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Equations, Functions, Critical Aspects and Mathematical Communication

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Abstract

The purpose of this paper is to present the mechanism for effective communication when the mathematical objects of learning are equations and functions. The presentation is based on data collected while the same object of learning is presented in two classes, and it includes two teachers and 45 students. Among other things, the data consists of video-recordings of lessons and tests. In the analysis, concepts relating to variation theory have been used as analytical tools. The results show that effective communication occurs in the classroom if it has the critical aspects in students learning as its starting point. The communication in the classroom succeeds or not if the aspects of the content supposed to be treated is the same as or different from the aspects of the content of the teacher's representation, and if the aspects of the content of the teacher's representation are the same as or different from the aspects discerned by the students. The results also show that the students cannot make sense of the difference between the highest/lowest value of a quadratic function and the maximum/minimum point; the difference between a quadratic equation and function; the students also have difficulties in solving a quadratic equation if it appears in a new context. The argument of the functions is identified as critical aspect in this study.

Keywords: communication, equations, functions, teaching, learning, dimensions of variation

1. Introduction

1.1 The Specific Problem

Mathematical knowledge is seen as an important requirement to develop society. Despite the increased interest in people with deeper mathematical knowledge, there is a constant stream of new articles which indicates that students have unsatisfactory knowledge in mathematics. Because equations and functions are often conveyed in symbols, oral and written communication about mathematical ideas is recognized as an important part of mathematics education. Students do not necessarily talk about mathematics naturally; teachers need to help them learn how to do so. The main questions in this paper are: How can theory help us to understand and support students' developing mathematical learning? It is possible to understand the mechanism for an effective communication which may lead to students' understanding pattern and structure, the logical analysis, and calculation with patterns and structures when working with algebra and functions during the classrooms lessons? My hypothesis is that through the communication that occurs in the classroom (e. g. listening, talking and writing), students are prompted to organize, re-organize and consolidate their mathematical understanding, as well as analyze, evaluate and build on the mathematical strategies of others.

The basic idea of the mathematical theory of communication, as developed by Claude Shannon:

> The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message that has been selected at another point. (Shannon, 1949, pp. 31)

The success or failure of communication is a matter of the relation between thought contents of speaker and hearer (Frege, 1918). Research on effective communication primarily focuses on a process-oriented approach where the focus is on the transfer of messages, coding and analysis (e.g., Nilsson & Waldemarson, 1990). In addition, there is a semiotic line where discussions take place about how messages interact with humans to create meaning (e.g., Morgan, 2006; O'Halloran, 2005), as well as a socio-cultural approach in which communication is defined as an activity which attempts to get an interlocutor (possibly oneself) to act or feel in a certain way (Sfard, 2002). Sfard (2002) found that communication is effective if communicative aims are met and if the discourse focus is clear. Sfard (2002) defined the concept discourse as a dynamic process denoting a specific act of communication, verbal or not, with others or with oneself, synchronic (e.g. communication face to face) or asynchronous (e.g. reading a book, writing). She also proposed the following definition of communication:

Communication is a collectively performed patterned activity in which action A of an individual is followed by action B of another individual so that:

1) A belongs to a certain well-defined repertoire of actions known as communicational
2) Action B belongs to a repertoire of re-actions that fit A, that is, actions recurrently observed in conjunction with A. This latter repertoire is not exclusively a function of A, and it depends, among others, on factors such as the history of A (what happened prior to A), the situation in which A and B are performed, and the identities of the actor and re-actor. (Sfard, 2008, pp. 86-87)

Pagin (2008) used the term communication as something that takes place in individual communicative events. The event is a process that starts with some inner state (the states must be mental, or private, or unobservable) of the sender and ends with some inner state of the receiver. A signal is transmitted between sender and receiver. The relevant inner state of the sender takes part in causing the signal, and the signal in turn takes part in causing the relevant inner state of the receiver. This is not sufficient for a definition of communication, but it indicates essential ingredients.

