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Conceptualizing Mathematics as Discourse in Different Educational Settings

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Abstract

In this work, we focus on a relatively new theory in mathematics education research, which views thinking as communication and characterizes mathematics as a form of discourse. We discuss how this framework can be utilized in different educational settings by giving examples from our own research to highlight the insights it provides in the context of mathematics education. We conclude with the affordances and constraints of the theory for mathematics education research and practice.

Keywords: educational theories, methodologies for educational research, mathematics education, discourse

1. Introduction

The field of mathematics education has been influenced by various theories with different characterizations of cognition and development, resulting in different implications for learning and teaching. There is extensive literature and cumulative knowledge generated over time on behaviorism, constructivism, social constructivism, and socio-cultural views as researchers have been testing the assumptions and applications of each theory. In this paper, we focus on Sfard's (2008) discursive framework, a relatively new theory, for which the process of generating cumulative knowledge is at its early stages. Our goal is to contribute to this knowledge base by bringing together three distinct studies based on this framework. In particular, our aim is to provide mathematics education researchers with more insights about the affordances and constraints of the framework for mathematics education research and practice.

2. Theoretical Framework

Consistent with the assumptions of socio-cultural theories, Sfard's (2008) discursive framework highlights the importance of communication and engagement in pre-established cultural practices in human development. In her framework, Sfard (2008) considers cognitive processes and interpersonal communication as facets of the same phenomenon and views thinking as communication (not necessarily verbal or visible). She rejects the dichotomies between thought and language as well as the separation between individual and social aspects of learning inherent in the constructivist approaches. (Note 1) She considers learning as moving towards a more sophisticated mathematical discourse through participation and conceptualizes mathematics as a discourse—a form of communication that is distinguishable by its word use, visual mediators, routines, and narratives. *Word use* refers to the use of mathematical vocabulary in participants' discourses and is a critical factor in mathematical communication. *Visual mediators* refer to all the visible objects that are created and acted on for mathematical communication. Some examples of visual mediators in mathematical discourse include graphs, symbols, diagrams, and written words. *Routines* are the "set of metarules that describe a repetitive discursive action" (Sfard, 2008, p. 208), where metarules "define the patterns in the activity of discursants" (Sfard, 2008, p. 201) and "guide the general course of communicational activities" (Sfard, 2001, p. 29). Routines are often tacit and they regulate *when* participants perform a particular activity and *how* they perform it (Sfard, 2008). The term routine is broad since it can refer to many different metarules defining the discourse-specific patterns in participants' actions. For example, one can focus on the routines regulating the intentions of participants (e.g., answering a question to please the teacher), or those regulating participation (e.g., raising hands before speaking in the classroom), or the routines regulating how participants think about mathematical objects (e.g., using the assumption of continuity when thinking about functions). *Narratives* refer to the set of spoken or written utterances about mathematical objects and their relationships that the participants consider as true or false in relation to their word use, visual mediators, and routines. The narratives participants consider as true in their mathematical discourses are called *endorsed narratives*. Some of the narratives endorsed by the experts of mathematical communities (e.g., mathematicians and teachers) include definitions, axioms, and theorems.

Participants' word use, visual mediators, routines, and narratives about mathematics can be idiosyncratic. The way learners use these elements can differ from their utilization by the experts of mathematical discourse (e.g., teachers, mathematicians). Sfard (2008) notes that one of the goals of school learning is to help students change these elements of their discourses so that they can better participate in the historically established activity of mathematics. Therefore, from this lens, change in one's mathematical discourse is an indicator of learning.

There are two possible ways to interpret Sfard's (2008) framework: as a way to think about cognition, mathematics, learning, and teaching (theoretical), and as a source that offers some analytical tools with which to analyze data (methodological). The theoretical and methodological aspects of the framework are complementary, compatible, and consistent. From this perspective, mathematics is viewed as a discourse; mathematical learning is formulated as change in discourse; the unit of analysis is discourse; and the claims are about participants' discourses that are characterized by their word use, visual mediators, routines (actions), and endorsed narratives. This contrasts with the behaviorist approaches where participants' actions and utterances are used as proxies for the habits, associations, and responses they form based on stimuli without an explicit focus on their thinking processes. Sfard's lens also contrasts with the constructivist approaches where the empirical explorations of participants' actions and utterances serve as proxies for their mental schema, mental processes, or tacit models, which cannot be observed directly.

