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Comparing Ninth-Grade Students' Approaches to Trigonometric Ratio Problems Through Real-World and Symbolic Contexts

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Abstract

This study investigates the problem-solving strategies employed by ninth-grade students when addressing symbolic and real-world contextual problems involving trigonometric ratios. Conducted with 46 ninth-grade students from a Turkish public high school, this research employed a worksheet consisting of six problems aligned with the Turkish ninth-grade mathematics curriculum. Three of these problems were based on real-world contexts, while the other three were conventional symbolic problems. The findings indicate that students exhibited proficiency in identifying similarity ratios using side length ratios. Additionally, the results revealed that students were more adept at solving real-world mathematical scenarios compared to purely symbolic tasks. This study offers significant insights into the problem-solving strategies of ninth-grade students when confronted with trigonometric ratio problems. It underscores crucial implications for mathematics curricula and pedagogy, highlighting the importance of preparing ninth-grade students for success in their future academic and professional endeavors. The study emphasizes the necessity for a balanced approach in teaching, integrating both real-world and symbolic problem-solving tasks to enhance students' mathematical understanding and application. By identifying the strengths and areas for improvement in students' problem-solving strategies, this research contributes to the development of more effective educational practices that address the diverse needs of learners.

Keywords: problem solving strategies, real-world contextual problems, symbolic problems, trigonometric ratios

1. Introduction

The employment of symbolic language is a fundamental characteristic of mathematical pedagogy. Many students perceive mathematics as a discipline primarily consisting of numbers, equations, and symbols (Eşlik, 2010). Historically, although mathematics is frequently regarded as an abstract discipline, its origins can be traced back to the lived experiences and curiosities of individuals across various epochs.

In this context, trigonometry, as a subject in mathematics education, effectively bridges algebraic, geometric, and graphical reasoning. Throughout the learning process, the foundational principles of trigonometry present numerous challenges to students, particularly in the middle grades when introduced as similarity ratios. The subject necessitates that students relate triangle diagrams and other geometric shapes in analytic space to numerical correlations and adeptly manipulate symbols (Blackett & Tall, 1991).

The National Council of Teachers of Mathematics (NCTM) (2000, p. 308) emphasizes in its guidelines that students should "use trigonometric relationships to determine lengths and angle measures". Furthermore, the Common Core State Standards Initiative (CCSSI) advocates for students to understand, through similarity, how side ratios in right-angled triangles correspond to the triangle's internal angles, ultimately leading to the establishment of trigonometric ratios for acute angles (CCSSI, 2010; Moyer, 2013). Recent research has investigated pedagogical strategies for teaching trigonometry, highlighting the significance of integrating real-world applications and hands-on learning experiences (Spangenberg, 2021).

In this context, trigonometry is indeed one of the earliest topics introduced in high school mathematics. Weber (2005, p. 91) asserts that since trigonometry integrates "algebraic, geometric, and graphical reasoning, it can serve as a pivotal bridge to understanding pre-calculus and calculus." Consequently, students encounter trigonometric relationships as they explore advanced mathematical concepts, including both geometry and calculus (Demir & Heck, 2013).

The aspects of mathematics intertwined with trigonometry are distributed across multiple school years in the

Turkish mathematics curriculum (Ministry of National Education [MoNE], 2018a; 2018b). This curriculum introduces sine, cosine, and tangent as relational concepts associated with an angle. Such an introduction implies that these concepts either establish the foundational understanding of trigonometry within the geometric context of right-angled triangles and similarity for middle school students, or they are introduced within the framework of geometric transformations using the unit circle concept and as functions of a real number for high school students.

Specifically, high school mathematics in Turkey emphasizes the sine, cosine, and tangent ratios as relationships between the sides of a right-angled triangle (MoNE, 2018a). Essentially, the high school curriculum in Turkey delves deeper into trigonometric relationships compared to the more foundational lessons taught in middle school (MoNE, 2018b). In understanding how to effectively teach these concepts, it is valuable to explore the diverse historical and cultural methods that have contributed to the development of trigonometric knowledge.

1.1 Methods of Teaching Concepts Related Trigonometric Relationships

Concepts related to trigonometry have been manifested in various forms across diverse cultures and historical epochs. The historical evolution of these concepts can be traced back to ancient civilizations such as Egypt and Babylon. However, the foundation of modern trigonometry was established by Hipparchus around 140 BC, particularly with his development of the table of chords (Duke, 2011). Over time, various societies devised unique methods to address challenges related to the motions of celestial objects, determining the heights of objects, or measuring time. These methods were influenced by their mathematical understanding, specific needs, and cultural practices.