1.2 Theoretical Assumptions

The theoretical framework used in this study is the variation theory (Marton & Booth, 1997; Marton & Tsui, 2004). There are several reasons for the choice of this theory. First, the object of learning is the focus on a teaching situation. The object of learning is formed of three components: the intended, the enacted and the lived object of learning. The intended object of learning refers to the part of the content that students should learn and which is supposed to be treated in the classroom. The enacted object of learning is what appears in the classroom and refers to what is possible for students to experience within a learning environment. The intended and enacted objects of learning can be compared to determine whether what is being taught matches what was intended to be taught. The students' initial level of capability to the appropriate object of learning as well as the way in which students understand the object of learning is the lived object of learning. Second, the central idea in variation theory is that to discern certain aspects of the object of learning, a person needs to experience variation corresponding to those aspects (Marton et al., 2004). Some of those aspects are critical aspects in students' learning. A critical aspect is the capability to discern aspects presented, for example in algebraic structures by experiencing them. To experience an equation or a function is to experience both its meaning, its structure (composition) and how these two mutually constitute each other. So neither structure nor meaning can be said to precede or succeed the other. If these aspects are not focused on in a teaching situation or in textbooks, they remain critical in the students' learning (C. Olteanu & L. Olteanu, 2010, 2011). In classroom situations, it is very important that the teacher is able to bring critical features of the object of learning into students' focal awareness. Third, learning theory of variation serves as a useful theoretical framework to help teachers plan and structure their lessons. It guides them to decide what aspects to focus on, which ones to vary simultaneously, and which to keep invariant or constant. Furthermore, it guides teachers to consciously design patterns of variation to bring about the desired learning outcomes. Fourth, the student's lived object of learning can be compared against categories of description as a means of assessing the level of learning achieved or against the enacted level of learning to determine whether the enacted object of learning is being transferred to the lived object of learning as expected.

1.3 Dimensions of Variation

Marton et al. (2004) argue that in order to discern different aspects of the object of learning, variation must be experienced in these aspects. An aspect is defined as the capability to discern the whole, the parts that form the whole, the relation between the parts, the transformation between the parts, and the relation part-whole for a mathematical concept or between different concepts (C. Olteanu & L. Olteanu, 2011). This category was empirically identified (C. Olteanu & L. Olteanu, 2010, 2011; Olteanu, 2012) and can be used to analyse the way in which the student can work out the meaning of the whole if s/he knows the meaning of the simple parts, the semantic significance of a finite number of syntactic modes of composition, and recognizes how it is built up out of simple parts. Marton, Runesson and Tsui (2004) have defined the patterns of variations which can facilitate students' discernment of critical features or aspects of the object of learning: (1) contrast (C) means that to discern a quality X, a mutually exclusive quality non X needs to be experienced simultaneously; (2) the meaning of separation (S) refers to the other dimensions of variation that need to be kept invariant or varying at a different rate in order to discern a dimension of variation that can take on different values,; (3) generalisation (G) means that to discern a certain value, X1, in one of the dimensions of variation X from other values in other dimensions of the variation, X1 needs to remain invariant while the other dimensions vary; (4) fusion (F) is to experience the simultaneity of two dimensions of variation. C. Olteanu and L. Olteanu (2011) have found a new dimension of variations named similarity (SI) and it is defined as the property of two or more expressions to adapt the same meaning.

Olteanu (2007) identified two ways to open up the dimensions of variation: convergent and divergent. With a convergent variation, different aspects are directed to the whole of the object of learning. These aspects consist of the objects' parts and the relationships between them. It seems that this variation leads to a positive development in student learning. A divergent variation means that the whole of the object of learning is presented first and afterwards the parts that constitute it, without first discerning the parts in question. The ways in which teachers/students experience the object of learning they meet in their worlds is analysed and described in this article in terms of a small number of qualitatively different categories. Among these categories, teachers can identify the aspects that are important for current understanding, possibly not as comprehensive as the teacher's own understanding but adequately powerful for current concerns.

1.4 A New Way of Understanding Why Communication Succeeds

In this article, the focus will be on explaining why communication succeeds, and how it succeeds, by having as basis variation theory (Marton et al., 1997; Marton et al., 2004). From a variation theoretical perspective, it is the object of learning that is the focus in a teaching situation. An object of learning has two constituent parts: the direct and indirect objects of learning. The former is defined in terms of content, that is, arithmetic, algebra, etc., and the latter refers to the specific capability that students are expected to develop, for example, being able to calculate, pronounce words, and discern the object of learning in novel situations. As mentioned above, the object of learning is formed of three components: the intended, the enacted and the lived object of learning. The intended object of learning becomes visible, for example, in the teachers finalised lesson plan. The enacted object of learning can be observed when the teacher carries out the lesson and it is later analyzed in terms of whether the object of learning was made attainable through actual patterns of variation and invariance, which were constituted by the teacher and the students. This is called the enacted object of learning, which is the object that has a real impact on student learning. The students' initial level of capability to comprehend the object of learning as well as the object of learning that students experienced and understood after the lesson can be analysed, for example, based on the students reasoning when they write different tests. In this way, the object of learning can be seen as an event and the term communication as something that takes place in individual as well as collective communicative events. C. Olteanu and L. Olteanu (2010) defined effective communication as:

> A process by which the teacher assigns and conveys meaning in an attempt to create shared understanding, […] the process of meaningful interaction among the intended, enacted, and lived objects of learning. (pp. 385)

The process of meaningful interaction among the intended, enacted and lived objects of learning is an indication of whether the communication in the classroom is successful or not (C. Olteanu & L. Olteanu, 2011). The general idea of success is this: a communicative event is successful just if the terminal state corresponds to the initial state (Pagin, 2008). This implies that, the communication in the classroom succeeds or not if the aspects of the content supposed to be treated in the classroom (the intended object of learning) are the same as or different from the aspects of the content of the teacher's representation (the enacted object of learning), and if the aspects of the content of the teacher's representation (the enacted object of learning) are the same as or different from the aspects discerned of students, i.e. the content of the student's representation (the lived object of learning). This means that, if you understand a new concept, or a new theory, or a hint, you interpret it in accordance with how it was meant, and if the interpretation is not so in accordance, it has resulted in misunderstanding rather than in understanding it. In order for there to be such a difference between interpreting correctly and interpreting incorrectly, what is in or not in accordance with what was meant must be established independently of the interpretation (Olteanu, 2012).

The study presented in this paper focuses on the same content, and the analysis of the data describes what the students discern, what are critical aspects and the communication that takes place in the classroom.

2. Method

The study was performed in two classes, selected from the Natural Science Program in upper secondary school, in Sweden. In both classes, the same textbook was used. A total of 45 students, 16 years old (25 males, 20 females) and two teachers (Anna and Maria) took part in the study. The teachers' taught the same course in mathematics. The data was collected in 10 steps (Figure 1). The students took a diagnostic test in the beginning of the course (Step 1); the lessons were videotaped (Step 2); the students wrote two tests during course and a diagnostic test after the course (Step 3–5); 8 students (four students in each class) were selected (in co-operation with the teachers) for an individual session, a post-test (contained tasks dealing with concepts that the students needed to develop further) and an interview followed (Step 6–9); the teachers looked at the video sequences analysed the students test, and concluded what each student could improve in her or his knowledge (Step 10).

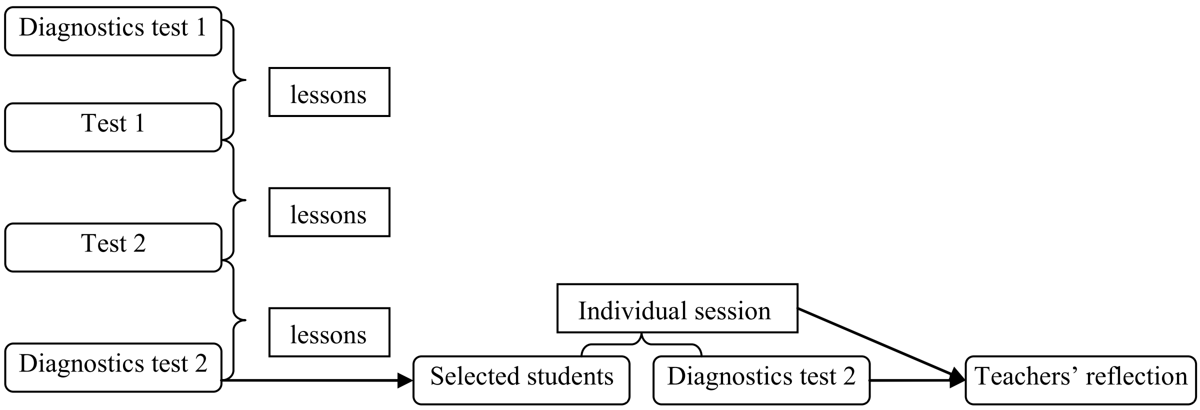

Figure 1. Data collections

By observing teaching in two classrooms in which they produced the same content and in a non-experimental situation, it becomes possible to analyze differences between the way in which they offer this content and student learning as a model of dimensions of variation. The tests took place on the same day and at the same time in the two classes and there were teachers who were guarding the students. The analysis, presented in this article, is based on video recorded lessons (12 lessons in each class) as well as the students' performance in test 2. The empirical data which this result is based on is marked in Figure 1 with blue.