Methodologically, the formulation and operationalization of discourse is a critical aspect of Sfard's (2008) theory that differentiates it from some other discursive approaches. There is no single theory of discourse analysis, the vastness and diversity of which lead to different methodological approaches. Traditionally, discourse analysis is used to mainly explore linguistic phenomena such as word meaning, intonation, grammar, phonetics, discourse flow, indexicality, linguistic sign processes, and various speech acts (Schiffrin, Tannen, & Hamilton, 2003). Although language and word use are critical aspects of Sfard's framework, her formulation of discourse is broader. As will be demonstrated by our studies, this broader conceptualization takes into account various features of mathematical activity that cannot be captured only through word use and enables the examination of the dynamic and interrelated nature of the four aforementioned discursive elements in mathematical thinking.

Next, we present three studies conducted by each of the authors separately. Each study uses Sfard's (2008) framework to address various issues in mathematics education. Since our main focus is to reflect on this framework, we summarize our studies briefly to discuss the insights Sfard's framework provided for each study rather than providing detailed accounts of the studies. Based on the summaries, we then provide an overall synthesis of the affordances and limitations of Sfard's framework for mathematics education research and practice.

3. Examining Classroom Discourse on Limits

There is extensive research on the limit concept and the challenges it presents to students and many researchers have explored student thinking about limits through a cognitivist framework. However, research on limits that takes into account the social context of learning and the roles teachers play in student learning is scarce. Güçler's (2013) study examined the classroom discourse on limits using Sfard's framework based on the assumptions that learning is essentially a social process and teachers play important roles in classroom communication. The study focused on one instructor's and his students' discourses on limits in a beginning-level calculus classroom at a large Midwestern university in the U.S. The data consisted of videotaped classroom observations where the instructor talked about limits; a diagnostic survey given to all the students at the end of their instruction on limits; and task-based interviews conducted with four students based on their responses to the survey. The participants' discourses were analyzed with respect to the four elements of discourse outlined by Sfard (2008).

The results of the study indicated that the participants used similar discursive features (e.g., words, graphs, and the limit notation) in quite different ways. The analysis of participants' *word use* revealed that the instructor used the word *limit* in an objectified manner (referring to limit as a number or a static mathematical object), whereas the students mainly used it in an operational manner (referring to limit as a process through dynamic motion). When the instructor used words operationally (e.g., "as the x values approach the limit point, the function values get closer and closer to the limit value"), he did so sparsely. Further, he clearly distinguished the process of finding the limit—the exploration of the function values as the x values approach the limit point—from the mathematical object obtained as a result of that process (e.g., "if the limit exists, it is equal to a number"). The analysis also showed that the instructor utilized operational word use only verbally whereas he wrote all of his objectified utterances on the board. In this study, the students adopted the spoken rather than written words of the instructor indicating the importance of the consistency between spoken and written words in teaching and the impact they may have on student learning.

The differences between participants' word use was also reflected in their *routines* of using particular metaphors about limits. Using the metaphor of discreteness as a routine in the discourse on limits supports the objectified view whereas using the metaphor of continuous motion as a routine supports the process view of the concept (Lakoff & Núñez, 2000). Such routines are often tacit but significantly influence the other elements of mathematical discourse such as word use and endorsed narratives (Sfard, 2008). For example, the instructor's dominantly objectified word use (e.g., "the limit is 1", "what is the sum of these limits?") was based on the routine of using the metaphor of discreteness, whereas the students' mainly operational word use (e.g., "the limit is approaching 1", "the limit is about movement") was based on their routines of using the metaphor of continuous motion. These distinct routines and word use also influenced participants' *endorsed narratives*. The instructor's word use and routines helped him endorse his primary narrative 'limit is a number', whereas the students' word use and routines helped them endorse their only narrative 'limit is a process'.

