,398	€ 45,845,629	€ 50,253,114	€ 54,996,944	Computed
	Tax	**-€ 7,055,000**	**-€ 7,803,096**	**-€ 8,613,852**	**-€ 9,491,089**	**-€ 10,438,384**	**-€ 11,461,407**	**-€ 12,563,278**	**-€ 13,749,236**	Computed
	Net Profit	**€ 21,165,000**	**€ 23,409,290**	**€ 25,841,555**	**€ 28,473,266**	**€ 31,316,548**	**€ 34,384,222**	**€ 37,689,835**	**€ 41,247,708**	Computed
Balance Sheets	Cash	€ 20,000,000	€ 21,000,000	€ 22,050,000	€ 23,152,500	€ 24,310,125	€ 25,525,631	€ 26,801,913	€ 28,142,008	Computed
	Net Working Capital	€ 125,000,000	€ 131,250,000	€ 137,812,500	€ 144,703,125	€ 151,938,281	€ 159,535,195	€ 167,511,955	€ 175,887,553	Computed
	Assets	**€ 144,000,000**	**€ 149,375,000**	**€ 153,443,750**	**€ 156,140,938**	**€ 157,397,984**	**€ 157,142,884**	**€ 155,300,028**	**€ 151,790,029**	Computed
	Capital Invested	**€ 289,000,000**	**€ 301,625,000**	**€ 313,306,250**	**€ 323,996,563**	**€ 333,646,391**	**€ 342,203,710**	**€ 349,613,896**	**€ 355,819,590**	Computed
	Equity	-€ 200,000,000	-€ 202,244,289	-€ 204,676,555	-€ 207,308,120	-€ 210,151,548	-€ 213,219,222	-€ 216,524,835	-€ 220,082,708	Computed
	Debt	-€ 89,000,000	-€ 99,380,711	-€ 108,629,695	-€ 116,688,297	-€ 123,494,842	-€ 128,984,488	-€ 133,089,060	-€ 135,736,882	Computed
	Net Financial Position	-€ 69,000,000	-€ 78,380,711	-€ 86,579,695	-€ 93,535,797	-€ 99,184,717	-€ 103,458,857	-€ 106,287,148	-€ 107,594,874	Computed
Ratios	%vc	60.000%	60.000%	60.000%	60.000%	60.000%	60.000%	60.000%	60.000%	Ratio
	%cm	40.000%	40.000%	40.000%	40.000%	40.000%	40.000%	40.000%	40.000%	Ratio
	%f	34.000%	33.676%	33.355%	33.038%	32.723%	32.411%	32.103%	31.797%	Ratio
	%EBIT	6.000%	6.324%	6.645%	6.962%	7.277%	7.589%	7.897%	8.203%	Ratio
	%Π	4.233%	4.459%	4.688%	4.919%	5.153%	5.388%	5.625%	5.863%	Ratio
	Δ%S		5.000%	5.000%	5.000%	5.000%	5.000%	5.000%	5.000%	Ratio
	Δ%CM		5.000%	5.000%	5.000%	5.000%	5.000%	5.000%	5.000%	Ratio
	Δ%EBIT		10.667%	10.325%	10.020%	9.745%	9.497%	9.271%	9.065%	Ratio
	Δ%Π		10.604%	10.390%	10.184%	9.986%	9.796%	9.614%	9.440%	Ratio
	DOL		213.333%	206.506%	200.400%	194.906%	189.938%	185.423%	181.302%	Ratio
	DFL		99.410%	100.628%	101.637%	102.468%	103.146%	103.696%	104.134%	Ratio
	DTL		212.076%	207.803%	203.681%	199.716%	195.914%	192.275%	188.797%	Computed
DOL Determinants	Δ%up		0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	Datum
	Δ%qms		5.000%	5.000%	5.000%	5.000%	5.000%	5.000%	5.000%	Datum
	Δ%uvc		0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	Datum
	Δ%qmvc		5.000%	5.000%	5.000%	5.000%	5.000%	5.000%	5.000%	Datum
	ε		0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	Computed
	Δ%F		4.000%	4.000%	4.000%	4.000%	4.000%	4.000%	4.000%	Datum
	Δ%EBIT		10.667%	10.325%	10.020%	9.745%	9.497%	9.271%	9.065%	Computed
	Check		0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	Computed
	DOL		213.333%	206.506%	200.400%	194.906%	189.938%	185.423%	181.302%	Computed
	PDOL		666.667%	632.530%	601.998%	574.531%	549.688%	527.113%	506.510%	Computed
	DOL Abatement Factor		-68.000%	-67.352%	-66.711%	-66.076%	-65.446%	-64.823%	-64.206%	Computed
DFL Determinants	i	2.000%	2.000%	2.000%	2.000%	2.000%	2.000%	2.000%	2.000%	Datum
	φ	5.933%	5.987%	5.932%	5.791%	5.585%	5.327%	5.030%	4.704%	Computed
	ρ	94.067%	94.013%	94.068%	94.209%	94.415%	94.673%	94.970%	95.296%	Computed
	α rate	25.000%	25.000%	25.000%	25.000%	25.000%	25.000%	25.000%	25.000%	Datum
	ν	75.000%	75.000%	75.000%	75.000%	75.000%	75.000%	75.000%	75.000%	Computed
	t_d	44.500%	49.139%	53.074%	56.287%	58.765%	60.494%	61.466%	61.675%	Computed
	DFL*	30.796%	32.948%	34.672%	36.015%	37.014%	37.692%	38.067%	38.148%	Computed
	ψ		99.399%	98.926%	98.488%	98.075%	97.681%	97.300%	96.929%	Computed
Determinants	Δ%i		0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	Computed
	Δ%φ		0.901%	-0.923%	-2.365%	-3.565%	-4.614%	-5.572%	-6.487%	Computed
	Δ%ρ		-0.057%	0.059%	0.149%	0.219%	0.273%	0.314%	0.344%	Computed
	Δ%α		0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	Computed
	Δ%ν		0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	Computed
	Δ%t_d		10.425%	8.008%	6.055%	4.401%	2.943%	1.607%	0.341%	Computed
	Δ%D		11.664%	9.307%	7.418%	5.833%	4.445%	3.182%	1.990%	Computed
	Δ%E		1.122%	1.203%	1.286%	1.372%	1.460%	1.550%	1.643%	Computed
	Δ%CI		4.369%	3.873%	3.412%	2.978%	2.565%	2.165%	1.775%	Computed
	T		100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	Computed
	A	6.308%	6.368%	6.306%	6.147%	5.915%	5.627%	5.297%	4.936%	Computed
	N = T x (1+A)		106.308%	106.368%	106.306%	106.147%	105.915%	105.627%	105.297%	Computed
	M		100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	100.000%	Computed
	P		106.346%	104.102%	102.303%	100.793%	99.472%	98.269%	97.133%	Computed
	Q = A x M x P x T		6.708%	6.629%	6.451%	6.196%	5.884%	5.529%	5.145%	Computed
	DFL		99.410%	100.628%	101.637%	102.468%	103.146%	103.696%	104.134%	Computed

I carried out two simulations: the first one, the Debt-to-Equity ratio has been kept constant over all the seven financial periods, so that $\Delta\%td_t$ be always equal to zero, while in the second one, the Equity is steady, leaving the debt and $\Delta\%td_t$ free to swing over time.

I shall present only the second simulation as it is more helpful to investigate a Business Case. The two simulations assume the values displayed in Table 2 over seven financial periods to create a stable environment. I have to point out that the model can examine only eight periods (1+7), distinguishing between:

1) steady-state (set at constant growth) and

2) one-off shock.

I have excluded by the analyses, for the time being:

1) long-term of the steady-state scenario (beyond period 7) and

2) steady-state jumps or shocks that modify permanently the steady-state.

Such contexts deserve an in-depth analysis that unfortunately falls outside the paper's objectives. After presenting the model, we can focus on one-off shocks by changing the parameters from 1 to 7 and 11 of Table 2 over the horizon of seven financial periods.

Table 2. Values assumed by the variables in simulations 1 and 2

#	Variable	Description	Constant Debt Ratio	Floating Debt Ratio
1	$\Delta\%up_t$	Unit price percentage variation for the period t compared to the previous period	0.00%	0.00%
2	$\Delta\%uvc_t$	Unit variable cost percentage variation for the period t compared to the previous period	0.00%	0.00%
3	$\Delta\%qms_t$	Quantity/mix percentage variation of Revenue for the period t compared to the previous period	5.00%	5.00%
4	$\Delta\%qmvc_t$	Quantity/mix percentage variation of Variable Costs for the period t compared to the previous period	5.00%	5.00%
5	$\Delta\%F_t$	Fixed Costs percentage variation for the period t compared to the previous period	4.00%	4.00%
6	i_t	Cost of Debt for the period t	2.00%	2.00%
7	α_t	Corporate Income Tax Rate for the period t	25.00%	25.00%
8	t_{d_t}	Debt-to-Equity ratio of the period t as a ratio between Debt and Equity	44.50%	floating
9	λ	Sensitivity of liquidity to the Revenue level	4.00%	4.00%
10	ω	Sensitivity of Net Working Capital to the Revenue level	25,00%	25,00%
11	γ	Sensitivity of Net Fixed Investment to the Revenue variation (0 when $\Delta\%S_t < 0$)	95.00%	95.00%
12	θ	Depreciation and Amortisation rate of Fixed Investment	10.00%	10.00%
13	δ_t	Dividend payout ratio for the period t	100.00%	100.00%

Figure 1 is obtained by the model using the previous parameters, which synthetically represents the main variables in the seven periods covered by the analysis. It is worth noting that DOL is descending almost linearly, starting from about 213% in period 1 to end around 181% in period 7. Linearity is only illusory depending on the limited time horizon; the downward trend depends on PDOL, determined by the ratio between the Contribution Margin and EBIT: since EBIT as a percentage of Revenue grows to 8.2% from 6.0% while the Contribution Margin percentage remains constant at 40,0% over time, it follows that their ratio decreases according to the following equation:

$$PDOL_{t+n} = \frac{1}{1 - \frac{FC_{t-1}}{CM_{t-1}} * \left(\frac{1+\Delta\%F_t}{1+\Delta\%S_t}\right)^n} \tag{27}$$
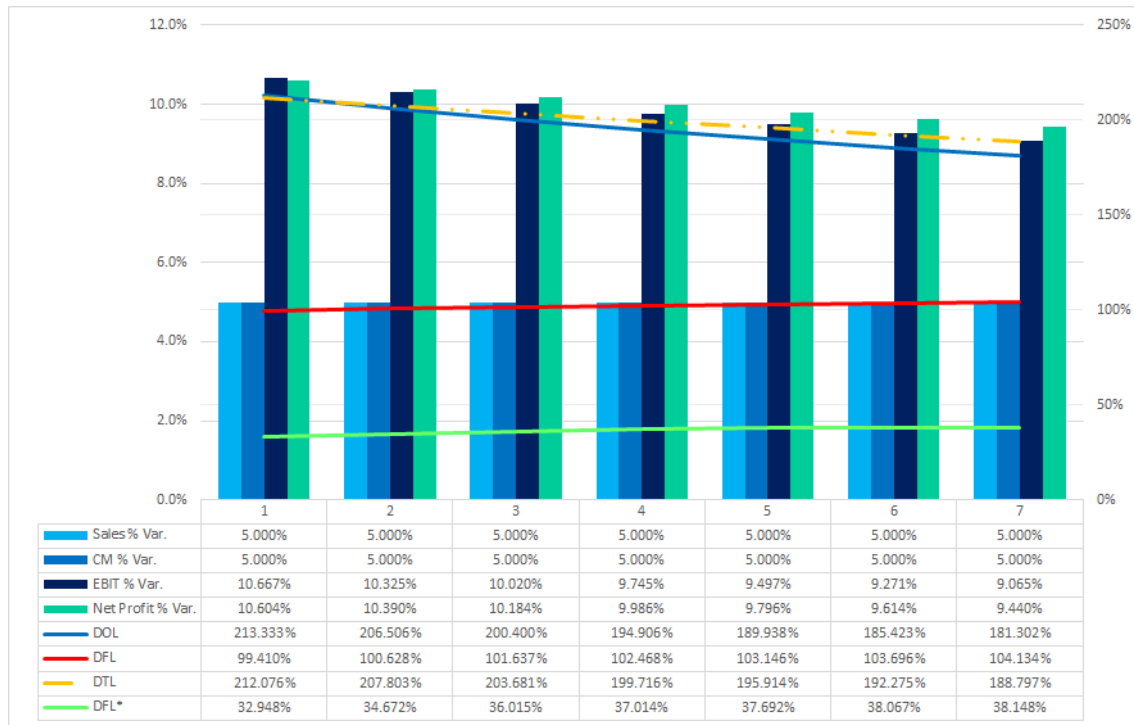
Figure 1. The trend of the parameters and variables examined: histogram scale is on the left while line scale is on the right

PDOL will tend to 100% in the long run (as n tends to infinity) provided that $CM_{t-1} > FC_{t-1}$ and $\Delta\%S_t > \Delta\%F_t$. Also, EBIT percentage variation $\Delta\%EBIT_t$, in the long run, tends to $\Delta\%S_t$ and the higher the difference between $\Delta\%S_t$ and $\Delta\%F_t$ the faster the process will be; in such a steady-state, also the ratio between EBIT and Revenue will tend to converge towards Contribution Margin and Revenue ratio. In steady-state, both DOL and DFL tend towards 100%. At the moment, it is sufficient to provide only some indications on the long run trend in steady-state as this is beyond the objective to examine a Business case. In the latter, we shall observe that the dynamic of the variables considered is so complex to move far away from the steady-state, becoming a simple curiosity without any practical use. DFL remains sufficiently constant and slightly increasing around 100% in seven periods. Apart from period 1, where DFL is lower than 100%, the Net Profit percentage variations will be greater than the EBIT percentage variations because DTL will assume values higher than DOL. Figure 1 shows that the debt ratio measured by DFL* is slightly increasing.

Table 3. Dissection of DOL and DFL in factors

	DOL & DFL Split		0	1	2	3	4	5	6	7
	$\Delta\%S$			5.000%	5.000%	5.000%	5.000%	5.000%	5.000%	5.000%
	PDOL	A		666.667%	632.530%	601.998%	574.531%	549.688%	527.113%	506.510%
DOL	S/CM	B	250.000%	250.000%	250.000%	250.000%	250.000%	250.000%	250.000%	
Split	-(VC*Δ%VC+F*Δ%F)/CM*Δ%S	C		-218.000%	-217.352%	-216.711%	-216.076%	-215.446%	-214.823%	-214.206%
	S/CM-(VC*Δ%VC+F*Δ%F)/CM*Δ%S	B+C=D		32.000%	32.648%	33.289%	33.924%	34.554%	35.177%	35.794%
	DOL	**AxD**		**213.333%**	**206.506%**	**200.400%**	**194.906%**	**189.938%**	**185.423%**	**181.302%**
	Δ%EBIT			10.667%	10.325%	10.020%	9.745%	9.497%	9.271%	9.065%
	1/Δ%EBIT			9.38	9.68	9.98	10.26	10.53	10.79	11.03
	N	A		106.308%	106.368%	106.306%	106.147%	105.915%	105.627%	105.297%
	N-1			6.308%	6.368%	6.306%	6.147%	5.915%	5.627%	5.297%
DFL	(N-1)/Δ%EBIT	B		59.134%	61.674%	62.929%	63.079%	62.285%	60.693%	58.430%
Split	-Q(1+Δ%S)			-7.043%	-6.961%	-6.773%	-6.506%	-6.178%	-5.806%	-5.402%
	-Q(1+Δ%S)/Δ%EBIT	C		-66.031%	-67.414%	-67.598%	-66.759%	-65.054%	-62.624%	-59.593%
	(N-1)/Δ%EBIT-Q(1+Δ%S)/Δ%EBIT	B+C		-6.897%	-5.740%	-4.668%	-3.679%	-2.769%	-1.931%	-1.162%
	DFL	**A+B+C**		**99.410%**	**100.628%**	**101.637%**	**102.468%**	**103.146%**	**103.696%**	**104.134%**

What is the dynamic determining such a stable trend of DOL and DFL? Table 3 shows the three arithmetic factors determining DOL and DFL; DOL dissects as follows:

A. PDOL which represents the Potential DOL of the current financial period, entirely determined by parameters of the previous financial period;

B. the inverse of the percentage Contribution Margin of the preceding financial period;

C. the ratio between Variable Costs and Fixed Costs variations on one side and the potential change of the Contribution Margin on the other one: such a ratio measures the evolution of profitability during the current financial period.