3. Results

3.1 The Intended Object of Learning

The intended object of learning can be seen in the teachers' planning of the course and contains the following items:

- Graphs of quadratic functions (parabola)
- Reflective properties of parabolas: symmetry axis, vertex, and the intersection with the x-axis
- Solving of quadratic equations
- Applications related to problems solving

3.2 The Enacted and Lived Object of Learning

3.2.1 Quadratic Functions

Both teachers introduced the new contents on the basis of graph interpretations of quadratic functions with the help of handheld calculators with graphic tools. The foci concentrated on building the ways of making sense of vertices, symmetry axis and the intersection with x-axel. To do this, they varied the structures of functions using different coefficients e.g.:

$$y_2 = x^2 + 2x - 3 \text{ and } y_3 = -x^2 - 6x + 2 \text{ (Maria)} \quad (1)$$

or

$$f(x) = x^2 - 4 \text{ and } f(x) = (x - 1)^2 - 9 \text{ (Anna)} \quad (2)$$

With the selected functions, students in Maria's class had the ability to distinguish two extreme points, namely the maximum and minimum points, but this was not possible for students in Anna's class because she chose only functions that have a minimum point. The students in Anna's class implicitly cannot make sense of which coefficient influences the function outcome. Despite the fact, teachers used different structures of functions they did not separate the aspect of the x^2-coefficient in the interpretation of the symmetry axis when a function was represented in general or particular form. The fusion between the intersection of the function with the x-axis, the

x^2- and x-coefficients was not realised. Also, the students did not have the possibility to distinguish the difference between the vertex and the highest respective, the lowest value of the quadratic functions in the beginning. These aspects were quickly pointed out later. The following extract from the video shows this.

Maria wrote on the blackboard: $y = x^2 + 5x – 5$

Teacher: What is the lowest value of the function?

The student does not answer.

The teacher writes ”the lowest value of the function” on the blackboard.

Teacher: What are we searching for?

Pontus: The connection between x and y.

Teacher: Yes, we can say it like this: we search the minimum point.

The teacher writes ”minimum point” on the blackboard.

Teacher: Where do we find it? Kurt?

Kurt: When x is zero.

Teacher: No, not when x is zero.

Kurt: No, mh…

Ulrika: When y is zero.

Teacher: No, but the lowest value of the function is the minimum point. Here I have the minimum point (the teacher points at a graph).

Anna shows the following picture on an overhead (Figure 2):

Teacher: So the problem we have here is actually what the highest height is for this curve, the highest height? How high does it reach?

Teacher: So, go in here (he points at the graph), x equals 5 and if I put together and read I get 4.8, or I calculate with this rule.

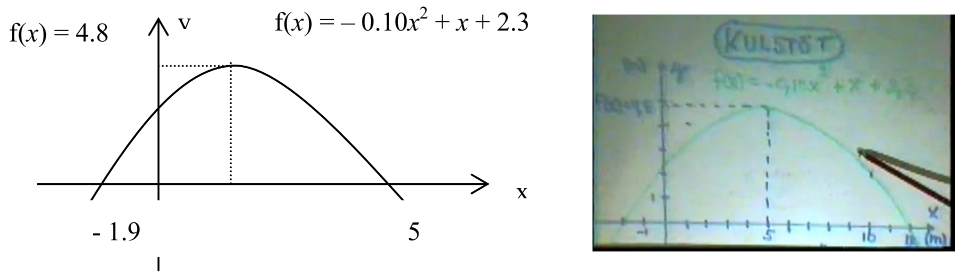

Figure 2. Shot putting

3.2.2 Quadratic Equations

The second aspect of the object of learning was to solve quadratic equations. Maria introduced the solving of a quadratic equation successively. First, she solved the equations in a particular case (e.g. $x^2 = 144$; $(x + 1)(x – 3) = 0$) and after that she introduced the formula (simplified form) by completing the square. Maria's communicative focus refers to the formula that was found and that it is only valid when the coefficient of x square is 1 and the equation is equal to zero. She practices the formula with the help of numerous examples e.g.

$$x^2 + 2x - 15 = 0 \tag{3}$$

or

$$2x - 3x^2 = -1 \tag{4}$$

This practice opens up a rich dimension of variation characterised by separation, generalisation and fusion. This can be shown in the following sequence from the video.