The analysis of the *visual mediators* in terms of graphs and symbolic notation revealed that the instructor used graphs as teaching aids and mainly used algebraic notation when representing functions. In contrast, the students relied heavily on graphs to make sense of the functions whose limits they explored. The instructor also used the limit notation to represent the limit of a given function but the students struggled with the notation and they often used it inaccurately (e.g., writing $\lim f(x) \to L \text{ as } x \to a$ rather than using the equal sign, which signifies that the limit of $f(x)$ is *equal to L*).

Despite such discrepancies between the instructor's and students' discourses on limits, the students in the study considered their discourses as compatible with the instructor's discourse suggesting that the distinct features of the instructor's discourse remained implicit for them. Such tacitness may result in possible miscommunication in the classroom in terms of the varying roles and uses of the four elements of discourse in different mathematical contexts.

For this study, Sfard's framework provided insights that could not be gained through theoretical lenses such as behaviorism and constructivism. The students adopted particular aspects of the instructor's discourse (e.g., operational word use and specific visual mediators such as graphs) and expressed their realizations of limit as compatible with the instructor's discourse indicating that learning is not only an individual but also a social process. Sfard's framework also provided unique insights for this study compared to other discursive approaches that conceptualize discourse merely as language. Sfard's broader formulation of discourse as consisting of word use, visual mediators, routines, and endorsed narratives helped explore the instances of communicational failures in the classroom by providing the tools with which to specify *when and how* participants' discourses differed from each other. By making their elements of discourse explicit topics of discussion in the classroom and highlighting the mathematical contexts in which particular discursive tools are used or abandoned, teachers can play critical roles in the development of students' mathematical discourses in the classroom. The framework can be useful for teachers in developing specific strategies to approach their students in terms of these four characteristics of mathematical discourse.

4. Language-Dependent Nature of Mathematics Learning

The second study (Kim et al., 2012) investigated and compared how native-English and native-Korean speaking university students, who received their education in the U.S. and in Korea respectively, thought about the concept of *infinity*. The primary motivation for this study was the lexical discontinuity of the Korean infinity discourse and the lexical continuity of the English infinity discourse between everyday and mathematical discourses. The four characteristics of students' discourses on *infinity* were identified using Sfard's analytic framework to explore how differences between the two languages affect the learning of mathematics. The students' discourse was scrutinized with an eye to the common characteristics as well as language-related differences.

University students from several calculus classes in the U.S. and Korea were recruited for the study. The comparisons of the U.S. and Korean groups were based on the analyses of students' responses to a survey and discourse analyses of paired interviews. After collecting the survey data, 132 English-speaking and 126

Korean-speaking university students' written survey responses were analyzed and categorized within each group and representatives from each group for the interview study were chosen. Twenty representatives in each group were interviewed in pairs, and discourse analyses of the interview data were conducted on the basis of the analytic framework. Finally, the written responses to mathematics problems on the survey were reanalyzed to make conjectures about the generality of the characteristics observed in the interviews.

The findings showed that the non-mathematical use of the word *infinity* by English speakers was processual, whereas Korean speakers' word use was more formal, mathematical, and structural. Processual use means employing expressions such as "increasing," "getting bigger," "keeps going on," and "gets higher," whereas structural use means using terms like "sets," "number of elements," "sizes," "values," and "last term." In developing their mathematical discourses on *infinity*, English speakers' mathematical uses of *infinity* were processual due to the processual nature of the non-mathematical uses of the word *infinity* in everyday English. For instance, the majority of English speakers employed *infinity* in conjunction with an infinite process to compare two infinite sets (e.g., "goes on and on") as a *routine*. In contrast, Korean speakers' mathematical uses of *infinity* were more consistent with, and possibly developed from, the structural discourse of mathematical textbooks, perhaps due to lack of proficiency or experience with the word in non-mathematical discourse. Unlike the English speakers, Korean students' word use on *infinity*, mainly learned in school and university, did not involve a successive transformation of a previously existing discourse and required the development of *infinity* as a new entity instead. Consistent with this lexical discontinuity between the non-mathematical and mathematical uses of *infinity*, Korean speakers' talk about *infinity* in their comparison routine was more formal and mathematical (e.g., cardinal numbers associated with sets). Thus, students' routines seemed to be grounded in the characteristics of their word use in non-mathematical discourse (i.e., processual uses of the word *infinity* in English versus structural uses due to lack of experience with the non-mathematical uses of the word *infinity* in Korean).