The difference between the previous point B. and point C. multiplied by point A. gives rise to the DOL of the current financial period. In the simulation performed, the inverse of the Contribution Margin is always equal to 250%, which means the inverse of 40%, the third term tends to increase slightly as the inverse of the ratio between Fixed Costs and Contribution Margin tends to grow up. DOL is decreasing for two reasons: the PDOL reduction already examined and the same trend of the ratio between Fixed Costs and Contribution Margin. In the long run of the steady-state, DOL tends to 100%, like PDOL.

DFL dynamic is more complicated but can be traced back to three factors:

A. the first one is N_t;

B. the second one is the variation induced by EBIT that modifies its allocation among Financial Charges, Taxes, and Net Profit owing to the fiscal impact of the current period;

C. the third one, centred on Q_t, is affected by the after-tax variations of Cost of Debt and financing mix.

The algebraic sum of the three terms mentioned above originates DFL: the first is slightly descending and slightly higher than 100%, the second and the third compensate algebraically each other, determining a continuously increasing negative difference which, added to the first term, generates an increasing DFL in a neighbourhood of 100%, that means the potential equilibrium value of DFL. In the absence of exogenous shocks, the DFL trend appears to be sufficiently stable.

6.2 Model Dynamic

Now comes the most important part: what happens if, in a particular financial period, the second period, for instance, $\Delta\%qms_t$ and $\Delta\%qmvc_t$, are set to vary synchronously from -100% to $+\infty$?

While the first kind of simulation involves the analysis in a steady-state with a floating debt ratio over multiple financial periods, the second kind of simulation is conducted within a single financial period by varying a parameter to simulate the changes that would undergo both Income Statement and Balance Sheet if the current financial period closed with different values of $\Delta\%qms_t$ and $\Delta\%qmvc_t$. The more $\Delta\%qms_t$ varies, the further the original operative point is moved away, the greater the variations induced on Income Statement and Balance Sheet will be, particularly on Debt: also, for this second kind of simulation, the Equity has been kept constant whilst the Debt is free to float. We use such a hypothesis to modify DOL and DFL indirectly, simulate their mutual relation, and identify the transmission mechanism of the $\Delta\%qms_t$ variations on DOL and DFL.

The model might generate a misrepresented response for $\Delta\%qms_t$ values at the borders of its domain. I do not think it is appropriate to address this problem since the final goal is to understand what would happen to DOL and DFL for infinitesimal variations of $\Delta\%qms_t$, compared to its current value: consequently, we can remove any concern about distortions.

Figure 2 shows the examined variables in the $\Delta\%qms_t$ domain from -100% to + 100%.

At the limits of the $\Delta\%qms_t$ domain, DOL and DFL asymptotically tend to predefined values even though different according to the positive or negative sign assumed by $\Delta\%qms_2$. DOL presents a discontinuity at Revenue percentage variation $\Delta\%S_t = 0$ from which we can obtain the discontinuity of $\Delta\%qms_t$ for each value supposed of $\Delta\%up_t$:

$$\Delta\%qms_t = -\frac{\Delta\%up_t}{1+\Delta\%up_t} \tag{28}$$

In the simulation under analysis, DOL presents a discontinuity at $\Delta\%qms_2 = 0\%$, having set the condition $\Delta\%up_2 = 0$: for values approaching zero from the left, DOL tends to $+\infty$ whilst from the right DOL tends to $-\infty$. DFL does not show any significant variation or discontinuity nearby DOL discontinuity, denying the existence of any negative relationship between the two variables exactly where $\Delta\%qms_2$ stresses DOL intensely. Figure 2 shows that also DFL reveals a discontinuity in correspondence with DOL=0 that, by definition, corresponds to the EBIT percentage variation $\Delta\%EBIT_t = 0$. It is necessary a deep-dive into DOL and DFL functions, for values of $\Delta\%qms_2$ inside the domain 0.00% and 4.00%, depicted in Figure 3, where DOL is monotonically increasing from negative to positive values, and its root corresponds to the following $\Delta\%qms_t$ solution:

$$\Delta\%qms_t = \frac{VC_{t-1}*[\Delta\%uvc_t*(1-\varepsilon_t)-\varepsilon_t]-S_{t-1}*\Delta\%up_t+ FC_{t-1}*\Delta\%F_t}{S_{t-1}*(1+\Delta\%up_t)-VC_{t-1}*(1+\Delta\%uvc_t)} \qquad (29)$$

where:
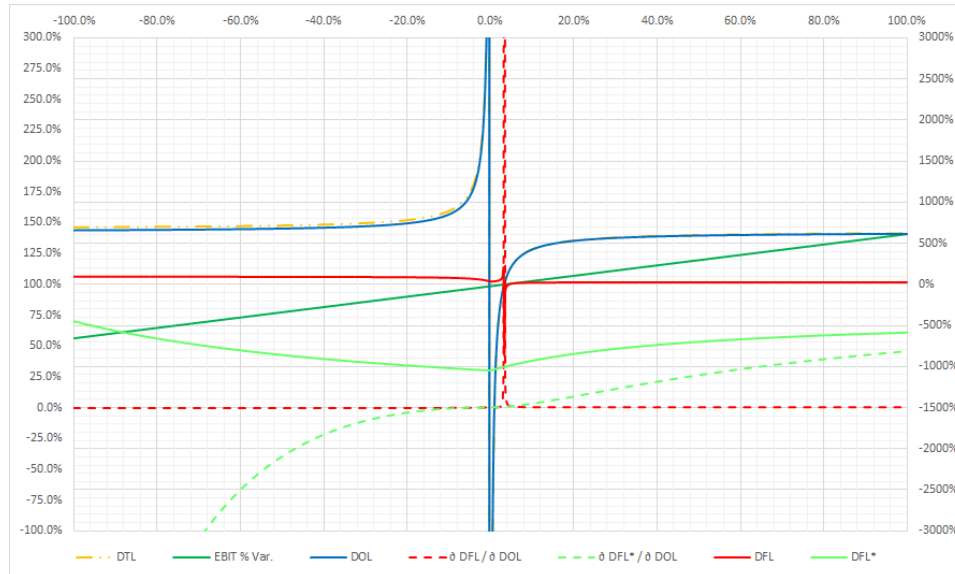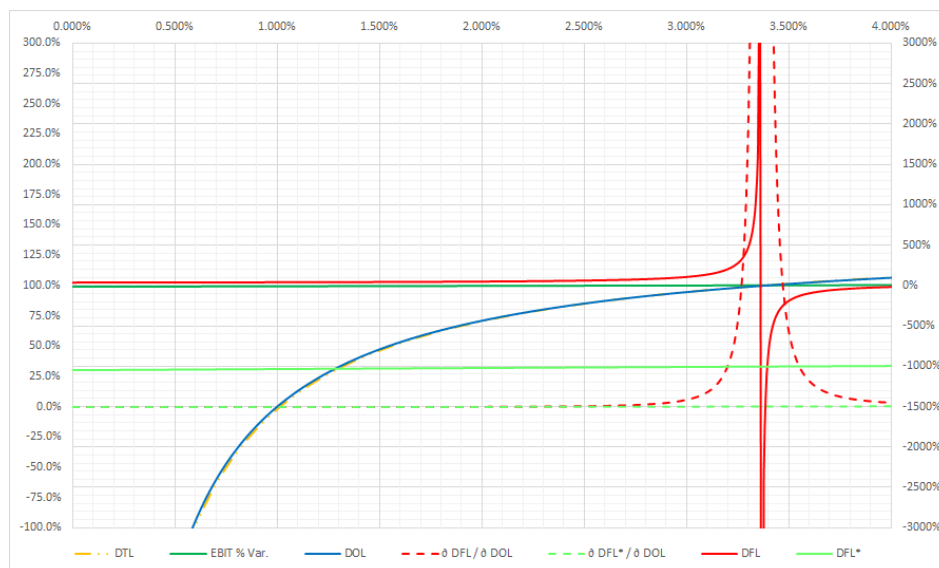
$\varepsilon_t= \Delta\%qms_t - \Delta\%qmvc_t$

Figure 2. The trend of the variables under analysis: DOL, DTL, and $\Delta\%EBIT_t$ scale on the right, all the other variables scale on the left ($\Delta\%qms_2$ domain between -100.00% and +100.00%)

We can simplify equation (29) in the present case in the following way:

$$\Delta\%qms_2 = \frac{FC_1}{CM_1} * \Delta\%F_2 = \frac{176,800,000}{210,000,000} * 4.00\% = 3.36762\% \qquad (29 \text{ bis})$$

On the left of the DOL root, DFL rises very quickly, tending to $+\infty$, whilst on its right, it tends to $-\infty$ recovering so rapidly to reach values greater than 100% for $\Delta\%qms_2$ equal to or higher than 4.431%.

Therefore DFL is substantially stable in the $\Delta\%qms_2$ domain from -100% to + 100% even though it shows a discontinuity at DOL root. Moreover, DFL shows an increasing trend in the neighbourhood of its discontinuity.
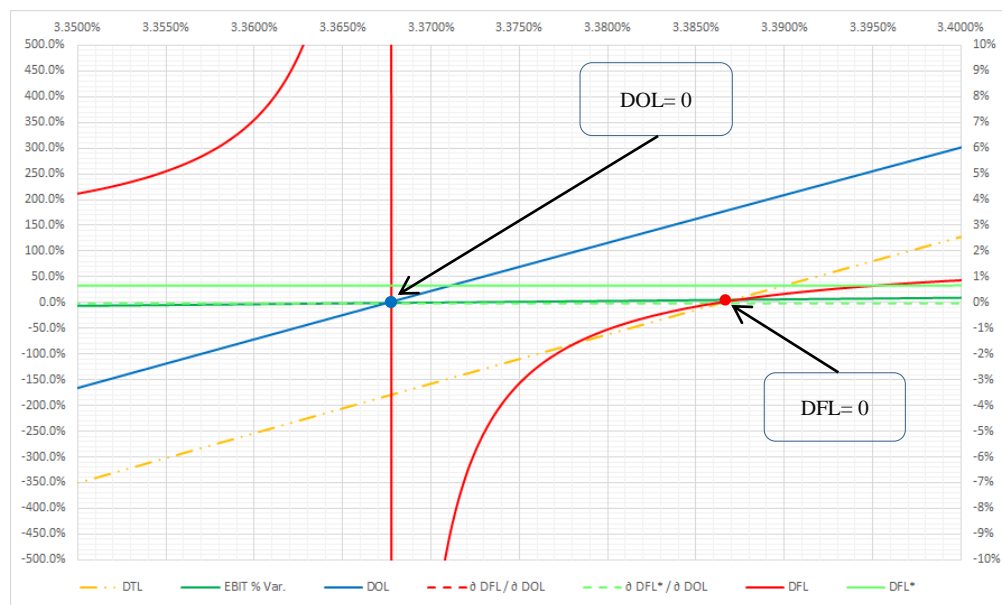

Figure 3. The trend of the variables under analysis: DOL, DTL, and $\Delta\%EBIT_t$ scale on the right, all the other variables scale on the left ($\Delta\%qms_2$ domain between 0.00% and 4.00%)

DFL has a root corresponding to the following Revenue percentage variation $\Delta\%S_t$:

$$\Delta\%S_t = \frac{1-\frac{Z_t}{D_{t-1}}*\left(E_{t-1}-\delta_{t-1}*\Pi_{t-1}\right)+\frac{Z_t}{DFL^*_{t-1}}*\left(K1_{t-1}*K2_t+K4_t\right)-\left(1-\frac{FC_{t-1}}{EBIT_{t-1}}*\Delta\%F_t\right)*[N_t+\frac{Z_t}{D_{t-1}}*EBIT_{t-1}*(1-\alpha_t)]}{PDOL_t*[N_t+\frac{Z_t}{D_{t-1}}*EBIT_{t-1}*(1-\alpha_t)]-\frac{Z_t}{DFL^*_{t-1}}*K1_{t-1}*(K2_t+K3_t)} \tag{30}$$

where:

$$Z_t = \frac{A_{t-1}*M_t*T_t}{1-i_t*(1-\alpha_t)} \tag{31}$$

from which we can obtain $\Delta\%qms_t$ using the following equation:

$$\Delta\%qms_t = \frac{\Delta\%S_t-\Delta\%up_t}{1+\Delta\%up_t} \tag{32}$$

In the specific case, the DFL root assumes the following $\Delta\%qms_t$ value:

$$\Delta\%qms_2 = 3.38650\% \tag{30 bis}$$

The functions examined so far do not show any negative impact of DOL on DFL, despite the presence of DOL in the denominator of two DFL factors, as already seen in equation (2 bis). All this because DFL is a variable dependent, mainly, but not exclusively, on $\Delta\%qms_t$; when this variable change, it exerts a series of changes on other variables, primarily on Q_t generating a chain reaction that offsets the impact of DOL on DFL. In particular, by examining Figure 3, we can point out that DOL increases monotonically, but this trend does not influence DFL at all, showing a rather well-marked constancy. Even in Figure 4, DFL shows a tendency "in phase" with DOL except at its discontinuity where DOL has its root.

In summary, it is more advisable to consider DOL and DFL as exogenously specified for the time being. It does not mean that there cannot be a corporate policy that uses DOL and DFL to keep their product constant over time or use one according to the other to have a Degree of Total Leverage DTL higher than or very close to 1. All that has nothing to do with the mathematical relationship existing between DOL and DFL; is the management prerogatives that makes it possible to obtain such a result.

Figure 4. The trend of the variables under analysis: DOL, DTL, and $\Delta\%EBIT_t$ scale on the right, all the other variables scale on the left ($\Delta\%qms_2$ domain between 3.350% and 3.400%)

Moving on, to examine DTL, which represents the ratio between the Net Profit and the Revenue percentage variations, we can glimpse that its discontinuity can be linked only to $\Delta\%S_t$ that generates a DOL discontinuity. DFL discontinuity in this way disappears, as can be seen from the following equation, remembering that $\Delta\%S_t$ in the present simulation is due uniquely to $\Delta\%qms_t$:

$$DTL_t = \frac{\Delta\%\Pi_t}{\Delta\%S_t} = N_t*\left(DOL_t+\frac{1}{\Delta\%S_t}\right)-Q_t*\left(1+\frac{1}{\Delta\%S_t}\right)-\frac{1}{\Delta\%S_t} \tag{33}$$

Also, DTL has two asymptotes at the boundaries of the $\Delta\%qms_t$ domain, different for negative and positive values, reflecting the same situation already identified for DOL and DFL. The roots of DFL and DTL coincide.