Historically, a prevalent perspective posits that trigonometry originated as an instrumental tool for astronomical models (Van Brummelen, 2010). The development of the trigonometric ratios sine and cosine is largely attributed to the need to resolve astronomical quandaries. In the broader context of history, a widely accepted notion posits that the genesis of trigonometry was rooted in calculations involving the lengths of shadows cast by objects (Swetz, 1995). Swetz (1995) elucidates that numerous ancient cultures relied on shadow observations, not only to determine optimal agricultural times but also to identify times for religious observances. Societies such as the Egyptians, Hindus, and Islamic communities determined the timing for their religious rituals and prayers based on the sun's position in the sky, ascertained by measuring shadow lengths. Essentially, the astronomical interests of these societies catalyzed the development of the sine and cosine concepts. Concurrently, curiosity regarding time and the shadows cast by objects spurred the conceptualization of the tangent and cotangent relationships (Swetz, 1995).

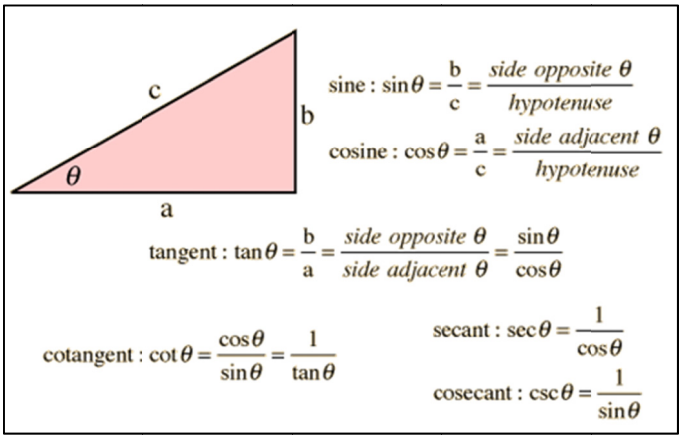


Figure 1. Trigonometric relationships with ratio method

Considering this historical context, two prevalent methods are employed in school mathematics to teach trigonometric relationships (Kendal & Stacey, 1996). Firstly, the ratio method defines trigonometric relationships based on the ratios of side pairs in a right-angled triangle, as illustrated in Figure 1. For example, the cosine of an angle corresponds to the ratio of the length of the "adjacent side" to the "hypotenuse." Similarly, the tangent of an angle is conceptualized as the ratio of the length of the "opposite side" to the length of the "adjacent side" (Van de Walle, Karp, & Bay-Williams, 2012).

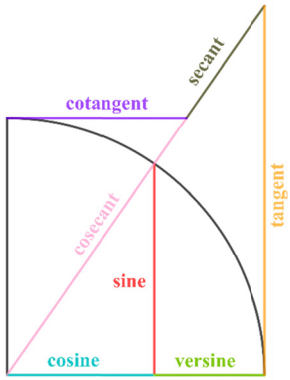


Figure 2. Trigonometric relationships on a unit circle with r = 1 (Rogers & Pope, 2016)

In the unit circle method, the sine and cosine values of an angle correspond to the x and y coordinates, respectively, of a point. This point is obtained by rotating the point (1,0) around the origin through the given angle on a unit circle (Kendal & Stacey, 1996; Rogers & Pope, 2016), as depicted in Figure 2.

Both methods have specific advantages over the other. For instance, the ratio method provides a direct and easily understandable approach to comprehending trigonometric relationships, particularly the fundamental ratios of sin, cos, and tan in right triangles (Delima, 2022). Additionally, the ratio method aids students in visually understanding the relationships between side lengths and angles in right triangles (Maknun et al., 2019).

On the other hand, the unit circle method allows for a deeper and more comprehensive understanding of trigonometric relationships. This method enables the generalization of trigonometric relationships to include both positive and negative angles (Altman & Kidron, 2016). Moreover, defining trigonometric relationships on the unit circle facilitates understanding the continuous and periodic nature of functions, which is advantageous in advanced mathematical studies and other disciplines (Mickey & McClelland, 2017). The unit circle method also assists students in visualizing trigonometric relationships and mentally constructing models, which is particularly beneficial for solving complex problems (Moore, LaForest, & Kim, 2016).