Moreover, these differences in the use of the word *infinity* in non-mathematical discourse between the U.S. and Korean groups seemed to result in differences in *endorsed narratives* and *visual mediators* in mathematical discourse. For instance, English speakers formulated interpretations in their own words to define *go to infinity*. Their responses were relatively consistent and in tune with the canonical interpretation of the word *infinity* in the process of unbounded or never-ending growth. In contrast, Korean speakers used more formal mathematical formulations about the notion of *go to infinity* (e.g., use of formal languages such as "converge" and "limit"). Their interpretations were more diverse and often inconsistent with the canonical interpretations. All these findings provide strong support for the conjecture that learning trajectories of English and Korean speakers can be somewhat different. Therefore, it is concluded from the study that the mathematical discourse of U.S. English speakers seemed to be growing continuously from their non-mathematical discourse on *infinity*, but there was no such continuity in the case of Korean speakers.

In this study, Sfard's framework was a promising way to explore the language-dependent nature of mathematics learning because each language has different developmental trajectories and infra-discursive lexical structures. According to the findings of this study, the framework can show how students' non-mathematical discourse has an impact not only on their later use of mathematical keywords, but also on other aspects of their mathematical discourse, such as endorsed narratives, routines, and visual mediators. Moreover, the framework can be used to analyze and compare language-dependent properties of students' mathematical discourse in word use, visual mediators, routines, and endorsed narratives in comparative international studies. Based on language-specific features of the discourse, the analytic framework can show how to develop a bottom-up processing for building scaffoldings in conjunction with linguistically different needs of students in particular and culturally different discursive needs of students in general.

In research on student thinking and development, Sfard's framework plays a valuable role in investigating how language is the source and foundation and the sole condition on which discursive transformations occur. In these discursive transformations, the framework can also serve as a foundation for a discourse-oriented approach to mathematics education because language, its characteristics, and the four distinctive features of students' discourse are so tightly interwoven into the fabric of their discursive development. While analyzing the four distinctive characteristics of discourses in comparative studies, the framework can reveal language-dependent means and discourse-dependent transformation methods to support meaningful learning for the growth of mathematical discourses.

5. Revisiting the van Hiele Model of Thinking through a Discursive Lens

Many researchers have used Sfard's discursive framework to investigate students' mathematical discourses in the areas of algebra, number sense, functions, and calculus. In her study, Wang (2011) focused on prospective teachers' geometric discourses in the context of quadrilaterals. In particular, she translated the five distinct levels of the van Hiele model (1959/1986) into discursive terms and viewed each van Hiele level as a level of geometric discourse. She hypothesized that using this approach would provide detailed analysis of students' geometric discourses in terms of (a) their use of geometric words and the meaning of those words (*word use*); and (b) the patterns in students' discourse-specific actions (*routines*) at each van Hiele level—an area that is not extensively examined in the van Hiele model. In addition, she used the framework to examine the development of prospective teachers' geometric thinking through the analysis of the changes in their geometric discourses—an area that needs more attention in the van Hiele model. The results of the study showed the differences in students' geometric discourses within a van Hiele level across different students, as well as within an individual student's discourse, and provided evidence that changes in prospective teachers' geometric thinking can be detected by the analysis of their discourses. In what follows, we provide some details and findings of the study.