6.3 DOL Partial Derivative and Boundary Limit

To investigate the shape of the functions in their domain and codomain, we must compute the partial derivatives of DOL and DFL with respect to $\Delta\%qms_t$. From the examination of Figures 2 to 4, it is clear that the DOL slope is always positive while DFL still has an indeterminate incline even though it seems to be positively curved in the neighbourhood of the DOL discontinuity.

Let it start with the examination of the partial derivative of DOL:

$$\frac{\partial DOL_t}{\partial \Delta\%qms_t} = \frac{VC_{t-1}*(\Delta\%uvc_t - \Delta\%up_t) - VC_{t-1}*(1+\Delta\%uvc_t)*(1+\Delta\%up_t)*\varepsilon_t + FC_{t-1}*\Delta\%F_t*(1+\Delta\%up_t)}{EBIT_{t-1}(\Delta\%S_t)^2} \tag{34}$$

In the case under analysis, it is clear that such a partial derivative will always be greater than 0 since (34) can be simplified to the following:

$$\frac{\partial DOL_t}{\partial \Delta\%qms_t} = \frac{FC_{t-1}*\Delta\%F_t}{EBIT_{t-1}(\Delta\%S_t)^2} \tag{34 bis}$$
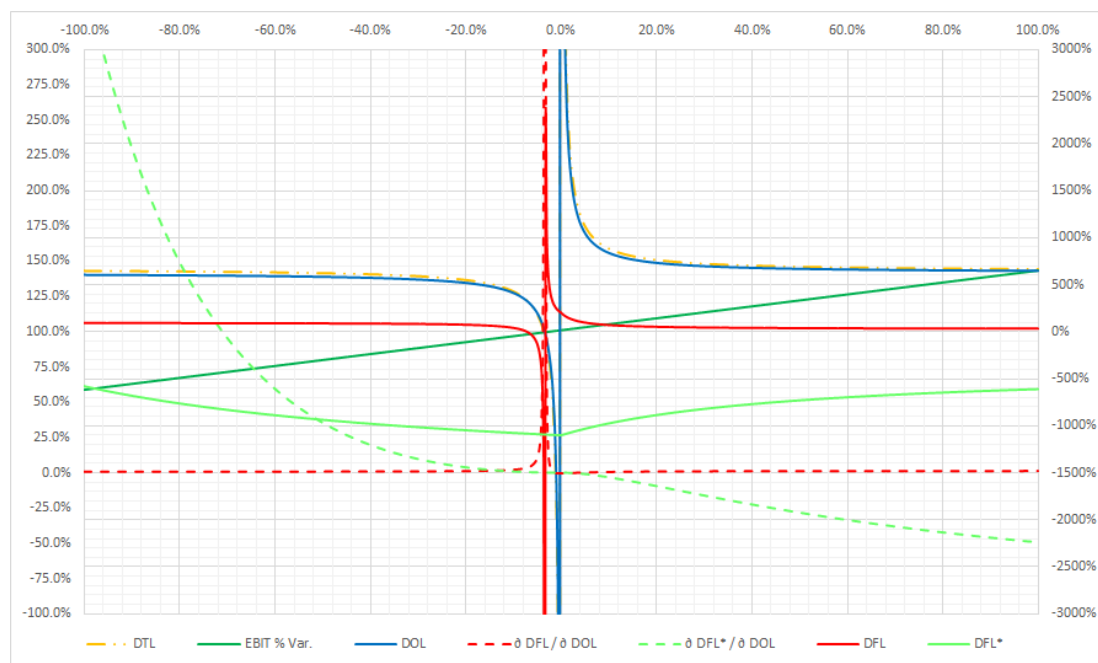
Figure 5. The trend of the variables examined with $\Delta\%F_2 = -4.00\%$: DOL, DTL, and $\Delta\%EBIT_t$ scale on the right, rest scale on the left. DOL and DFL curves appear "in phase"

To be positive, equation (34) must meet the following conditions:
1) $EBIT_{t-1} > 0$

2) $\Delta\%F_t > \frac{VC_{t-1}}{FC_{t-1}} * \left[(1 + \Delta\%uvc_t) * \varepsilon_t - \frac{(\Delta\%uvc_t - \Delta\%up_t)}{(1+\Delta\%up_t)}\right]$

Since condition 2. in the specific case eventuates in $\Delta\%F_t > 0$, both conditions proved for any value of the $\Delta\%qms_t$ domain; therefore, in this case, DOL will always have a positive slope. If condition 1. is verified, but condition 2. not then, DOL would appear with a negative slope, as in Figure 5, where instead of the parameter $\Delta\%F_2 = +4.00\%$, as in Figure 1, we have entered the value $\Delta\%F_2 = -4.00\%$.

Such a change in the DOL slope generates, in turn, a change in the DFL slope over the entire $\Delta\%qms_t$ domain, making it bent negatively. We have to point out that the starting DFL shape was negative for negative values of $\Delta\%qms_t$ and positive for the positive ones, while the DOL one is always positive. It can be said that DOL and DFL are "in phase" when both have the same slope, either positive or negative, and "out of phase" when the slopes are opposite. All this is in a relationship between DOL and DFL within a single financial period or infra-annual.

Consequently, starting from a situation in which DOL and DFL are "out of phase" for the negative domain of $\Delta\%qms_t$ and "in phase" for the positive one, by changing the sign to the parameter $\Delta\%F_t$ we reach a situation in which both DOL and DFL curves are "in phase" and negatively bent on the whole $\Delta\%qms_t$ domain.

6.4 DFL Partial Derivatives

The partial derivative of DFL with respect to $\Delta\%qms_t$ is the following:

$$\frac{\partial DFL_t}{\partial\Delta\%qms_t} = \frac{[1-N_t+Q_t*(1+\Delta\%S_t)]}{(DOL_t*\Delta\%S_t)^2} * \frac{\partial\Delta\%EBIT_t}{\partial\Delta\%qms_t} +$$

$$-\frac{A_{t-1}*M_t*T_t*(1+\Delta\%up_t)}{DOL_t*\Delta\%S_t*DFL^*_{t-1}} * [DFL^*_t * K1_{t-1} * (K2_t + K3_t) + \psi_t * (1 + \Delta\%qms_t) * \frac{\partial DFL^*_t}{\partial\Delta\%qms_t}] \qquad (35)$$

For the definition of:

$$\frac{\partial\Delta\%EBIT_t}{\partial\Delta\%qms_t} \qquad (36)$$

and

$$\frac{\partial DFL^*_t}{\partial\Delta\%qms_t} \qquad (37)$$

please refer to Appendix A. At the operative point $\Delta\%qms_t = 5.00\%$, DOL and DFL appear "in phase" and monotonically increasing, assuming the following values:

$$\frac{\partial DOL_t}{\partial\Delta\%qms_t} = 8520.4819\% \qquad (38)$$

$$\frac{\partial DFL_t}{\partial\Delta\%qms_t} = 72.1205\% \qquad (39)$$

At this point, it is evident that we can obtain the partial derivative of DFL with respect to DOL in an analytical way or, exploiting the inverse function theorem, by the ratio between (39) and (38), reaching the following result in correspondence of $\Delta\%qms_t = 5.00\%$:

$$\frac{\partial DFL_t}{\partial\Delta\%qms_t} / \frac{\partial DOL_t}{\partial\Delta\%qms_t} = \frac{\partial DFL_t}{\partial DOL_t} = 0.8464\% \qquad (40)$$

The analytical expression of $\partial DFL_t / \partial DOL_t$ is the following:

$$\frac{\partial DFL_t}{\partial DOL_t} = \frac{[1-N_t+Q_t*(1+\Delta\%S_t)]*\frac{\partial\Delta\%EBIT_t}{\partial DOL_t}}{(DOL_t*\Delta\%S_t)^2} - \frac{(1+\Delta\%S_t)*\frac{\partial Q_t}{\partial DOL_t}+Q_t*\frac{\partial(1+\Delta\%S_t)}{\partial DOL_t}}{DOL_t*\Delta\%S_t} \qquad (41)$$

where:

$$\frac{\partial Q_t}{\partial DOL_t} = \frac{A_{t-1}*M_t*T_t}{DFL^*_{t-1}} * (DFL^*_t * \frac{\partial\psi_t}{\partial DOL_t} + \psi_t * \frac{\partial DFL^*_t}{\partial DOL_t}) \qquad (42)$$

For more details, please refer to Appendix A.

From the equation (40), we can infer that this result would appear to be independent by having it achieved from two derivatives calculated with respect to $\Delta\%qms_t$: the same solution would have been obtained by deriving DOL and DFL with respect to other determinants of DOL although the checks are still in progress for the time being. It seems useless to search partial derivatives for DOL and DFL with respect to other variables such as $\Delta\%F_t$ or $\Delta\%up_t$. Also, equation (41) assumes the same value as (40) at $\Delta\%qms_t = 5.00\%$, confirming that DOL and DFL are "in phase" at that specific operative point.

Figure 6 (a) shows trends of the two functions around the operative point $\Delta\%qms_t = 5.00\%$. In Figure 6 (b), we report DOL, DFL, and $\partial DFL_t / \partial DOL_t$ in the $\Delta\%qms_t$ domain between -4.00% and + 8.00%. The partial derivative $\partial DFL_t / \partial DOL_t$ assumes negative values to the left of the ordinate axis and positive values to its right: this definitively confirms that DOL and DFL slopes are "in phase" and positive for $\Delta\%qms_t > 0$ and "out of phase" with positive DOL and negative DFL for $\Delta\%qms_t < 0$.
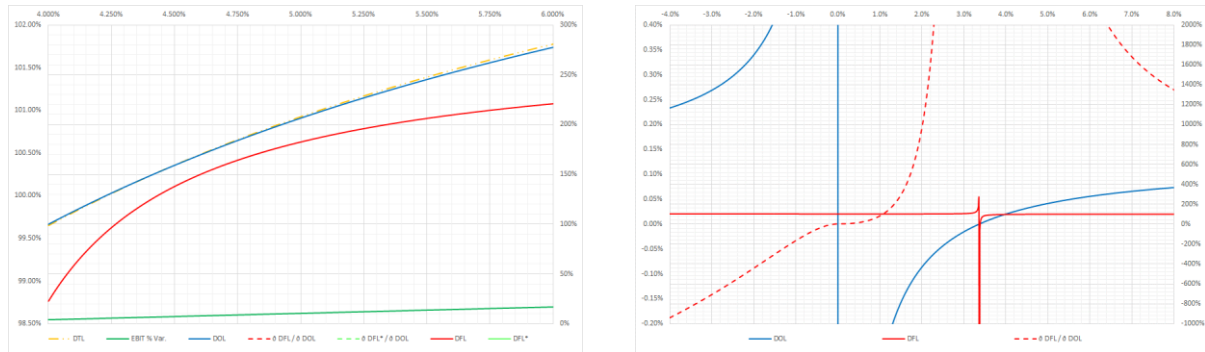
Figure 6(a) and (b). (a) on the left, the trend of the examined variables: DOL, DTL, and $\Delta\%EBIT_t$ scale on the right, rest scale on the left. (b) on the right, the trend of the examined variables: DOL and DFL scale on the right, rest scale on the left

What are the consequences of the $\partial DFL_t/\partial DOL_t$ relationship "in phase" or "out of phase"? For the moment, they do not seem significant: the value of $\partial DOL_t/\partial\Delta\%qms_t$ is far greater than the value of $\partial DFL_t/\partial\Delta\%qms_t$; consequently, a tiny variation of $\Delta\%qms_t$ does not have sizable consequences on DTL value even with a negative DFL slope and positive DOL slope, at least in ordinary conditions. In general, it would be preferable to operate where both $\partial DFL_t/\partial DOL_t$ and $\partial DOL_t/\partial\Delta\%qms_t$ are positive, but this appears relevant only in those circumstances where DOL is in a neighbourhood of its root which coincides with both $\Delta\%EBIT_t = 0$ and the DFL discontinuity, a condition generating potentially unpredictable situations even though DTL seems affected only by DOL discontinuity.

6.5 DFL Boundary Limits

We should remember that:

- DOL_t, Q_t, and $\Delta\%S_t$ are functions that depend on the variable $\Delta\%qms_t$,
- other variables are parameters as they do not vary with respect to $\Delta\%qms_t$ even though they can assume different values,
- still, others are just parameters as they refer to conditions of the period *t-1*.

In the present model under analysis, we have assumed that some variables are equal to 0, such as $\Delta\%up_t$ and $\Delta\%uvc_t$, while others are equal to 1, such as M_t and T_t.

What shape would DFL assume when these parameters change?

Changing the values of $\Delta\%up_t$ and $\Delta\%uvc_t$ would have a double effect of changing both DOL and DFL while changing the parameters that affect M_t and T_t, i.e. changing the Cost of Debt i_t and the Tax Rate α_t respectively compared to the previous period, the effect would be to change only DFL leaving DOL unaltered.

Are M_t and T_t able to significantly influence the form of DFL? What values must M_t and T_t assume to bend DFL positively? And in what $\Delta\%qms_t$ domain? The whole one or just a part of it?

The following condition μ_t for M_t must be satisfied to bend DFL positively at the operative point $\Delta\%qms_t = 5.00\%$:

$$\mu_t = \frac{(1-N_t)*\frac{\partial\Delta\%EBIT_t}{\partial\Delta\%qms_t}}{A_{t-1}*T_t*(1+\Delta\%up_t)*\frac{DFL_t^*}{DFL_{t-1}^*}*\Delta\%EBIT_t*[K1_{t-1}*(K2_t+K3_t)+\frac{\psi_t*(1+\Delta\%qms_t)}{DFL_t^*}*\frac{\partial DFL_t^*}{\partial\Delta\%qms_t}-\frac{\psi_t*(1+\Delta\%qms_t)}{\Delta\%EBIT_t}*\frac{\partial\Delta\%EBIT_t}{\partial\Delta\%qms_t}]} \tag{44}$$

from which we obtain that for $\partial DFLt/\partial\Delta\%qms_t$ to be greater than 0, it is necessary that:

$$i_t > \mu_t * i_{t-1} \tag{45}$$

having verified that the denominator of equation (44) be negative.

For T_t, we need to verify the condition τ_t:

$$\tau_t = \frac{\frac{\partial\Delta\%EBIT_t}{\partial\Delta\%qms_t}}{(1+A_{t-1})*\frac{\partial\Delta\%EBIT_t}{\partial\Delta\%qms_t}+A_{t-1}*M_t*(1+\Delta\%up_t)*\frac{DFL_t^*}{DFL_{t-1}^*}*\Delta\%EBIT_t*[K1_{t-1}*(K2_t+K3_t)+\frac{\psi_t*(1+\Delta\%qms_t)}{DFL_t^*}*\frac{\partial DFL_t^*}{\partial\Delta\%qms_t}-\frac{\psi_t*(1+\Delta\%qms_t)}{\Delta\%EBIT_t}*\frac{\partial\Delta\%EBIT_t}{\partial\Delta\%qms_t}]} \tag{46}$$

from which we obtain that for $\partial DFLt/\partial\Delta\%qms_t$ to be greater than 0, it is necessary that:

$$\alpha_t > 1 - \tau_t * (1- \alpha_{t-1}) \tag{47}$$

having verified that the denominator of equation (46) be not negative.

DFL will appear bent positively, at the operative point $\Delta\%qms_t = 5.00\%$, for values of i_t or α_t higher than the boundary limits defined respectively in equations (45) and (47). From Table 4, we can verify that i_t and α_t are already exceeding their lower limits in the financial period 2, set respectively at 1.9625% and 24.9087 %; in fact, the slope of DFL is positive in the operative point $\Delta\%qms_t = 5.00\%$ and equal to 72.1205%, as already seen from equation (39).