Maria solved the equation (4):

Teacher: What do you say about this then? Is it ready for using the formula? Is it written in the form $x^2 + px + q = 0$?

Erik: No…

Teacher: No, it is not. Firstly, this is the term x^2. It must be positive (points at x^2 in $x^2 + px + q = 0$) and it is not. Then start by making it positive and then I move this across (points at $-3x^2$ on the right side) and so it becomes $3x^2$, and then you have to gather all on one side for it to be equal to zero, so then I continue to move across (points at 2x and the right side), then it becomes minus two x and what does minus 1 become then?

Ulrika: Plus one.

Maria calculates on the blackboard.

Teacher: Is it ready for the formula now?

Ulrika: No

Teacher: No, it is not, because 3 is not allowed to be in front of x^2, there can only be x^2 so what should I do with the 3?

Amelie: divide

Teacher: Yes, divide, and then I divide all terms.

The communication is characterised by a strong focus on the critical aspects in students learning and this is reflected in students' ways of making sense of how to solve a quadratic equation. In the test, only 4 of 19 students could not give a correct answer to solve the equation:

$$x^2 + 6x + 5 = 0 \tag{5}$$

The 4 students had errors in their calculation and not in the interpretation of the formula. For example:

$$x = -3 \pm \sqrt{9-5} \;\Rightarrow\; x = -3 \pm \sqrt{-4} \quad \text{or} \quad x = 3 \pm \sqrt{9-5}$$

Even though Maria's focus was clear, there were students who wondered "Do we have to know all these methods?" and this points out to the absence of fusion between different methods used to solve a quadratic equation.

Anna introduced the same content by using particular cases (for example $x^2 = 4$; $(x + 2)(x - 4) = 0$) and after this she wrote the formula (simplified form) on the blackboard and only practiced on one example. The communication is vague and the focus is on the procedure for applying the formula. The following sequence shows this.

Teacher: Then, if you look in your formulae collection, and you don it?, not need to do that because I will write it for you, you will see that something like this is written… (the teacher writes the formula on the blackboard)

Teacher: What is the coefficient for x, well it is p, what is half of p, well it is half p, with reversed sign minus half p (points at $-p/2$).

Teacher: Yes, and it was actually it (points at 2) so I can take the one I got…I take the square of the symmetry axis, can't I? The square of the symmetry axis and then (points at –5 on the blackboard)…

That was received by the students with negative words. Anna's communication of the object of learning could not give the possibility to distinguish the conditions in which the formula can be applied and neither the importance of the *x*- coefficient and the constant term in practicing the formula to the students. She pointed out these two conditions quickly in a review lesson before the test. Also, she pointed out "I take the square of the symmetry axis" and this led to serious confusions for the students in the future. The space of variation opened up was poor and lacked some patterns of variation. The students' way of making sense of how to solve quadratic equations reflects in the way in which they wrote the solutions in the test. 9 of 18 students could not solve the equation (5) because they did not understand the formula, e.g.

$$x(x + 6) + 5 = 0 \;\Rightarrow\; x = 0 \text{ and } x = -6.$$

The students' way of making sense of how to solve a quadratic equation reflects more in the students' lived object of learning when they use the formula to solve an equation. One of the problems in the test was as follows:

A rectangular garden has an area of 825 m^2. The longer side of the garden is 8 m longer than the shorter side. Calculate the perimeter of the garden. (Problem 1)

10 of 19 students in Maria's class and 11 of 18 students in Anna's class could not give a satisfactory solution because of trouble with translating the text into a mathematical equation or with solving the quadratic equations (Table 1).

Table 1. Some examples of students' solving strategies

$x(x + 8) = 825$	A= x(x – 8) = x2 – 8x = 825	$x(x + 8) = 825$	$x(x + 8) = 825$
$x^2 + 8x = 825$	P = 4x +16	$x^2 + 8x = 825$	$x^2 + 8x = 825$
$x^2 + 8x = 825/8$		$\sqrt{825} \approx 29$	$x^2 = 825 - 8x$
$x + x = \sqrt{103.25}$	$x = \frac{8}{2} \pm \sqrt{\left(\frac{-8}{2}\right)^2 + 0}$	$29 + 4 = 33$	guess the solution
$2x = 10{,}2$		$29 - 4 = 25$	
$x = 5{,}1$			

Here, it is important to point out that working with applications related to problems solving that lead to a quadratic equation could not be observed in the enacted object of learning in Maria's and Anna's class.