The study started with 63 prospective elementary teachers in a mid-western university in the U.S., who participated in a pre- and post- van Hiele geometry test (Usiskin, 1982). The pre-test was administered at the beginning of the semester and the post-test was completed 10 weeks later. Of those prospective teachers, 20 participated in a pre-and post- interview soon after the pre-and post-tests. The interviews were designed to explore prospective teachers' geometric thinking through one-on-one interactions. All interview tasks were designed to elicit participants' geometric thinking about quadrilaterals and were aligned with the van Hiele geometric test items. One interviewer conducted all the interviews, which began with the same tasks and initial interview questions. However, some of the interview prompts varied based on the interviewees' responses to the tasks and questions (Wang, 2011). Each participant was assigned to a particular van Hiele level according to the van Hiele geometry test grading guidelines (Usiskin, 1982). Participants' interview analyses were merged with their van Hiele test results to determine the differences in their geometric discourses within a van Hiele level, and to identify the changes in their geometric discourses throughout the study.

The findings showed the differences in participants' geometric discourses at a given van Hiele level as well as the differences in the discourse of an individual student. For example, two participants' van Hiele test results suggested that they were at Level 3 (Theoretical) according to the van Hiele model – a level during which students can identify polygons by their definitions and have developed some informal deductive reasoning (van Hiele, 1959/1986). However, the analysis of their interviews in terms of their use of the words 'parallelogram', 'rectangle', 'rhombus', and 'square' shed light on how they used these words and what they meant when those words were used. For example, when participants were asked to identify and group quadrilaterals, one participant demonstrated a *complete* classification of the hierarchy among parallelograms viewing parallelograms, rectangles, squares and rhombi as interconnected based on their definitions (a salient characteristic of Level 3 thinking), whereas another participant showed a *partial* classification of the hierarchy by viewing parallelograms and rectangles as parallelograms, excluding squares and rhombi (Wang, 2011). In this case, both participants were able to use definitions to identify and classify parallelograms but the analysis of their word use and routines revealed different degrees of understanding of the definitions as part of Level 3 thinking while they were moving towards a higher van Hiele level (Gutierrez, Jaime, & Fortuny, 1991).

In a different case, a participant's pre- and post-test results showed no change in her van Hiele levels (i.e., Level 2-2, a level in which learners begin to recognize figures by their properties). On the other hand, the analyses of her word use during the interviews revealed different meanings she generated when using the word *parallelogram,* and the observations of her routines indicated that she developed some informal deductive reasoning at the time of the post-interview (Wang, 2011). For instance, the participant referred to *parallelograms* as "all polygons with pairs of parallel sides" (i.e., parallelograms, hexagons, etc.) in the pre-interview, and used the word according to the mathematical properties of a parallelogram (i.e., 'four-sided figure', and 'parallel sides', etc.) in the post-interview. In terms of her justification routines, the participant (incorrectly) used Angle-Side-Side to show two triangles were congruent but this routine suggested that she was moving towards Level 3 in the van Hiele model as she began to understand proofs and use deductive reasoning (Wang, 2011).

The van Hiele model continues to be the best-known theoretical framework to understand students' geometric thinking at different levels. Wang's approach to examine the van Hiele model with the characteristics of Sfard's mathematical discourse (word use and routines) demonstrated the details of these prospective teachers' geometric thinking across various van Hiele levels as well as within a level. Sfard's framework not only enhanced our understanding of the range of word use and routines inherent in each van Hiele level, but also helped demonstrate the benefit of revisiting the van Hiele model with a discursive lens to gain further insights about students' development of geometric thinking.

6. Affordances, Implications, and Limitations of the Framework

In this section, we first present our synthesis of the affordances of Sfard's (2008) framework for mathematics education research in light of the studies presented above. We then discuss the implications of the framework for practice and conclude with some limitations of the framework.

Formulating learning as a culturally influenced, dynamic, continuous process, Sfard's framework helped us analyze this learning process dynamically and continuously (e.g., routines). Through this lens, we could analyze not only the end result but also the process of learning, particularly when examining developmental trajectories of learners (e.g., from everyday to mathematical discourse).