In Figure 7(b), we can see that DOL always has a positive slope, while a tiny reduction of the Cost of Debt i_t to 1,960% from 2,000% can bend DFL negatively for each value of the domain $\Delta\%qms_t > 0$ and making it completely "out of phase" with respect to DOL for the entire $\Delta\%qms_t$ domain and this is also confirmed by the negative trend of $\partial DFL_t/\partial DOL_t$ [dashed curve in Figure 7 (a) and (b)].
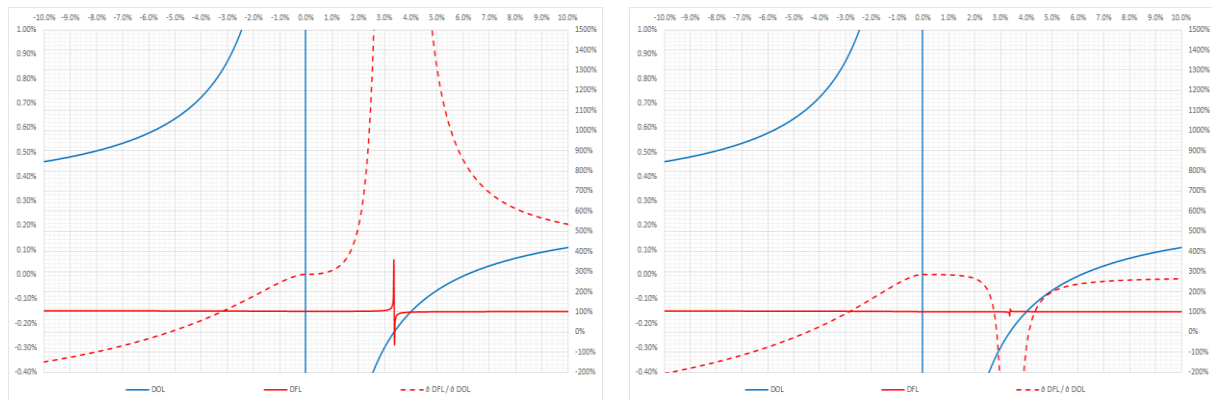
Figure 7(a) and (b). DOL and DFL scale on the right, rest scale on the left; (a) on the left, trend of the variables as in Figure 2; DOL and DFL curves always appear "out of phase" for negative values of $\Delta\%qms_t$ and "in phase" for positive values; (b) trend of the variables examined with $i_t = 1.960\%$: DFL and DOL are always "out of phase" in the entire domain of $\Delta\%qms_t$

In Figure 8(b), we can see that DOL always has a positive slope, while a tiny reduction in the Tax Rate α_t from 25.000% to 24.908% can bend DFL negatively for each value of the domain $\Delta\%qms_t > 0$ and making it completely "out of phase" with respect to DOL for the entire $\Delta\%qms_t$ domain and this is also confirmed by the negative trend of $\partial DFL_t/\partial DOL_t$ [dashed curve in Figure 8 (a) and (b)].
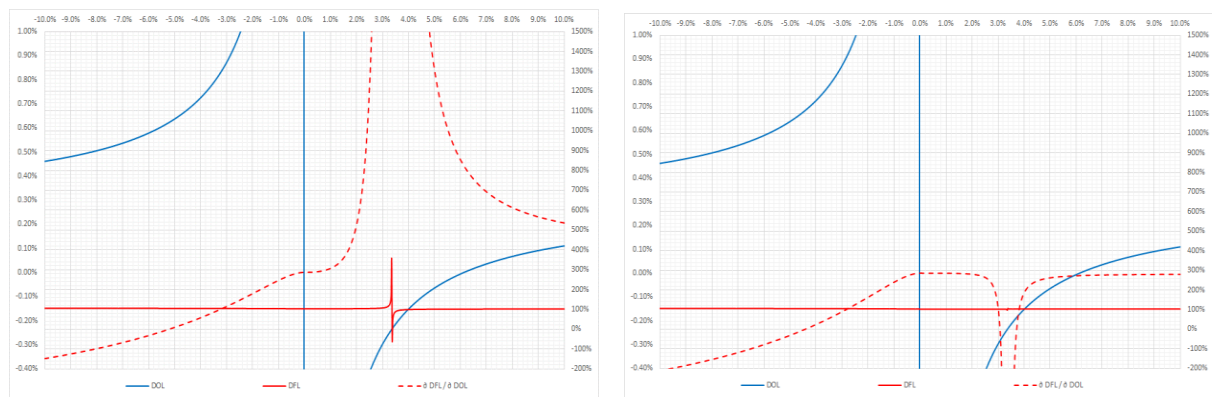

Figure 8(a) and (b). DOL and DFL scale on the right, rest scale on the left; (a) DOL and DFL curves always appear "out of phase" for negative values of $\Delta\%qms_t$ and "in phase" for the positive ones; (b) trend of the variables examined with $\alpha_t = 24.908\%$: DFL and DOL are always "out of phase" in the entire $\Delta\%qms_t$ domain

Now i_t or α_t could be modified more deeply, one variable at a time, to verify the effect on DFL: let us first bring i_t from 2.00 % to 2.50%, we measure the impact on DFL, take it back to 2.000% and then modify α_t from 25.00% to 30.00%. We report the effects in Figure 9 (a) and (b): now DOL and DFL appear "in phase" over the whole $\Delta\%qms_t$ domain.
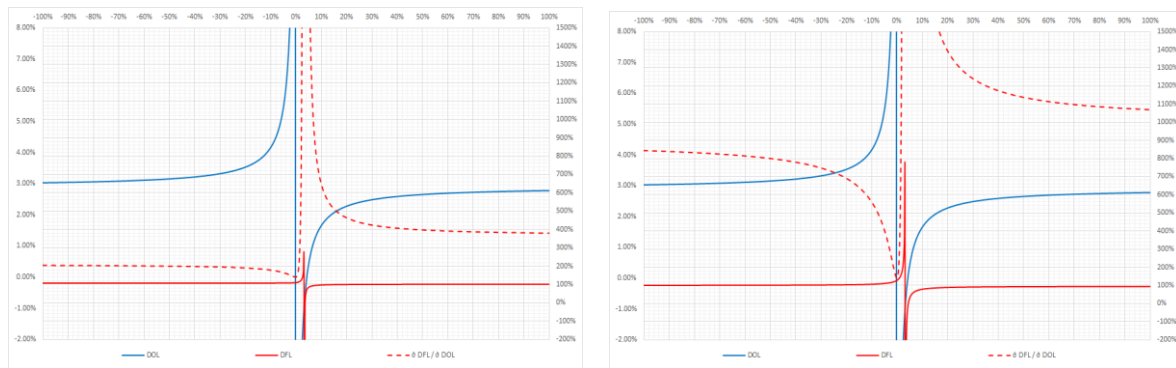
Figure 9(a) and (b). DOL and DFL scale on the right, rest scale on the left; (a) DOL and DFL curves always appear "in phase" for any value of $\Delta\%qms_t$ with $i_t = 2.500\%$; (b) DFL and DOL are "in phase" in the entire spectrum of $\Delta\%qms_t$ with $\alpha_t = 30.00\%$

Consequently, some values allow to modify a relationship between DOL and DFL "in phase" or "out of phase" on the whole $\Delta\%qms_t$ domain.

The consequences are manifold.

First of all, we are not dealing with an arithmetic relationship between DOL and DFL at a specific operative point as in a real Business Case; it is a rather complicated relationship between functions of which we have currently examined a particular case generated by the $\Delta\%qms_t$ variation over its entire domain under the same conditions of other variables and initial parameters at *t-1*.

Many variables and parameters can impact the relationship between DOL and DFL:

a.   some variables and or parameters directly influence DOL and through it indirectly influence DFL;

b.   other variables and or parameters directly affect only DFL;

c.   other variables and or parameters directly affect both DOL and DFL.

Given the complexity of the variables and parameters involved, it seems completely unrealistic to consider only the existence of a negative or "out of phase" relationship between DOL and DFL. The macroeconomic, market, and business conditions together with the management decisions could create "in phase" or "out of phase" relationships between DOL and DFL; the analysis of the DOL and DFL curves is required in every single financial period to discover such a relationship by computing the partial derivatives of DOL and DFL, starting from the complete availability of the business parameters originated in the past financial period and or planned for future periods. Such an assessment is only possible from an internal company perspective and referrable to every single financial period. Comparing different periods does not make any sense as DOL and DFL curves could change shape and concavity between them.

We cannot perform econometric analyses of DOL and DFL starting from the Financial Statement data, lacking sufficient data to trace the entire DOL and DFL curves: we cannot infer curves from a single point; the approach must be as follows:

•   it starts from the shape of DOL and DFL curves,

•   the operative point on the curves is determined,

•   the relationship between DOL and DFL established within the financial period is examined,

•   the task is repeated for the next financial period since the curves may have changed,

•   the operative point will be almost for sure different and

•   perhaps, the slopes of DOL and DFL could have changed from "in phase" to "out of phase" or vice versa.

6.6 Topological Perspective

In light of the above considerations, it is necessary to examine a very particular aspect of the relationships so far presented. It is clear from Figures 5, 7 (b), and 8 (b) that DOL and DFL flip around their points of discontinuity: these might be considered poles of the two functions. We need to investigate the behaviour of DOL and DFL roots and discontinuities since the DOL root corresponds to the DFL discontinuity.

From Table 4, we can point out that the DOL slope is always positive in any financial period from 1 to 7, even though $\partial DOL_t/\partial \Delta\%qms_t$ decreases continuously. The DFL slope has a dissimilar behaviour being positive in the first three periods and becoming negative in the subsequent four ones: accordingly, the DOL-DFL relationship turns from "in phase" into "out of phase", and that is confirmed by the partial derivative $\partial DFL_t/\partial DOL_t$, calculated independently both with the analytical method and with the inverse function theorem. Why does such a reversal of DFL occur in a steady-state environment, and what are the reasons?

A possible qualitative answer can come from the topological examination of the DOL and DFL functions briefly depicted here:

1) DOL discontinuity is always equal to zero, having set $\Delta\%up_t = 0$.

2) DOL roots decrease progressively, and this depends on the difference between $\Delta\%F_t$ and $\Delta\%qms_t$: if both were equal to 5.00%, the root would settle at 4.25 %; with 4.00%, it would settle lower at 3.40%, provided that $\Delta\%up_t = \Delta\%uvc_t = 0$.

3) DFL discontinuity depends on the previous point.

4) Also, DFL roots decrease progressively, albeit more slowly if $\Delta\%F_t = \Delta\%qms_t$ and the faster, the smaller their values are: with 4.00%, the flip occurs in the third financial period while with 3.00%, DFL and DOL are always "out of phase" for seven periods and with 5.00% they are "in phase" in all the seven periods.

5) If the DOL root comes first of DFL one, the curves appear "in phase" as in the first three periods; in the fourth, the situation is inverted, and DFL root precedes DOL one: this causes the phase inversion; we can find this situation also in the subsequent financial periods.

6) Boundary conditions μ_t and τ_t fulfil only in the first three financial periods, not in the subsequent four ones. That can explain the DFL flip from positive to negative. Conditions μ_t and τ_t indicate that $i_t = 2.00\%$ and $\alpha_t = 25.00\%$ are no longer sufficient to bend DFL positively. Both μ_t and τ_t show an increasing trend over the seven financial periods under analysis.

Table 4. Detailed analysis of DOL and DFL in the financial periods 1 to 7

Index	Note	1	2	3	4	5	6	7	Note
Δ%qms		5.0000%	5.0000%	5.0000%	5.0000%	5.0000%	5.0000%	5.0000%	Datum
DOL Discontinuity		0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	Computed
DOL = 0 (DFL Discontinuity)		3.40000%	3.36762%	3.33555%	3.30378%	3.27231%	3.24115%	3.21028%	Computed
DFL = 0 t_t floating		3.43675%	3.38650%	3.33732%	3.28919%	3.24207%	3.19593%	3.15072%	Computed
DFL Disc.ty > DOL Disc.ty		Yes	Yes	Yes	Yes	Yes	Yes	Yes	Computed
DFL = 0 > DOL = 0		Yes	Yes	Yes	No	No	No	No	Computed
DFL = 0 - DOL = 0		0.0368%	0.0189%	0.0018%	-0.0146%	-0.0302%	-0.0452%	-0.0596%	Computed
DOL Status		Positive	Positive	Positive	Positive	Positive	Positive	Positive	Computed
∂DOL/∂Δ%qms		9066.6667%	8520.4819%	8031.9755%	7592.4884%	7195.0120%	6833.8133%	6504.1568%	Computed
Δ%F lower boundary		0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	Computed
DFL Status		Positive	Positive	Positive	Negative	Negative	Negative	Negative	Computed
∂DFL/∂Δ%qms		146.0802%	72.1205%	6.5225%	-51.4996%	-102.6999%	-147.7852%	-187.4072%	Computed
Phase DOL/DFL Status		In Phase	In Phase	In Phase	Out of Phase	Out of Phase	Out of Phase	Out of Phase	Computed
M boundary		96.1977%	98.1269%	99.8278%	101.4043%	102.9322%	104.4744%	106.0899%	Computed
Denominator		-43.7126%	-41.0486%	-38.0246%	-34.8288%	-31.5887%	-28.3899%	-25.2885%	Computed
i boundary		1.9240%	1.9625%	1.9966%	2.0281%	2.0586%	2.0895%	2.1218%	Computed
Boundary direction		Lower	Lower	Lower	Lower	Lower	Lower	Lower	Computed
T boundary		100.2499%	100.1217%	100.0109%	99.9149%	99.8318%	99.7596%	99.6969%	Computed
Denominator		665.0046%	631.7612%	601.9330%	575.0196%	550.6145%	528.3836%	508.0498%	Computed
α boundary		24.8126%	24.9087%	24.9918%	25.0638%	25.1262%	25.1803%	25.2273%	Computed
Boundary direction		Lower	Lower	Lower	Lower	Lower	Lower	Lower	Computed
∂DFL/∂DOL		1.6112%	0.8464%	0.0812%	-0.6783%	-1.4274%	-2.1626%	-2.8813%	Computed
∂DFL/∂DOL as Inverse Function		1.6112%	0.8464%	0.0812%	-0.6783%	-1.4274%	-2.1626%	-2.8813%	Computed
DFL = 0 to identify <>0	Support	3.43675%	3.38650%	3.33732%	3.28919%	3.24207%	3.19593%	3.15072%	Computed
∂Δ%EBIT/∂Δ%qms	Support	666.6667%	632.5301%	601.9985%	574.5305%	549.6883%	527.1133%	506.5098%	Computed
∂Δ%S/∂DOL	Support	1.1029%	1.1736%	1.2450%	1.3171%	1.3899%	1.4633%	1.5375%	Computed
∂Q/∂DOL	Support	0.2489%	0.2485%	0.2490%	0.2504%	0.2527%	0.2558%	0.2596%	Computed
∂DFL*/∂Δ%qms	Support	79.6699%	77.2274%	75.7965%	75.3013%	75.7092%	77.0258%	79.2935%	Computed
∂DFL*/∂DOL	Support	0.8787%	0.9064%	0.9437%	0.9918%	1.0522%	1.1271%	1.2191%	Computed
∂EBIT/∂DOL	Support	€ 2,205,882	€ 2,464,649	€ 2,745,277	€ 3,049,396	€ 3,378,747	€ 3,735,196	€ 4,120,736	Computed
A	Support	€ 202,244,289	€ 204,676,555	€ 207,308,266	€ 210,151,548	€ 213,219,222	€ 216,524,835	€ 220,082,708	Computed
B	Support	€ 301,625,000	€ 313,306,250	€ 323,996,563	€ 333,646,391	€ 342,203,710	€ 349,613,896	€ 355,819,590	Computed
C	Support	€ 200,000,000	€ 210,000,000	€ 220,500,000	€ 231,525,000	€ 243,101,250	€ 255,256,313	€ 268,019,128	Computed
D	Support	€ 178,835,000	€ 178,835,000	€ 178,835,000	€ 178,835,000	€ 178,835,000	€ 178,835,000	€ 178,835,000	Computed
A_q'	Δ%qms	€ 143,565,990	€ 150,744,289	€ 158,281,504	€ 166,195,579	€ 174,505,358	€ 183,230,626	€ 192,392,157	Computed
B_q'	Δ%qms	€ 572,500,000	€ 601,125,000	€ 631,181,250	€ 662,740,313	€ 695,877,328	€ 730,671,195	€ 767,204,754	Computed
A_d'	DOL	€ 1,583,448	€ 1,769,199	€ 1,970,642	€ 2,188,947	€ 2,425,366	€ 2,681,235	€ 2,957,988	Computed
B_d'	DOL	€ 6,314,338	€ 7,055,059	€ 7,858,356	€ 8,728,895	€ 9,671,663	€ 10,691,998	€ 11,795,607	Computed
Z	Support	6.40364%	6.46501%	6.40155%	6.24087%	6.00523%	5.71259%	5.37741%	Computed

When DFL root topologically precedes the DOL one, the former must necessarily bend negatively. Now the DFL root moves to the left faster than the DOL one, and such a shift might depend on several factors:

1) the downward trend of A_{t-1}, a parameter that measures the distribution of EBIT between Financial Charges and EBT: its decline reveals that EBT share grows over time and squeezes the one allocated to Financial Charges;

2) the downward trend of the investment sensitivity ψ_t caused by the reduction of $K4_t$ not sufficiently offset by the increase of the product of $K1_{t-1}$ multiplied by $K2_t$;

3) from the debt swing;

4) the failure to fulfil conditions μ_t and τ_t, even though i_t and α_t are constant over the seven financial periods under analysis.