Numerous studies have investigated the complexities of trigonometric relationships and their integration into school mathematics. Doğan (2001) examined high school students' perceptions of trigonometry and found that when presented with verbal problems, students often exhibited confusion regarding trigonometric concepts. They frequently made errors in problems involving trigonometric equations, identities, the unit circle, and trigonometric relationships. However, they demonstrated greater proficiency in problems involving geometric shapes, which they approached using symbolic representations of trigonometric relationships. This differential performance suggests that students tend to memorize trigonometric relationships as symbolic formulas but struggle to apply them in practice.

Similarly, Taş (2013) explored students' comprehension levels of trigonometric concepts. The study revealed that while students had a firm grasp of tangent functions conceptually, they struggled when expressing the tangent and cotangent functions on a unit circle. Some students exhibited rote learning rather than conceptual understanding, especially when working with right-angled triangles. When it came to representing trigonometric relationships, the study indicated a preference among students for an algebraic approach over a geometric one. Building on this knowledge, Eşlik (2010) conducted research on trigonometric ratios, focusing on the impact of integrating contextual problems with student-centered teaching. The findings indicated that this enriched instruction method led to enhanced student comprehension. Furthermore, the study shed light on students' perspectives, revealing which aspects of the instruction they found most engaging and meaningful. Lastly, in a comparative study, Kendal and Stacey (1997) compared the teaching of trigonometry using ratio relations versus unit circle relations. They found that students taught via the ratio relations method outperformed those taught through the unit circle approach. Additionally, the ratio relations method proved more enduring in terms of retention, suggesting it was a more effective teaching strategy.

From another perspective, the ratio method for right-angled triangles can be readily contextualized using real-world scenarios, such as a ladder leaning against a wall or determining the height of an object based on its shadow length (Demir & Heck, 2013). Such real-world contextual problems play a crucial role in mathematics education. They provide students with the opportunity to connect concepts and understand the practical

significance of what they are learning (Eşlik, 2010). These tangible problems present unique and valuable learning opportunities, fostering a deeper understanding of mathematical concepts and enhancing motivation for more profound learning (Van de Walle et al., 2012). Thus, employing problem-solving techniques in teaching trigonometry can provide insights into the tangible nature of mathematics, even in the early stages of conceptual foundation-building. In this context, understanding the broader importance of problem-solving skills is essential for developing effective mathematical instruction.

1.2 Importance of Solving Problems

Problem-solving is essential for enhancing mental abilities such as reasoning, experiential learning, and comprehension. It also fosters mathematical thinking, not only in trigonometry but across various mathematical concepts (Van de Walle et al., 2012). Given the importance of this skill, engaging with structured problems enables students to deepen their understanding of trigonometric concepts. To achieve a profound understanding, the goal is to determine the fewest steps required to reach the optimal solution (Anderson, 1993).

Contemporary research highlights the significance of diverse problem-solving strategies in mathematics education, emphasizing student-centered methods and the integration of technology. Employing an effective strategy that leads to the optimal solution is paramount for efficient problem-solving. Allen (2017) outlines several strategies for problem-solving. For problem comprehension, these strategies include clarifying the issue, pinpointing key elements, visualizing with diagrams, analyzing specific examples, and considering extreme cases. For problem simplification, he recommends reframing the problem, tackling it in segments, and rephrasing it. For problem resolution, he suggests strategies such as guess-check-revise, questioning assumptions, drawing analogies from similar problems, using deductive and inductive reasoning, and working in reverse.

In addition to Allen's recommendations, Posamentier and Krulik (2015, p. 17) advocate for strategies such as "logical reasoning, pattern recognition, and accounting for all possibilities." Schoenfeld (2014) also identifies strategies such as arguing through contrapositive or contradiction and establishing subgoals. Incorporating appropriate materials is also a strategic approach to mathematical problem-solving (MoNE, 2018a). In addition to the significance of problem-solving, the nature of the problems presented to students and their types also influence student responses and the cognitive load required for solving these problems. Understanding the classification of mathematical problems is essential to effectively apply these problem-solving strategies.

1.3 Classification of Mathematical Problems

In the literature concerning types of mathematical problems, various classifications are proposed. Some researchers categorize mathematical problems as either 'routine' or 'non-routine,' based on the type of thinking and effort required for their solution (Asman & Markovits, 2009; Schoenfeld, 2016). Others have classified problems as 'structured' and 'ill-structured' (Brookhart & Nitko, 2019; Leung & Silver, 1997). This distinction between structured and ill-structured problems pertains to the clarity of the given and the sought components in a problem, that is, whether these elements are explicitly presented or not.