The framework helped us highlight the importance of the relationships among different discursive elements rather than characterizing discourse as consisting only of language. Although language is a critical element of the framework, a comprehensive examination of mathematical discourse is possible when language is explored in relation to the other features of discourse (e.g., visual mediators, routines). Consistently, this lens helped us incorporate not only the quantitative/syntactic but also the qualitative/semantic features of language in our analyses.

We believe the framework is applicable to different educational contexts whether the focus is on teaching, learning, different grade levels or learning settings (classroom or individual). Moreover, the framework lends itself to the analysis of different mathematical topics (e.g., number sense, geometry, calculus concepts), indicating a richness in terms of its scope as a theory.

The framework helped us avoid oversimplified views of learning and teaching and allowed for rich descriptions through its focus on the contextual, communal, cultural, dialogical, and dynamic nature of participants' discourses in mathematics. By doing so, it accounted for possible differences in individual students' thinking as well as the social and cultural context of their learning. Such an approach, together with the formulation of thinking as a form of communication, dissolves the dichotomy between individual and social aspects of thinking and learning.

In terms of implications for practice, Sfard's framework provides insights for teachers to enhance communication in their classrooms by attending explicitly to particular aspects of mathematical discourse that can remain implicit for their students (Güçler, 2013). The framework provides insights regarding the importance of culture and language in making sense of particular mathematical concepts and teachers' awareness of the influences of culture and students' uses of everyday language on mathematical learning (Kim et al., 2012). The framework also underlines the importance of teachers' awareness about the development of their students' discourses in terms of the discursive variations across students as well as within an individual student (Wang, 2011).

No single theory can address the complexity and multiplicity of issues surrounding mathematics education. Every conceptual model and theory brings forward particular aspects of and approaches to education while obscuring or ignoring others (Schoenfeld, 2007). The multiplicity of theories in our field have merits in terms of providing various approaches with which we can interpret phenomena but we need to be careful not to use a theory in our research uncritically (Lerman, 2010). It is in that spirit we now outline some limitations of Sfard's (2008) framework and, when possible, offer adaptations of or extensions to the theory.

Sfard (2001, 2008) has focused on the analysis of routines that underlie participatory regularities of discourse more than the routines that underlie the object-level narratives of mathematics (e.g., use of metaphors as a routine to talk about mathematical objects). We want to note that the broad description of routines in the framework does not prohibit but instead supports the analysis of metaphors (e.g., Güçler's (2013) study), so the examination of routines underlying the object-level narratives is more about possible interpretations or extensions of the framework rather than constituting a limitation of the framework.

Another critical issue to consider is whether the elements in the framework account for all types of mathematical activity. For example, Sfard (2008) considers visual mediators as the main form of medium that supports mathematical discourse but there are other modes of thinking (e.g., kinesthetic and auditory) that can play roles in mathematical learning (Presmeg, 1997). It may be important to extend the notion of visual mediators to be able to explore body language, gestures, and motor actions as part of mathematical communication. Further, it may also be important to extend the notion of mediators in mathematical discourse beyond visual mediators to account for auditory, kinesthetic, and tactile perceptions and thinking that cannot be captured only in visual mode (see Güçler et al. (2013) for an example of such extension).

Finally, Sfard (2008) mainly focuses on the learners' progressions from everyday to mathematical discourses. For student learning, such a view may prioritize the role of formal or scholarly mathematical discourse over other forms of mathematical discourses (e.g., lyrical discourses, metaphor-based discourses, or gesture-based discourses as a form of non-verbal mathematical discourse), which may have different purposes and functions in overall human development since learning can occur beyond school settings.

Although Sfard's framework may not be able to address all of the issues regarding mathematics education, we believe it is a useful analytical lens with which we can gain more information about the learning and teaching of mathematics in different contexts in a way that takes into account the complexities inherent in those processes.

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Note

Note 1. Although constructivists extensively focus on individuals' thinking processes, they distinguish between thinking, which they view as an internal activity, from communicating, which they view as an interpersonal activity. Many constructivists (and some social constructivists) emphasize individual construction as the main factor in development and consider language as occurring as a result thinking (Miller, 2002) rather than highlighting the symbiotic relationship between them.