Debt swing provides a further boost to the reduction of DFL root, but this is not the critical factor since such a reduction also occurs in the simulation with constant debt where the shift of the DFL root to the left slows down even though the contribution of A_{t-1} is more sizable owing to the lesser use of debt that shrinks the Financial Charges and increases the EBIT share allocated to EBT. Conditions μ_t and τ_t do not determine the inversion of DFL slope, they only state what minimum values i_t and α_t should assume to bend DFL positively, and they are only thresholds even though they grow over time.

It is possible to stop the shift of all roots to the left under the following conditions:

1) $\Delta\%up_t = \Delta\%uvc_t = 0$;

2) $\Delta\%qms_t = \Delta\%qmvc_t = \Delta\%F_t$;

3) When the sensitivity of the Growth Assets is equal to $\gamma = 100\%$;

4) In the absence of depreciation with $\theta = 0\%$, an unrealistic situation.

The first two conditions translate into DOL = 100% while DFL is slightly increasing with values below 100%, higher with constant than floating debt. Furthermore, in the latter case, the shift does not stop immediately in period 1, being delayed to period 3.

Given such a premise, it is possible to conjecture that the natural relationship between DOL and DFL in the long run of the steady-state be negative even though we should verify this statement with the long-term analysis of the DOL-DFL nexus.

So far, the results obtained depend on the variable ψ_t, which allows us to automatically vary the Assets by linking them to the Revenue and its variation over time. In this way, we can simulate the impact that Revenue growth has on the Balance Sheet and evaluate the effect of some corporate decisions such as:

•   the changes that Revenue and its variation produce on the Invested Capital,

•   the debt rate,

•   the Cost of Debt,

•   the payout ratio,

•   the Tax Rate.

We have not assessed the need for further investments to ensure greater market competitiveness, providing parameters 9 to 12 of Table 2 fully meet this imperative.

Such assumptions have generated a DFL that remains sufficiently constant both over time and within each financial period as in period 2; it is susceptible to variations caused by DOL that, in turn, are induced by $\Delta\%qms_t$, or by other factors such as $\Delta\%F_t$, $\Delta\%up_t$ or $\Delta\%uvc_t$, and the debt swing. Even removing the floating debt hypothesis, we would have reached the same conclusions through a more simple analytical way but not automatically extensible to real Business Cases.

7. Non-Recurring Shocks

The model presented in the previous paragraph allows to accurately measure the behaviour of DOL and DFL over seven financial periods in the absence of exogenous shocks and to evaluate such a relationship by artificially varying $\Delta\%qms_t$ within a single financial period. In this way, it was possible to investigate the

inter-annual and infra-annual relationship established between DOL and DFL. Some variables, taken as parameters, can change the shape of DOL and or DFL when they are suitably changed.

Before examining the Business Case, we need to verify the DOL-DFL nexus in case of a single one-off shock in a particular financial period of the analysis to check the direction, intensity, and duration of its impact. As already mentioned, we shall not investigate the long-term effects of the steady-state and the steady-state jumps over the horizon of seven financial periods and beyond.

We shall vary the variables shown in Table 2 from 1 to 7 and 11 one at a time in the financial period 2, and we report the effect of each variation over the time horizon comparing it to the steady-state situation already presented in Tables 3 and 4.

We report the output in Tables 5 and 6 that contain the original steady-state data. To facilitate the reading increasing values in the Tables have been shown in green (or light grey) and decreasing values in pink (or dark grey) compared to the steady-state simulation. All the shocks will worsen the steady-state $\Delta\%qms_t$ apart, which leads to an equivalent worsening of $\Delta\%qmvc_t$.

Only the variations of the Cost of Debt i_t and the Tax Rate α_t can modify DFL in two periods, leaving the DOL unchanged whilst its effect on DFL disappears in the financial period 4, just two periods after the shock.

The $\Delta\%qms_t = \Delta\%qmvc_t$ variation increases from 5.00% to 10.00% and produces a simultaneous increase of DOL and DFL in the shock period followed by a worsening of DOL and a DFL improvement in the subsequent periods.

The worsening of $\Delta\%F_t$, $\Delta\%up_t$, or $\Delta\%uvc_t$ worsens DOL and simultaneously improves DFL in the shock period, followed by an improvement of both DOL and DFL.

The worsening of i_t and α_t worsens DFL only in the shock period and its improvement in the following period leaving the DFL value unchanged from the second period after the shock. In such cases, DOL remains unaffected.

Higher investment needs, which translate into higher γ, leave the DOL unchanged because the higher depreciation is offset within the fixed costs keeping $\Delta\%F_t$ untouched even though they modify the financial needs causing a reduction of DFL during the shock period followed by its improvement in the following ones.

There is "no case" of an infra-annual inversion of $\partial DFL_t/\partial DOL_t$ that remains positive in the shock period.

Having defined the infra-annual DOL-DFL relationship, we must accurately specify also the inter-annual DOL-DFL relationship. When the inter-annual DOL-DFL variations occur in the same direction, either increasing or decreasing, the nexus is "in sync"; on the contrary, it is "out of sync".

What are the consequences deriving from the shock analysis? If the impact does not concern DOL, it seems to be reabsorbed by DFL, but, in this case, no inter-annual "out of sync" relationship between DOL and DFL is noted, being DFL dependent on DOL and not vice versa.

In the case of a $\Delta\%qms_t = \Delta\%qmvc_t$ increase, there is an immediate "in sync" DOL-DFL variation, but this relationship turns to be "out of sync" in the subsequent periods. All the other variations show an "out of sync" relationship between DOL and DFL immediately followed by an "out of sync" in the following period, $\Delta\%up_t$ apart that shows an "in sync" relationship. The previous conclusions depend on the direction and magnitude of the shocks considered and the initial conditions of the financial period 0. The DOL-DFL relationship should not be considered exclusively "out of sync", also in this case.

Table 5. Infra-annual analysis of one-off shocks

Scenario	Shock Value	Index	1	2	3	4	5	6	7	Note
Steady		DOL	213.33333%	206.50602%	200.39969%	194.90610%	189.93765%	185.42267%	181.30196%	Ratio
		DFL	99.41041%	100.62826%	101.63715%	102.46779%	103.14643%	103.69553%	104.13427%	Ratio
		DTL	212.07553%	207.80342%	203.68055%	199.71597%	195.91390%	192.27501%	188.79748%	Computed
		$\Delta\%qms$	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	Datum
		DOL Discontinuity	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	Computed
		DOL = 0 (DFL Discontinuity)	3.40000%	3.36762%	3.33555%	3.30378%	3.27231%	3.24115%	3.21028%	Computed
		DFL = 0 t_d floating	3.43675%	3.38650%	3.33732%	3.28919%	3.24207%	3.19593%	3.15072%	Computed
		DFL = 0 - DOL = 0	0.0368%	0.0189%	0.0018%	-0.0146%	-0.0302%	-0.0452%	-0.0596%	Computed
		$\partial DOL/\partial\Delta\%qms$	9066.6667%	8520.4819%	8031.9755%	7592.4884%	7195.0120%	6833.8133%	6504.1568%	Computed
		$\partial DFL/\partial\Delta\%qms$	146.0802%	72.1205%	6.5225%	-51.4996%	-102.6999%	-147.7852%	-187.4072%	Computed
		$\partial DFL/\partial DOL$	1.6112%	0.8464%	0.0812%	-0.6783%	-1.4274%	-2.1626%	-2.8813%	Computed
		$\partial DFL/\partial DOL$ as Inverse Function	1.6112%	0.8464%	0.0812%	-0.6783%	-1.4274%	-2.1626%	-2.8813%	Computed
Quantity Shock Mix	$\Delta\%qms =$ $\Delta\%qmvc =$ 10,00%	DOL	213.33333%	419.51807%	178.03089%	174.51881%	171.27973%	168.28316%	165.50296%	Ratio
		DFL	99.41041%	101.51579%	102.53503%	103.15909%	103.66944%	104.08164%	104.40925%	Ratio
		DTL	212.07553%	425.87708%	182.54403%	180.03200%	177.56473%	175.15188%	172.80040%	Computed
		$\Delta\%qms$	5.00000%	10.00000%	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	Datum
		DOL Discontinuity	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	Computed
		DOL = 0 (DFL Discontinuity)	3.40000%	3.36762%	3.18393%	3.15361%	3.12357%	3.09382%	3.06436%	Computed
		DFL = 0 t_d floating	3.43675%	3.38650%	3.16272%	3.11678%	3.07181%	3.02776%	2.98460%	Computed
		DFL = 0 - DOL = 0	0.03675%	0.01888%	-0.02121%	-0.03682%	-0.05177%	-0.06607%	-0.07976%	Computed
		$\partial DOL/\partial\Delta\%qms$	9066.66667%	2130.12048%	6242.47157%	5961.50444%	5702.37848%	5462.65277%	5240.23706%	Computed
		$\partial DFL/\partial\Delta\%qms$	146.08018%	4.36881%	-65.16887%	-109.24781%	-148.32865%	-182.91434%	-213.46591%	Computed
		$\partial DFL/\partial DOL$	1.61118%	0.20510%	-1.04396%	-1.83255%	-2.60117%	-3.34845%	-4.07359%	Computed
		$\partial DFL/\partial DOL$ as Inverse Function	1.61118%	0.20510%	-1.04396%	-1.83255%	-2.60117%	-3.34845%	-4.07359%	Computed
Fixed Costs Shock	$\Delta\%F = 8,00\%$	DOL	213.33333%	-219.51807%	229.20828%	220.55994%	212.92858%	206.14507%	200.07580%	Ratio
		DFL	99.41041%	114.91119%	103.06633%	103.83878%	104.44648%	104.91767%	105.27557%	Ratio
		DTL	212.07553%	-252.25083%	236.23657%	229.02677%	222.39640%	216.28260%	210.63093%	Computed
		$\Delta\%qms$	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	Datum
		DOL Discontinuity	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	Computed
		DOL = 0 (DFL Discontinuity)	3.40000%	6.73524%	3.46384%	3.43085%	3.39817%	3.36581%	3.33375%	Computed
		DFL = 0 t_d floating	3.43675%	6.95862%	3.47340%	3.42398%	3.37558%	3.32816%	3.28169%	Computed
		DFL = 0 - DOL = 0	0.03675%	0.22338%	0.00957%	-0.00687%	-0.02260%	-0.03765%	-0.05207%	Computed
		$\partial DOL/\partial\Delta\%qms$	9066.66667%	17040.96386%	10336.66261%	9644.79559%	9034.28655%	8491.60527%	8006.06417%	Computed
		$\partial DFL/\partial\Delta\%qms$	146.08018%	755.26506%	42.04497%	-28.83925%	-90.70462%	-144.59108%	-191.44167%	Computed
		$\partial DFL/\partial DOL$	1.61118%	4.43206%	0.40676%	-0.29901%	-1.00400%	-1.70275%	-2.39121%	Computed
		$\partial DFL/\partial DOL$ as Inverse Function	1.61118%	4.43206%	0.40676%	-0.29901%	-1.00400%	-1.70275%	-2.39121%	Computed
Unit Price Shock	$\Delta\%up = -4,00\%$	DOL	213.33333%	-7011.29518%	352.25957%	323.06208%	299.72308%	280.64088%	264.74888%	Ratio
		DFL	99.41041%	106.41448%	107.06859%	107.54835%	107.87117%	108.07273%	108.17932%	Ratio
		DTL	212.07553%	-7461.03330%	377.15935%	347.44792%	323.31480%	303.29626%	286.40353%	Computed
		$\Delta\%qms$	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	Datum
		DOL Discontinuity	0.00000%	4.16667%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	Computed
		DOL = 0 (DFL Discontinuity)	3.40000%	14.85291%	3.70616%	3.67087%	3.63591%	3.60128%	3.56698%	Computed
		DFL = 0 t_d floating	3.43675%	15.34083%	3.74569%	3.69200%	3.63943%	3.58795%	3.53751%	Computed
		DFL = 0 - DOL = 0	0.03675%	0.48792%	0.03953%	0.02113%	0.00353%	-0.01333%	-0.02947%	Computed
		$\partial DOL/\partial\Delta\%qms$	9066.66667%	912515.06024%	20180.76554%	17844.96612%	15977.84661%	14451.27077%	13179.91070%	Computed
		$\partial DFL/\partial\Delta\%qms$	146.08018%	50.96047%	260.80315%	130.74431%	20.50089%	-72.93106%	-152.10317%	Computed
		$\partial DFL/\partial DOL$	1.61118%	0.00558%	1.29234%	0.73267%	0.12831%	-0.50467%	-1.15405%	Computed
		$\partial DFL/\partial DOL$ as Inverse Function	1.61118%	0.00558%	1.29234%	0.73267%	0.12831%	-0.50467%	-1.15405%	Computed
Unit variable cost Shock	$\Delta\%uvc = 4,00\%$	DOL	213.33333%	-590.48193%	257.16899%	244.83253%	234.19783%	224.93589%	216.79737%	Ratio
		DFL	99.41041%	110.56168%	104.95764%	105.64460%	106.15859%	106.53306%	106.79451%	Ratio
		DTL	212.07553%	-652.84672%	269.91850%	258.65234%	248.62112%	239.63108%	231.52769%	Computed
		$\Delta\%qms$	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	Datum
		DOL Discontinuity	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	Computed
		DOL = 0 (DFL Discontinuity)	3.40000%	9.96555%	3.54845%	3.51466%	3.48119%	3.44803%	3.41519%	Computed
		DFL = 0 t_d floating	3.43675%	10.41362%	3.56416%	3.51276%	3.46244%	3.41315%	3.36487%	Computed
		DFL = 0 - DOL = 0	0.03675%	0.44806%	0.01570%	-0.00190%	-0.01875%	-0.03488%	-0.05032%	Computed
		$\partial DOL/\partial\Delta\%qms$	9066.66667%	23701.20482%	12573.51910%	11586.60245%	10735.82673%	9994.87121%	9343.78928%	Computed
		$\partial DFL/\partial\Delta\%qms$	146.08018%	184.28459%	79.08685%	-9.08486%	-85.23137%	-150.87959%	-207.38826%	Computed
		$\partial DFL/\partial DOL$	1.61118%	0.77753%	0.62900%	-0.07841%	-0.79390%	-1.50957%	-2.21953%	Computed
		$\partial DFL/\partial DOL$ as Inverse Function	1.61118%	0.77753%	0.62900%	-0.07841%	-0.79390%	-1.50957%	-2.21953%	Computed
Interest rate Shock	$\Delta\%i = 50,00\%$	DOL	213.33333%	206.50602%	200.39969%	194.90610%	189.93765%	185.42267%	181.30196%	Ratio
		DFL	99.41041%	66.14550%	138.28634%	102.46779%	103.14643%	103.69553%	104.13427%	Ratio
		DTL	212.07553%	136.59443%	277.12540%	199.71597%	195.91390%	192.27501%	188.79748%	Computed
		$\Delta\%qms$	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	Datum
		DOL Discontinuity	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	Computed
		DOL = 0 (DFL Discontinuity)	3.40000%	3.36762%	3.33555%	3.30378%	3.27231%	3.24115%	3.21028%	Computed
		DFL = 0 t_d floating	3.43675%	3.91452%	2.81074%	3.28919%	3.24207%	3.19593%	3.15072%	Computed
		DFL = 0 - DOL = 0	0.03675%	0.54690%	-0.52480%	-0.01458%	-0.03024%	-0.04522%	-0.05956%	Computed
		$\partial DOL/\partial\Delta\%qms$	9066.66667%	8520.48193%	8031.97554%	7592.48838%	7195.01203%	6833.81330%	6504.15678%	Computed
		$\partial DFL/\partial\Delta\%qms$	146.08018%	2041.57586%	-1991.62136%	-51.49960%	-102.69990%	-147.78518%	-187.40725%	Computed
		$\partial DFL/\partial DOL$	1.61118%	23.96080%	-24.79616%	-0.67830%	-1.42738%	-2.16256%	-2.88135%	Computed
		$\partial DFL/\partial DOL$ as Inverse Function	1.61118%	23.96080%	-24.79616%	-0.67830%	-1.42738%	-2.16256%	-2.88135%	Computed
Tax Rate Shock	$\Delta\%\alpha = 20,00\%$	DOL	213.33333%	206.50602%	200.39969%	194.90610%	189.93765%	185.42267%	181.30196%	Ratio
		DFL	99.41041%	28.34138%	181.37919%	102.46779%	103.14643%	103.69553%	104.13427%	Ratio
		DTL	212.07553%	58.52665%	363.48334%	199.71597%	195.91390%	192.27501%	188.79748%	Computed
		$\Delta\%qms$	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	Datum
		DOL Discontinuity	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	Computed
		DOL = 0 (DFL Discontinuity)	3.40000%	3.36762%	3.33555%	3.30378%	3.27231%	3.24115%	3.21028%	Computed
		DFL = 0 t_d floating	3.43675%	4.51261%	2.23345%	3.28919%	3.24207%	3.19593%	3.15072%	Computed
		DFL = 0 - DOL = 0	0.03675%	1.14499%	-1.10210%	-0.01458%	-0.03024%	-0.04522%	-0.05956%	Computed
		$\partial DOL/\partial\Delta\%qms$	9066.66667%	8520.48193%	8031.97554%	7592.48838%	7195.01203%	6833.81330%	6504.15678%	Computed
		$\partial DFL/\partial\Delta\%qms$	146.08018%	4078.75144%	-4341.07927%	-51.49960%	-102.69990%	-147.78518%	-187.40725%	Computed
		$\partial DFL/\partial DOL$	1.61118%	47.86996%	-54.04747%	-0.67830%	-1.42738%	-2.16256%	-2.88135%	Computed
		$\partial DFL/\partial DOL$ as Inverse Function	1.61118%	47.86996%	-54.04747%	-0.67830%	-1.42738%	-2.16256%	-2.88135%	Computed
Net investment Fixed debt	$\lambda = 200,00\%$	DOL	213.33333%	206.50602%	200.39969%	194.90610%	189.93765%	185.42267%	181.30196%	Ratio
		DFL	99.41041%	84.99948%	104.79003%	105.22133%	105.54798%	105.78703%	105.95275%	Ratio
		DTL	212.07553%	175.52904%	209.99889%	205.08279%	200.47535%	196.15312%	192.09442%	Computed
		$\Delta\%qms$	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	5.00000%	Datum
		DOL Discontinuity	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	0.00000%	Computed
		DOL = 0 (DFL Discontinuity)	3.40000%	3.36762%	3.33555%	3.30378%	3.27231%	3.24115%	3.21028%	Computed
		DFL = 0 t_d floating	3.43675%	3.56518%	3.31080%	3.26394%	3.21802%	3.17302%	3.12890%	Computed
		DFL = 0 - DOL = 0	0.03675%	0.19756%	-0.02474%	-0.03984%	-0.05429%	-0.06813%	-0.08138%	Computed
		$\partial DOL/\partial\Delta\%qms$	9066.66667%	8520.48193%	8031.97554%	7592.48838%	7195.01203%	6833.81330%	6504.15678%	Computed
		$\partial DFL/\partial\Delta\%qms$	146.08018%	716.96743%	-92.21595%	-142.35659%	-186.13343%	-224.28845%	-257.48335%	Computed
		$\partial DFL/\partial DOL$	1.61118%	8.41463%	-1.14811%	-1.87497%	-2.58698%	-3.28204%	-3.95875%	Computed
		$\partial DFL/\partial DOL$ as Inverse Function	1.61118%	8.41463%	-1.14811%	-1.87497%	-2.58698%	-3.28204%	-3.95875%	Computed