3.2.3 Quadratic Equations and Functions

The third component of the object of learning was to connect the concept of function with that of equation and this was the principal purpose of contributing to solving problems that could give the students possibility to make sense of the intersections with the x-axis and the highest/lowest value of a quadratic function.

At this moment the teachers simultaneously used both the graph interpretation of a function and the algebraic calculation to identify the intersections with *x*- axis, symmetry axis and the highest/lowest value of a quadratic function. I will also point out that the teachers had used the simplified form to solve the quadratic equations, but now they used functions that were represented in the general form. For example:

$$f(x) = -\ 0.10x^2 + x + 2.3 \text{ (Anna) and} \tag{6}$$

$$y = 1.2 + 0.9x - 0.25x^2 \text{ (Maria)} \tag{7}$$

In the enacted object of learning Anna pointed out that there is a correspondence between the dependent and independent variables in the representation of a function, and he separated the interaction with the *x*- axis from the other points on the graph. In Maria's enacted object of learning the correspondence between *x* and *y* cannot be identified in the communication and furthermore, she mixes up the concepts of functions and equations. The following video sequence shows this:

Maria wrote the following on the blackboard:

$$y_2 = x^2 + 2x - 3 \tag{8}$$

Teacher: It is still a quadratic equation (after a while) or function.

The students' way of making sense of the concept of function and equation influenced the students' ways of working with realistic problems. In the enacted object of learning, the focus was unclear and many times incoherent, which led to an increase in the amount of trouble that the students had in this area. The following sequences show how the teachers were working with problems solving.

Anna wrote the following on the blackboard:

Shot putting (Figure 1):

$f(x) = -\ 0.10x^2 + x + 2.3$

Teacher: I am interested in studying the height above the ground.

The teacher shows the following on the OH:

Highest height:

$F(5) = -\ 0.10 \cdot 5^2 + 5 + 2.3$

$F(5) = 4.8$ m

Length: 11.9 + 0 = 11.9 m

Teacher: The symmetry axis shows that x = 5 and that you get as half the coefficient of x with reversed sign.

Maria wrote the following on the blackboard:

$$y = 1.2 + 0.9x - 0.25x^2$$

Teacher: How high above the ground does the water jet reach highest?

Teacher: This 1.8, here we have the symmetry axis. x is equal to 1.8….. We know that if x is 1.8 we have the highest height and then we can calculate it.

She writes on the blackboard:

$$y_{max} = 1.2 + 0.2 \cdot 1.8 - 0.25 \cdot 1.8^2 = 2.01$$

At this moment we can see that both teachers used the general form of a quadratic function, and they found the highest value of the function on the basis of the function symmetry line. The critical point is that the teachers' focus is not clear. This makes them and the students talk about two different objects even though they use the same words, namely the *x* coordinate, which gives the highest value of the function and the symmetry axis. Firstly, the symmetry axis is obtained by using the simplified form of a quadratic equation. Secondly, the calculation of the functions value in the point in which the symmetry line intersects the *x*-axis is confounded with the calculation of the value of the equation in the same point. Neither of the teachers had a clear focus on this nor does it reflect in the students' lived object of learning, that is, in the test. For example, the students had to solve the following problem:

Pelle stands on a rock next to a lake and casts a stone over the lake. After *t* seconds, the height of the stone above the water level is:

$$h(t) = 8.5 + 9.8t - 4.9t^2 \tag{9}$$

a. When is the stone 10 m above the water level?

b. Calculate the greatest height above the water level that the stone has. (Problem 2)

14 of 19 students in Maria's class, and 13 of 18 students in Anna's class could not give a satisfactory answer to the first question. 10 of 19 students in Maria's class, and 13 of 18 students in Anna's class could not give a satisfactory answer to the second question. The students difficulties, was that they mixed up the value of the functions with the variable value as a consequence of the way of making sense of the concept of function, or they could not solve the quadratic equation. For example:

$$10 = 8.5 + 9.8t - 4.9t^2 \;\Rightarrow 0 = 0.85 + 0.98t - 0.49t^2$$

or

$$10 = 8.5 + 9.8t - 4.9t^2 \;\Rightarrow\; 1.5 = 9.8t - 4.9t^2 \;\Rightarrow\; 1.5 = \sqrt{4.9t^2} \;\Rightarrow\; 1.5 = 2.21t \;\Rightarrow\; t = 0.68$$