Table 6. Inter-annual analysis of one-off shocks

We have seen that:

- the inter-annual relationship between DOL and DFL in steady-state seems to be "out of sync";

- the infra-annual relationship in steady-state could be indifferently "in phase" or "out of phase", depending on contingent situations, with a tendency to become "out of phase" in the long run;

- the partial derivative $\partial DFL_t/\partial DOL_t$ shows certain inertia to maintain the positive sign, that means "in phase", in the shock period, but this could depend on the magnitude of the shock itself, not sufficient to exceed the minimum threshold to flip the relation even though in some cases it anticipates or slows down the infra-annual flip;

- there is an inter-annual DOL-DFL variation in case of one-off shocks, not always "out of sync", in the shock period and the following one;

- Some one-off shocks only impact DFL leaving DOL unchanged and revealing that the latter is unrelated to the former.

In conclusion, the model presented allows probing the mathematical and economic relationship between DOL and DFL. The infra-annual and inter-annual relationships of the DOL-DFL nexus depend on the temporary circumstances of the current and previous financial period and cannot be considered exclusively "out of phase" and "out of sync" because even "in phase" and "in sync" conditions can be admitted.

8. Business Case and Discussions

Table 7 shows Income Statements, Balance Sheets, Ratios, and all the parameters and variables that determine DOL and DFL for the financial period 0 and the following seven periods. Figure 10 shows the percentage variation of the fundamental ratios in the seven periods under analysis.

From the analysis of data and graphical trends, we cannot evaluate the relationship between DOL and DFL: in periods 2 and 3, they seem to vary "out of sync", while in subsequent periods, they would appear to move "in sync". Some facts are evident:

1) Contribution Margin, EBIT, and Net Profit in seven periods show an improvement in percentage terms while the debt halves despite the Invested Capital show a growth dynamic, a sign that the company is self-financing by higher profits and by not distributing dividends in the financial periods 2 and 3 and partial payouts in 4 to 7.

2) The company shows a brilliant Revenue growth not always followed by a good EBIT trend in the first three financial periods; in the four following periods, the situation improves with a DOL continually above 100%.

3) The DFL trend appears to fluctuate over time, but we have to point out that only in periods 2, 4, and 7 is higher than 100%, helping accelerate the Net Profit dynamic compared to the EBIT one.

DOL trend in the first three periods is due to a negative dynamic of constantly decreasing unit prices $\Delta\%up_t$ combined with either an increasing dynamics of unit variable costs $\Delta\%uvc_t$ or exceeding fixed costs $\Delta\%F_t$ which prevents the achievement of DOL values higher than 100%. On the other hand, in the financial periods from 4 to 7, the unit price reduction $\Delta\%up_t$ is offset by a greater unit variable cost reduction $\Delta\%uvc_t$, together with a more careful fixed costs dynamic $\Delta\%F_t$, which tends to stick, sometimes without exceeding, the variation of $\Delta\%qms_t$. The result is to keep DOL consistently above 100%.

DFL dynamic is harder to understand. In particular, DFL does not appear close to 100% in financial periods 1, 3, and 5 that we have to investigate better. In financial period 1, the increase in the Tax Rate α_t generates a push to reduce the DFL only partially offset by a reduction of the Cost of Debt i_t; in financial period 3, the Cost of Debt and Tax Rate increase together with a significant debt increase even though the debt ratio falls. In financial period 5, a sizable tax rebate in the previous period generates an anomaly in the Tax Rate: such a rebate creates an abnormal increase of $\Delta\%\alpha_t$ that reaches 1352%. The significant fluctuation of DFL in period 4 is due to such a tax rebate which helps DFL reaching 280%.

Table 7. Corporate data, ratios, and analytic data determining DOL and DFL

Data Set	Variable or Parameter	0	1	2	3	4	5	6	7	Note
Income statements	Sales	€ 512,761,932	€ 717,276,815	€ 773,078,195	€ 976,787,291	€ 1,135,935,072	€ 1,305,167,587	€ 1,477,706,569	€ 1,626,337,706	Datum
	Varibale Costs	-€ 295,266,187	-€ 412,226,830	-€ 461,879,386	-€ 616,332,156	-€ 660,922,948	-€ 737,213,031	-€ 823,182,894	-€ 889,512,271	Datum
	Contribution Margin	**€ 217,495,745**	**€ 305,049,985**	**€ 311,198,809**	**€ 360,455,135**	**€ 475,012,124**	**€ 567,954,556**	**€ 654,523,675**	**€ 736,825,434**	Computed
	Fixed Costs	-€ 146,016,780	-€ 210,460,503	-€ 221,499,346	-€ 253,822,395	-€ 331,023,177	-€ 380,223,808	-€ 414,787,761	-€ 459,717,014	Datum
	Depreciation & Amortization	-€ 23,505,220	-€ 29,745,172	-€ 22,963,059	-€ 28,122,109	-€ 42,945,935	-€ 63,285,816	-€ 79,777,010	-€ 81,799,905	Datum
	Total Fixed Costs	-€ 169,522,000	-€ 240,205,674	-€ 244,462,405	-€ 281,944,504	-€ 373,969,112	-€ 443,509,624	-€ 494,564,771	-€ 541,516,919	Computed
	EBIT	**€ 47,973,745**	**€ 64,844,311**	**€ 66,736,404**	**€ 78,510,631**	**€ 101,043,012**	**€ 124,444,932**	**€ 159,958,904**	**€ 195,308,516**	Computed
	Financial Charges	-€ 18,599,977	-€ 4,415,686	-€ 3,460,518	-€ 5,453,667	-€ 5,318,674	-€ 3,788,710	-€ 2,114,394	-€ 1,443,104	Datum
	EBT	€ 29,373,767	€ 60,428,625	€ 63,275,887	€ 73,056,964	€ 95,724,339	€ 120,656,222	€ 157,844,509	€ 193,865,412	Computed
	Tax	**€ 0**	**-€ 24,219,185**	**-€ 16,271,302**	**-€ 20,382,893**	**-€ 749,341**	**-€ 13,720,510**	**-€ 22,511,871**	**-€ 27,669,311**	Datum
	Net Profit	€ 29,373,767	€ 36,209,439	€ 47,004,585	€ 52,674,071	€ 94,974,997	€ 106,935,712	€ 135,332,638	€ 166,196,101	Computed
Balance sheets	Cash	€ 279,766,910	€ 251,123,841	€ 206,401,337	€ 171,853,040	€ 158,546,822	€ 142,634,654	€ 234,550,067	€ 431,774,354	Datum
	Net Working Capital	€ 168,056,166	€ 223,258,929	€ 252,022,522	€ 315,060,302	€ 302,627,795	€ 327,351,242	€ 306,488,276	€ 292,538,631	Datum
	Assets	€ 144,138,123	€ 142,106,336	€ 161,808,263	€ 216,232,808	€ 278,127,369	€ 341,329,813	€ 365,534,216	€ 319,605,135	Datum
	Capital Invested	**€ 591,961,198**	**€ 616,489,106**	**€ 620,232,122**	**€ 703,146,150**	**€ 739,301,986**	**€ 811,315,709**	**€ 906,572,559**	**€ 1,043,918,119**	Computed
	Equity	-€ 348,396,856	-€ 370,738,585	-€ 355,801,407	-€ 408,475,478	-€ 503,450,476	-€ 586,642,438	-€ 695,241,148	-€ 827,604,090	Datum
	Debt	-€ 243,564,342	-€ 245,750,521	-€ 264,430,715	-€ 294,670,672	-€ 235,851,510	-€ 224,673,270	-€ 211,331,410	-€ 216,314,029	Datum
	Net Financial Position	€ 36,202,567	€ 5,373,320	-€ 58,029,378	-€ 122,817,632	-€ 77,304,688	-€ 82,038,616	€ 23,218,657	€ 215,460,324	Computed
Ratios	%vc	57.583%	57.471%	59.745%	63.098%	58.183%	56.484%	55.707%	54.694%	Ratio
	%cm	42.417%	42.529%	40.255%	36.902%	41.817%	43.516%	44.293%	45.306%	Ratio
	%f	33.061%	33.489%	31.622%	28.864%	32.922%	33.981%	33.468%	33.297%	Ratio
	%EBIT	9.356%	9.040%	8.633%	8.038%	8.895%	9.535%	10.825%	12.009%	Ratio
	%Π	5.729%	5.048%	6.080%	5.393%	8.361%	8.193%	9.158%	10.219%	Ratio
	Δ%S		39.885%	7.780%	26.350%	16.293%	14.898%	13.220%	10.058%	Ratio
	Δ%CM		40.256%	2.016%	15.828%	31.781%	19.566%	15.242%	12.574%	Ratio
	Δ%EBIT		35.166%	2.918%	17.643%	28.700%	23.160%	28.538%	22.099%	Ratio
	Δ%Π		23.271%	29.813%	12.062%	80.307%	12.594%	26.555%	22.806%	Ratio
	DOL		88.169%	37.507%	66.955%	176.148%	155.459%	215.874%	219.712%	Ratio
	DFL		66.175%	1021.730%	68.365%	279.817%	54.375%	93.052%	103.197%	Ratio
	DTL		58.346%	383.220%	45.774%	492.893%	84.531%	200.876%	226.736%	Computed
DOL Determinants	Δ%up		-2.972%	-1.478%	-2.249%	0.593%	-2.590%	-7.000%	-3.655%	Datum
	Δ%qms		44.170%	9.397%	29.258%	15.607%	17.953%	21.742%	14.233%	Computed
	Δ%uvc		-3.616%	2.539%	3.502%	-7.039%	-5.484%	-8.077%	-5.793%	Computed
	Δ%qmvc		44.850%	9.271%	28.925%	15.355%	18.015%	21.473%	14.702%	Datum
	ε		-0.680%	0.126%	0.333%	0.252%	-0.062%	0.269%	-0.469%	Computed
	Δ%F		41.696%	1.772%	15.332%	32.639%	18.595%	11.512%	9.494%	Computed
	Δ%EBIT		35.166%	2.918%	17.643%	28.700%	23.160%	28.538%	22.099%	Computed
	Check		0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	Computed
	DOL		88.169%	37.507%	66.955%	176.148%	155.459%	215.874%	219.712%	Computed
	PDOL		453.364%	470.434%	466.310%	459.116%	470.109%	456.390%	409.182%	Computed
	DOL Abatement Factor		-80.552%	-92.027%	-85.642%	-61.633%	-66.931%	-52.700%	-46.305%	Computed
DFL Determinants	i	7.637%	1.797%	1.309%	1.851%	2.255%	1.686%	1.001%	0.667%	Computed
	φ	38.771%	6.810%	5.185%	6.946%	5.264%	3.044%	1.322%	0.739%	Computed
	ρ	61.229%	93.190%	94.815%	93.054%	94.736%	96.956%	98.678%	99.261%	Computed
	α rate	0.000%	40.079%	25.715%	27.900%	0.783%	11.372%	14.262%	14.272%	Computed
	ν	100.000%	59.921%	74.285%	72.100%	99.217%	88.628%	85.738%	85.728%	Computed
	t_d	69.910%	66.287%	74.320%	72.139%	46.847%	38.298%	30.397%	26.137%	Computed
	DFL*	41.145%	39.863%	42.634%	41.907%	31.902%	27.692%	23.311%	20.721%	Computed
	ψ		74.449%	93.345%	89.725%	90.411%	95.511%	98.694%	104.626%	Computed
	Δ%i		-76.471%	-27.167%	41.424%	21.846%	-25.222%	-40.669%	-33.321%	Computed
	Δ%φ		-82.436%	-23.853%	33.962%	-24.223%	-42.161%	-56.583%	-44.102%	Computed
	Δ%ρ		52.200%	1.743%	-1.857%	1.808%	2.343%	1.777%	0.591%	Computed
	Δ%α		4007899354%	-35.840%	8.498%	-97.194%	1352.657%	25.418%	0.073%	Computed
	Δ%ν		-40.079%	23.972%	-2.942%	37.611%	-10.672%	-3.261%	-0.012%	Computed
	Δ%t_d		-5.183%	12.119%	-2.934%	-35.060%	-18.248%	-20.631%	-14.013%	Computed
	Δ%D		0.898%	7.601%	11.436%	-19.961%	-4.740%	-5.938%	2.358%	Computed
	Δ%E		6.413%	-4.029%	14.804%	23.251%	16.524%	18.512%	19.038%	Computed
	Δ%CI		4.143%	0.607%	13.368%	5.142%	9.741%	11.741%	15.150%	Computed
	T		59.921%	123.972%	97.058%	137.611%	89.328%	96.739%	99.988%	Computed
	A	63.322%	7.307%	5.469%	7.465%	5.556%	3.140%	1.340%	0.744%	Computed
	N = T x (1+A)		97.864%	133.031%	102.366%	147.883%	94.291%	99.776%	101.327%	Computed
	M		23.529%	72.833%	141.424%	121.846%	74.778%	59.331%	66.679%	Computed
	P		72.129%	99.835%	88.196%	68.825%	82.909%	83.079%	93.003%	Computed
	Q = A x M x P x T		6.439%	6.587%	6.621%	8.615%	3.077%	1.497%	0.831%	Computed
	DFL		66.175%	1021.730%	68.365%	279.817%	54.375%	93.052%	103.197%	Computed