We also can see that the function is denoted with *h* and the independent variable is called *t*. Since the generalisation in the several ways of representing a function symbolically was absent in the construction of the enacted object of learning, the students had trouble in fully understanding whether 10 m referred to an independent or a dependent variable. In problem 2b the students' way of making sense of the concept of equation and function led them to calculate *h*(0.68) or *h*(0) or *h*(- 4.9).

4. Conclusions

The analysis of the enacted object of learning referring to solving a quadratic equation shows that the communication in Anna's class was not effective and it is based on the fact that the communication is obstructed because Anna used central notions in the objectified way while the students failed to do so. If the communication is effective, for example when Maria treated the quadratic equation, the students get the possibility to make sense of critical aspects when solving a quadratic equation and this improves the students' learning. Also, Maria opened up convergent dimensions of variation as she presented how to solve quadratic equations. This aspect could not be identified in Anna's class. Even so, we can see that the students in both classes had difficulties to solve quadratic equations when they appeared in new situations (e.g. solving the problems 1 and 2). In Maria's class, it can clearly be seen that the number of students having difficulties with solving quadratic equations written in general form increased drastically. This phenomenon can be understood by the absence of generalisation and fusion as patterns of variation in the enacted object of learning.

In a problem situation, the function is often written in general form and the students must determine the maximum/minimum point or the highest/lowest value. At this moment, it is necessary that the students understand that the x coordinate corresponding to the intersection of the symmetry axis with the x -axis is obtained on the basis of the simplified form of a quadratic function, and this value must be put in the general form to calculate the maximum/minimum point or the highest/lowest value of the function. It is also important for the students to understand that a function can be denoted in different ways and also to understand the connection between dependent and independent variables. In both classes, the students cannot make sense of how to calculate the highest respective the lowest value of a quadratic function. This is a consequence of the presentation of different aspects in the simplified form and general form of a quadratic function. This presentation was not clear communicated and x changed meaning several times. The dimension of variations called similarity was not offered to the students during the lessons. Also, the teachers opened up divergent dimensions of variation. The teachers did not specify that a function can have a highest/lowest value without having a maximi/minimi point and that this depends on the domain of definition for the function.

The teacher's inability to create an effective communication leads to the fact that the teacher's efforts in differentiating, fusing, and generalising some important concepts are not understood by the students. In other words, not only the ways in which they communicate are important, but also the fact that they are able to do so while handling a specific mathematical issue. The mechanism for an effective of communication which may lead to students' understanding pattern and structure, the logical analysis, and calculation with patterns and structures when working with algebra and functions during the classrooms lessons can be summarized as follows: (1) identify critical aspects of the object of learning, as reflected by students' prior or actual understanding; (2) identify what kinds of pattern of variation can best be used to help students discern the critical aspects and their relationships; (3) plan learning experiences by making use of appropriate patterns of variation; (4) carefully analyse what kinds of assessment can be used to provide feedback to the students. In this way, they assign and convey meaning in an attempt to create shared understanding and develop an effective communication in the classroom.

5. Further Implications to the Mathematics Field

The direct object of learning in this study was the quadratic equations and functions, and the indirect one was to develop the students' capabilities to solve quadratic equations, to make sense of the properties which characterize the quadratic functions and to use this knowledge in novel situations. To develop these capabilities teachers need to communicate aspects of the content in the classroom in accordance with how it was meant and focus on the whole, the parts that form the whole, the relation between the parts, the transformation between the parts, and the relation part-whole for a mathematical concept or between different concepts. In this way, it is possible for the teacher to create a meaningful interaction among the intended, enacted and lived objects of learning, which is to create a successful communication in the classroom.

These studies indicate the need to plan the mathematical content of the starting point in the students exhibited critical aspects and open up the convergent dimensions of variation by contrast, separation, generalization, fusion and similarity. This is possible if teachers are constantly working with an iterative process in which they can discuss with each other and reflect on the implementation of lessons in relation to what students discern and what dimensions of variation are created in the classroom.

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