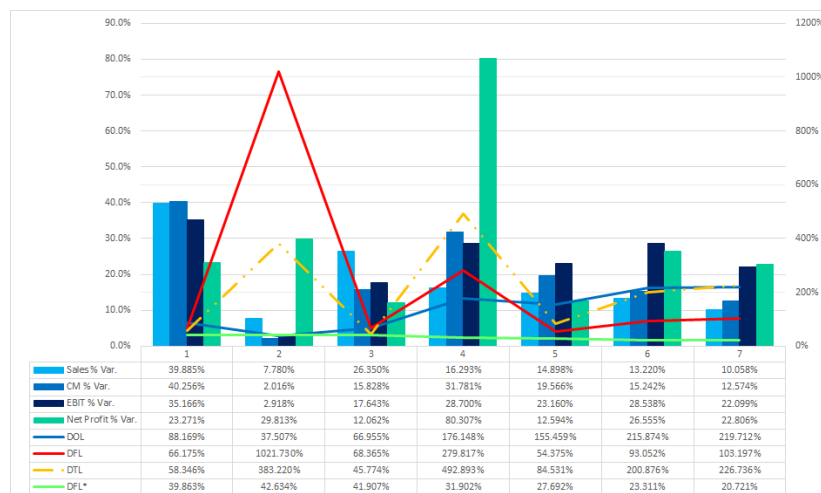|	1	2	3	4	5	6	7
Sales % Var.	39.885%	7.780%	26.350%	16.293%	14.898%	13.220%	10.058%
CM % Var.	40.256%	2.016%	15.828%	31.781%	19.566%	15.242%	12.574%
EBIT % Var.	35.166%	2.918%	17.643%	28.700%	23.160%	28.538%	22.099%
Net Profit % Var.	23.271%	29.813%	12.062%	80.307%	12.594%	26.555%	22.806%
DOL	88.169%	37.507%	66.955%	176.148%	155.459%	215.874%	219.712%
DFL	66.175%	1021.730%	68.365%	279.817%	54.375%	93.052%	103.197%
DTL	58.346%	383.220%	45.774%	492.893%	84.531%	200.876%	226.736%
DFL*	39.863%	42.634%	41.907%	31.902%	27.692%	23.311%	20.721%

Figure 10. The trend of the parameters and variables examined: histograms scale on the left, lines on the right

We have to investigate DOL and DFL trends in-depth by using the data in Tables 8 and 9, from which we can observe the following facts:

1) DOL slope is always positive because both the EBIT is positive over time and the Fixed Costs percentage variation $\Delta\%F_t$ is higher than the threshold required;

2) DOL discontinuities always precede DFL ones, but in the financial period 2, 4, and 7, DFL roots precede DOL ones, and that bends the DFL slope from positive to negative;

3) In the financial periods 2, 4 and 7, DFL shows a negative slope and becomes "out of phase" with respect to DOL, while in periods 1, 3, 5, and 6, it appears "in phase". We get the demonstration by two independent values of $\partial DFL_t / \partial DOL_t$, both identical and negative in the financial periods 2, 4 and 7.

4) Unlike paragraph 6, where i_t and α_t were constant, in the Business Case under analysis, these values are free to float and, therefore they could be the reason for the inversion of the DFL slope. The thresholds μ_t and τ_t in periods 2, 4, and 7 are not verified and, that could have caused the DFL slope to flip from positive to negative. Table 9 shows all the data to check the situations observed.

Table 10 shows how far i_t and α_t are from their boundary conditions. For example, in all the financial periods, apart from period 4, the Cost of Debt i_t seems to diverge more than the Tax Rate α_t to allow DFL to reach a positive slope that we have to remember is negative in financial periods 2, 4, and 7.

Table 8. Analysis of the factors determining DOL and DFL

	DOL & DFL Split		0	1	2	3	4	5	6	7
DOL Split	$\Delta\%S$			39.885%	7.780%	26.350%	16.293%	14.898%	13.220%	10.058%
	PDOL	A		453.364%	470.434%	466.310%	459.116%	470.109%	456.390%	409.182%
	S/CM	B	235.757%	235.134%	248.419%	270.987%	239.138%	229.801%	225.768%	
	-(VC*$\Delta\%$VC+F*$\Delta\%$F)/CM*$\Delta\%$S	C		-216.309%	-227.161%	-234.061%	-232.620%	-206.069%	-182.501%	-172.073%
	S/CM-(VC*$\Delta\%$VC+F*$\Delta\%$F)/CM*$\Delta\%$S	B+C=D		19.448%	7.973%	14.358%	38.367%	33.069%	47.300%	53.695%
	DOL	AxD		**88.169%**	**37.507%**	**66.955%**	**176.148%**	**155.459%**	**215.874%**	**219.712%**
	$\Delta\%$EBIT			35.166%	2.918%	17.643%	28.700%	23.160%	28.538%	22.099%
DFL Split	1/$\Delta\%$EBIT			2.84	34.27	5.67	3.48	4.32	3.50	4.53
	N	A		97.864%	133.031%	102.366%	147.883%	94.291%	99.776%	101.327%
	N-1			-2.136%	33.031%	2.366%	47.883%	-5.709%	-0.224%	1.327%
	(N-1)/$\Delta\%$EBIT	B		-6.074%	1132.004%	13.413%	166.841%	-24.650%	-0.784%	6.006%
	-Q(1+$\Delta\%$S)			-9.008%	-7.099%	-8.365%	-10.018%	-3.536%	-1.695%	-0.914%
	-Q(1+$\Delta\%$S)/$\Delta\%$EBIT	C		-25.615%	-243.305%	-47.415%	-34.907%	-15.265%	-5.940%	-4.137%
	(N-1)/$\Delta\%$EBIT-Q(1+$\Delta\%$S)/$\Delta\%$EBIT	B+C		-31.689%	888.699%	-34.001%	131.934%	-39.916%	-6.724%	1.869%
	DFL	A+B+C		**66.175%**	**1021.730%**	**68.365%**	**279.817%**	**54.375%**	**93.052%**	**103.197%**

Table 9. Infra-annual analysis of DOL and DFL

Index	Note	1	2	3	4	5	6	7	Note
$\Delta\%$qms		44.17017%	9.39678%	29.25761%	15.60743%	17.95306%	21.74159%	14.23346%	Datum
DOL Discontinuity		3.06338%	1.50044%	2.30092%	-0.58950%	2.65886%	7.52688%	3.79366%	Computed
DOL = 0 (DFL Discontinuity)		36.24719%	8.73051%	25.01672%	10.10680%	13.09625%	15.11756%	8.77994%	Computed
DFL = 0 t_q floating		38.69418%	4.07285%	26.33377%	4.86784%	15.08443%	15.49338%	8.65608%	Computed
DFL Disc.ty > DOL Disc.ty		Yes	Yes	Yes	Yes	Yes	Yes	Yes	Computed
DFL = 0 > DOL = 0		Yes	No	Yes	No	Yes	Yes	No	Computed
DFL = 0 - DOL = 0		2.4470%	-4.6577%	1.3170%	-5.2390%	1.9882%	0.3758%	-0.1239%	Computed
DOL Status		Positive	Positive	Positive	Positive	Positive	Positive	Positive	Computed
∂DOL/$\partial\Delta\%$qms		898.3412%	5154.3899%	1330.4136%	2114.7862%	2184.3818%	1740.2895%	1924.2533%	Computed
$\Delta\%$F lower boundary		0.01454%	-6.77560%	-10.46583%	17.09877%	5.14730%	2.33560%	2.95839%	Computed
DFL Status		Positive	Negative	Positive	Negative	Positive	Positive	Negative	Computed
∂DFL/$\partial\Delta\%$qms		373.2303%	-134158.8863%	726.1463%	-2481.5277%	775.9461%	84.4934%	-42.0226%	Computed
Phase DOL/DFL Status		In Phase	Out of Phase	In Phase	Out of Phase	In Phase	In Phase	Out of Phase	Computed
M boundary		-6.0822%	346.2111%	42.9096%	670.0050%	-141.3980%	-9.6615%	107.8204%	Computed
Denominator		-155.8731%	-41.7827%	-22.9437%	-37.2881%	-19.2537%	-9.9739%	-4.9884%	Computed
i boundary		-0.4645%	6.2208%	0.5615%	12.4002%	-3.1887%	-0.1629%	1.0788%	Computed
Boundary direction		Lower	Lower	Lower	Lower	Lower	Lower	Lower	Computed
T boundary		66.8754%	98.3263%	102.6347%	98.8758%	97.8701%	98.3089%	99.4841%	Computed
Denominator		663.6996%	445.3991%	405.3386%	527.6866%	487.2412%	438.2350%	407.3294%	Computed
α boundary		33.1246%	41.0819%	23.7576%	28.7105%	2.8961%	12.8704%	14.7044%	Computed
Boundary direction		Lower	Lower	Lower	Lower	Lower	Lower	Lower	Computed
∂DFL/∂DOL		41.5466%	-2602.8083%	54.5805%	-117.3418%	35.5225%	4.8551%	-2.1838%	Computed
∂DFL/∂DOL as Inverse Function		41.5466%	-2602.8083%	54.5805%	-117.3418%	35.5225%	4.8551%	-2.1838%	Computed
DFL = 0 to identify <>0	Support	38.69418%	4.07285%	26.33377%	4.86784%	15.08443%	15.49338%	8.65608%	Computed
$\partial\Delta\%$EBIT/$\partial\Delta\%$qms	Support	443.8515%	437.9443%	416.0181%	521.7544%	476.8632%	430.8239%	405.2278%	Computed
$\partial\Delta\%$S/∂DOL	Support	10.8008%	1.9114%	7.3474%	4.7567%	4.4594%	5.3439%	5.0069%	Computed
∂Q/∂DOL	Support	0.2508%	0.2903%	0.4085%	0.6162%	0.3041%	0.1758%	0.0430%	Computed
∂DFL*/$\partial\Delta\%$qms	Support	16.9446%	63.9841%	27.2635%	34.6596%	45.9757%	39.5280%	19.7665%	Computed
∂DFL*/∂DOL	Support	1.8862%	1.2414%	2.0492%	1.6389%	2.1047%	2.2713%	1.0272%	Computed
∂EBIT/∂DOL	Support	€ 23,702,820	€ 5,509,517	€ 20,868,363	€ 19,369,934	€ 22,058,276	€ 30,807,430	€ 33,685,686	Computed
A	Support	€ 370,738,585	€ 355,801,407	€ 408,475,478	€ 503,450,476	€ 586,642,438	€ 695,241,148	€ 827,604,090	Computed
B	Support	€ 616,489,106	€ 620,232,122	€ 703,146,150	€ 739,301,986	€ 811,315,709	€ 906,572,559	€ 1,043,918,119	Computed
C	Support	€ 212,932,203	€ 283,981,966	€ 277,635,546	€ 409,632,695	€ 481,836,956	€ 536,138,478	€ 648,197,910	Computed
D	Support	€ 334,529,146	€ 308,796,822	€ 355,801,407	€ 408,475,478	€ 479,706,726	€ 559,908,510	€ 661,407,989	Computed
A_q'	$\Delta\%$qms	€ 124,829,976	€ 202,766,260	€ 193,905,189	€ 393,878,485	€ 416,948,069	€ 454,480,666	€ 553,368,518	Computed
B_q'	$\Delta\%$qms	€ 381,281,815	€ 1,045,249,347	€ 663,781,209	€ 954,677,958	€ 1,092,494,987	€ 1,059,906,079	€ 958,283,669	Computed
A_d'	DOL	€ 13,895,608	€ 3,933,856	€ 14,574,805	€ 18,624,979	€ 19,087,692	€ 26,115,233	€ 28,757,572	Computed
B_d'	DOL	€ 42,442,873	€ 20,278,818	€ 49,892,846	€ 45,143,001	€ 50,013,922	€ 60,904,008	€ 49,800,288	Computed
Z	Support	9.02481%	6.66264%	7.60839%	12.80322%	3.76775%	1.81788%	0.89823%	Computed

Table 11 presents a qualitative analysis of the infra and inter-annual relationships of DOL and DFL. DOL trend has a superior economic understanding as it is primarily affected by the dynamics of $\Delta\%uvc_t$, which appears to be the variable that most determines both a negative and a positive trend. For DFL, the essential dynamic is due to Tax Rate shocks; also, the dynamic of New Fixed Investments measured by γ and $\Delta\%i_t$ appear significant but do not determine the same inter-annual swings of the Tax Rate α_t: we could argue that DFL in the Business Case under analysis is more affected by fiscal impacts than by financial ones.

Table 10. Test of boundary conditions for i_t and α_t

Boundary Conditions	1	2	3	4	5	6	7
i	1.797%	1.309%	1.851%	2.255%	1.686%	1.001%	0.667%
i boundary	-0.464%	6.221%	0.562%	12.400%	-3.189%	-0.163%	1.079%
Boundary Direction	Lower	Lower	Lower	Lower	Lower	Lower	Lower
Verified/Not Verified	YES	NO	YES	NO	YES	YES	NO
Divergence	125.850%	-375.352%	69.659%	-449.876%	289.090%	116.284%	-61.700%
α rate	40.079%	25.715%	27.900%	0.783%	11.372%	14.262%	14.272%
α boundary	33.125%	41.082%	23.758%	28.711%	2.896%	12.870%	14.704%
Boundary Direction	Lower	Lower	Lower	Lower	Lower	Lower	Lower
Verified/Not Verified	YES	NO	YES	NO	YES	YES	NO
Divergence	17.352%	-59.759%	14.847%	-3567.617%	74.532%	9.758%	-3.027%

Table 11. DOL, DFL, and the primary variables determining trends, shocks, infra, and inter-annual relationships

	Period	1	2	3	4	5	6	7
	DOL Threshold	< 100%	< 100%	< 100%	> 100%	> 100%	> 100%	> 100%
	Trend	-	Decrease	Increase	Increase	Decrease	Increase	Increase
	$\Delta\%qms$			Fundamentally Neutral				
DOL	$\Delta\%up$ & $\Delta\%uvc$	Positive	Negative	Negative	Positive	Positive	Positive	Positive
Impact	$\Delta\%F$	Negative	Positive	Positive	Negative	Neutral	Positive	Positive
	Shock prevailing factor	$\Delta\%F$	$\Delta\%uvc$	$\Delta\%uvc$	$\Delta\%uvc$	$\Delta\%uvc$	$\Delta\%uvc$	$\Delta\%uvc$
	$\partial DOL/\partial\Delta\%qms$ Grade				Positive			
	Grade Reason		$EBIT_{t-1}$ is always positive, while $\Delta\%F_t$ is always higher than its lower boundary					
	DFL Threshold	< 100%	> 100%	< 100%	> 100%	< 100%	< 100%	> 100%
	Trend	-	Increase	Decrease	Increase	Decrease	Increase	Increase
	$\Delta\%i$	Positive	Positive	Negative	Negative	Positive	Positive	Positive
	$\Delta\%\alpha$	Negative	Positive	Negative	Positive	Negative	Negative	Negative
DFL	$\Delta\%D$	Increase	Increase	Increase	Decrease	Decrease	Decrease	Increase
Impact	$\Delta\%CI$	Increase	Increase	Increase	Increase	Increase	Increase	Increase
	DFL*	Decrease	Increase	Decrease	Decrease	Decrease	Decrease	Decrease
	Shock prevailing factor	$\Delta\%\alpha$	λ	$\Delta\%i$	$\Delta\%\alpha$	$\Delta\%\alpha$	$\Delta\%i$	λ
	$\partial DFL/\partial\Delta\%qms$ Grade	Positive	Negative	Positive	Negative	Positive	Positive	Negative
	Grade Reason	$\Delta\%i$	$\Delta\%i$	$\Delta\%i$	$\Delta\%\alpha$	$\Delta\%i$	$\Delta\%i$	$\Delta\%i$
DOL / DFL Status Infra-annual		In Phase	Out of Phase	In Phase	Out of Phase	In Phase	In Phase	Out of Phase
DOL / DFL Status Inter-annual		-	Out of Sync	Out of Sync	In Sync	In Sync	In Sync	In Sync

9. Conclusions and Discussions

Having investigated the Business Case, it is possible to draw some conclusions on the DOL-DFL nexus developed and examined so far. First of all, there is no evidence that DFL can influence DOL, making the relationship asymmetrical: both Cost of Debt i_t and Tax Rate α_t can affect DOL neither within the financial period in which the shock occurs nor in the following ones.

Second, we have to investigate the DOL-DFL nexus both at the infra-annual and inter-annual levels: the former measures the impact that the DOL exerts on DFL within the financial period calculable by the partial derivative $\partial DFL_t/\partial DOL_t$. This relationship indifferently assumes a negative or positive sign based on the contingent situation and by the DFL topology related to the DOL one: there may be a topological bias that privileges a negative relationship in steady-state; such a supposed bias requires specific investigation. We have to point out that all the cases investigated until now show a partial derivative $\partial DOL_t/\partial\Delta\%qms_t$ always positive; consequently, the algebraic sign of $\partial DFL_t/\partial DOL_t$ depends on the $\partial DFL_t/\partial\Delta\%qms_t$ one. We should investigate Business cases or *ad hoc* models in which economic conditions are less favourable or just adverse. What happens if $\partial DOL_t/\partial\Delta\%qms_t$ assumes a negative value over a long time horizon?

We have investigated the inter-annual relationship between DOL and DFL as a simple direction of the DOL-DFL variation compared to the previous financial period; even here, we have no evidence of an exclusively negative relation since the positive ones are also possible.

The third aspect we have addressed is the possibility to link the Net Profit Π_t in a mathematically and economically convincing way to some factors that can be easily derived algebraically with respect to other economic quantities, allowing us to delve into the relationship among them and the systematic risk β linked to the Net Profit through a few simple algebraic and stochastic elaborations: see Mandelker et al. (1984) and equation (3 bis). I hope that the present paper has enlarged the way to obtain many other essential insights and contributions to link the systematic risk to its real determinants: see, for instance, the fundamental papers by Gahlon et al. (1982), Brenner et al. (1977 and 1978).

Fourth, we cannot rule out the possibility that firms, able to influence the factor and product markets in a macroeconomic framework favourable, can also tune the DOL-DFL nexus. Such an oligopolistic power should match with a considerable planning effort. Even if there is no oligopolistic dominance on the factor and product markets, a firm may still be able to influence DOL and DFL to improve its profitability but, an effort is required to compensate for the lower degree of freedom in the number of variables addressable by the managers that should adopt an even greater planning capability. Empirical tests should aim at identifying how many and how long firms are systematically able to achieve a Degree of Total Leverage DTL greater than or close to 1 over a significant time horizon.

Many firms may consider some determinants of Net Profit Π_t as exogenously specified, including DOL and DFL but excluding Revenue growth. That consideration is not always acceptable and advisable; it is not a policy a company pursues in the long term for the sake of the company's profitability. The role of management is precisely to develop the company profitably: how it accomplishes such a task is irrelevant in the present paper, the important thing is to have shaped the tools to measure both the final result and how it is achieved: in this respect, the equation (3 bis) seems an interesting way.

The DOL-DFL nexus requires further investigation, both theoretical to achieve improved mathematical models and empirical to appraise company performance and management conduct over long time horizons. A substantial enhancement might be an in-depth assessment of how DOL, DFL, Revenue growth, and their determinants can impact both risk and systematic risk β in the short and long run.

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Appendix A

Partial derivatives

The present Appendix is a deep dive into some partial derivatives that do not worth including in paragraph 6 to keep the reader focused on the main topic.

The first partial derivative to inspect is the equation (35):

$$\frac{\partial DFL_t}{\partial \Delta\%qms_t} = \frac{[1-N_t+Q_t*(1+\Delta\%S_t)]}{(DOL_t*\Delta\%S_t)^2} * \frac{\partial \Delta\%EBIT_t}{\partial \Delta\%qms_t} +$$

$$-\frac{A_{t-1}*M_t*T_t*(1+\Delta\%up_t)}{DOL_t*\Delta\%S_t*DFL^*_{t-1}} * [DFL^*_t * K1_{t-1} * (K2_t + K3_t) + \psi_t * (1 + \Delta\%qms_t) * \frac{\partial DFL^*_t}{\partial \Delta\%qms_t}] \tag{35}$$

based on several partial derivatives.

For the first one, the solution is the following:

$$\frac{\partial \Delta\%EBIT_t}{\partial \Delta\%qms_t} = PDOL_t + \frac{S_{t-1}*\Delta\%up_t - VC_{t-1}*\Delta\%uvc_t}{EBIT_{t-1}} \tag{A.1}$$

To get the second partial derivative:

$$\frac{\partial DFL^*_t}{\partial \Delta\%qms_t} = \frac{(1-\alpha_t)}{1-i_t*(1-\alpha_t)} * \{\frac{\partial \Delta\%EBIT_t}{\partial \Delta\%qms_t} - i_t * CI_{t-1} * [(1 + \Delta\%S_t) * \frac{\partial \psi_t}{\partial \Delta\%qms_t} + \psi_t * \frac{\partial(1+\Delta\%S_t)}{\partial \Delta\%qms_t}]\} \tag{A.2}$$

the starting point is the definition of DFL^*_t:

$$DFL^*_t = \frac{D_t}{D_t+E_t} = 1 - \frac{E_{t-1}-Div_{t-1}+\frac{\{EBIT_t-i_t*[CI_{t-1}*\psi_t*(1+\Delta\%S_t)-(E_{t-1}-Div_{t-1})]\}*(1-\alpha_t)}{1-i_t*(1-\alpha_t)}}{CI_{t-1}*\psi_t*(1+\Delta\%S_t)} \tag{A.3}$$

where:

$$Div_{t-1} = \delta_{t-1} * \Pi_{t-1} \tag{A.4}$$

stands for the dividend approved for the period *t-1* based on the payout ratio δ_{t-1}.

Besides, we must solve two other derivatives to complete the equation (A.2), of which the first one is the following:

$$\frac{\partial \psi_t}{\partial \Delta\%qms_t} = \frac{K1_{t-1}*K3_t - K4_t}{(1+\Delta\%S_t)*(1+\Delta\%qms_t)} \tag{A.5}$$

The second one is quite simple:

$$\frac{\partial(1+\Delta\%S_t)}{\partial \Delta\%qms_t} = 1 + \Delta\%up_t \tag{A.6}$$

We must unroll the partial derivative presented in equation (41):

$$\frac{\partial DFL_t}{\partial DOL_t} = \frac{[1-N_t+Q_t*(1+\Delta\%S_t)]*\frac{\partial \Delta\%EBIT_t}{\partial DOL_t}}{(DOL_t*\Delta\%S_t)^2} - \frac{(1+\Delta\%S_t)*\frac{\partial Q_t}{\partial DOL_t}+Q_t*\frac{\partial(1+\Delta\%S_t)}{\partial DOL_t}}{DOL_t*\Delta\%S_t} \tag{41}$$

The first derivative to solve is the following:

$$\frac{\partial \Delta\%EBIT_t}{\partial DOL_t} = \frac{\partial (DOL_t * \Delta\%S_t)}{\partial DOL_t} = \Delta\%S_t + DOL_t * \frac{\partial \Delta\%S_t}{\partial DOL_t} \tag{A.7}$$

It follows that:

$$\frac{\partial (1+\Delta\%S_t)}{\partial DOL_t} = \frac{\partial \Delta\%S_t}{\partial DOL_t} = \frac{EBIT_{t-1}*(\Delta\%S_t)^2}{FC_{t-1}*\Delta\%F_t - VC_{t-1}*\{(1+\Delta\%uvc_t)*\left[\frac{\Delta\%up_t}{(1+\Delta\%up_t)}+\varepsilon_t\right]-\Delta\%uvc_t\}} \tag{A.8}$$

The last partial derivative to solve that appears inside equation (41) is the following:

$$\frac{\partial Q_t}{\partial DOL_t} = \frac{A_{t-1}*M_t*T_t}{DFL^*_{t-1}} * \left(DFL^*_t * \frac{\partial \psi_t}{\partial DOL_t} + \psi_t * \frac{\partial DFL^*_t}{\partial DOL_t}\right) \tag{42}$$

Equation (42) needs two other partial derivatives to be solved where the first one is quite simple:

$$\frac{\partial \psi_t}{\partial DOL_t} = \frac{\left(K1_{t-1}*K3_t - K4_t\right)*\frac{\partial \Delta\%S_t}{\partial DOL_t}}{(1+\Delta\%S_t)^2} \tag{A.9}$$

while the last one is complex:

$$\frac{\partial DFL^*_t}{\partial DOL_t} \tag{A.10}$$

and it needs to be split into several parts, starting from the definition of DFL^*_t found in the equation (A.3):

$$DFL^*_t = \frac{D_t}{D_t + E_t} = 1 - \frac{E_{t-1} - Div_{t-1} + \frac{\{EBIT_t - i_t*[CI_{t-1}*\psi_t*(1+\Delta\%S_t) - (E_{t-1}-Div_{t-1})]\}*(1-\alpha_t)}{1-i_t*(1-\alpha_t)}}{CI_{t-1}*\psi_t*(1+\Delta\%S_t)} = 1 - \frac{A}{B} = 1 - A*(B)^{-1} \tag{A.3}$$

where:

$$B = CI_{t-1} * \psi_t * (1 + \Delta\%S_t) \tag{A.11}$$

and:

$$A = E_{t-1} - Div_{t-1} + \frac{\{EBIT_t - i_t*[CI_{t-1}*\psi_t*(1+\Delta\%S_t) - (E_{t-1}-Div_{t-1})]\}*(1-\alpha_t)}{1-i_t*(1-\alpha_t)} = D + \frac{(1-\alpha_t)}{1-i_t*(1-\alpha_t)} * [EBIT_t - i_t*(B-D)] \tag{A.12}$$

where:

$$D = E_{t-1} - Div_{t-1} \tag{A.13}$$

Going back to the equation (A.10), eventually, we can solve it in the following way:

$$\frac{\partial DFL^*_t}{\partial DOL_t} = \frac{\partial [1-A*(B)^{-1}]}{\partial DOL_t} = -\frac{B*\frac{\partial A}{\partial DOL_t} - A*\frac{\partial B}{\partial DOL_t}}{B^2} \tag{A.14}$$

where:

$$\frac{\partial A}{\partial DOL_t} = \frac{(1-\alpha_t)}{1-i_t*(1-\alpha_t)} * \left(\frac{\partial EBIT_t}{\partial DOL_t} - i_t * \frac{\partial B}{\partial DOL_t}\right) \tag{A.15}$$

and:

$$\frac{\partial B}{\partial DOL_t} = CI_{t-1} * K1_{t-1} * (K2_t + K3_t) * \frac{\partial \Delta\%S_t}{\partial DOL_t} \tag{A.16}$$

Equation (A.15) needs the following solution to be solved as well:

$$\frac{\partial EBIT_t}{\partial DOL_t} = EBIT_{t-1} * \left(\Delta\%S_t + DOL_t * \frac{\partial \Delta\%S_t}{\partial DOL_t}\right) \tag{A.17}$$

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