Pehkonen (1997) employed the terms 'open' and 'closed' problems instead of structured and ill-structured. Besides the aforementioned classifications, other classifications have considered multiple criteria simultaneously (Anderson, 1993; Foong, 2000; Orton & Frobisher, 2004). For instance, Anderson (1993) considered both the complexity of the mathematical content and whether the problem was well-structured in his classification.

Orton and Frobisher (2004) examined the complexity level of problems and their connection to real life, categorizing them into routine problems, environmental problems (real-world or everyday life problems), and process problems (problems situated within a mathematical context rather than real-life situations). Conversely, Foong (2000) proposed a more comprehensive and systematic classification, considering whether the aim of the problems was open or closed, the time required for their solutions, and the complexity of their mathematical content (Figure 3).

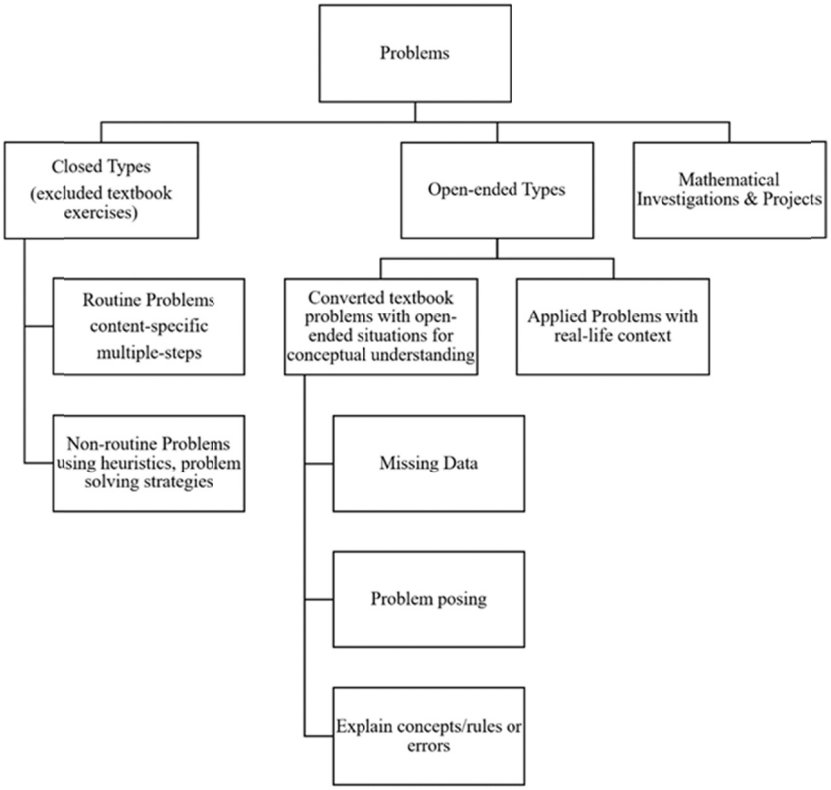


Figure 3. Classification scheme for mathematical "Problems"

In summary, researchers frequently recognize the advantages of the unit circle method in teaching trigonometry. Unlike the ratio method, which engages students only with acute and positive angles, the unit circle method enables the exploration of obtuse and negative angles. Additionally, while the ratio method restricts students to positive values for sine and cosine (as the ratio of two lengths is always positive), the unit circle method permits negative values as well (Weber, 2005; 2008; Moore, 2013).

Consequently, it is not clear which of these methods is superior for teaching and learning trigonometry. Real-world contexts can enhance understanding and reasoning in trigonometric relationships, whether presented through the ratio or unit circle method. By solving well-structured contextual problems, students can deepen their comprehension of trigonometric relationships and consciously select suitable strategies for effective problem-solving. However, in the early stages of high school, particularly in Turkey, these concepts are often not introduced in their formal and symbolic forms as functions. Instead, they are presented as trigonometric ratios within the framework of similar and right-angled triangles.

Thus, this study aimed to investigate ninth graders' approaches to solving problems related to the foundational concepts of trigonometric relationships and the strategies they employed. The research questions addressed in this study include:

- How do ninth-grade students respond to symbolic versus real-world contextual problems involving trigonometric ratios?
- What strategies do ninth-grade students use when solving symbolic and real-world contextual problems involving trigonometric ratios?

2. Method

This study utilized a case study design within the domain of qualitative research, aiming to comprehensively understand students' problem-solving processes (Creswell & Plano Clark, 2017). By employing the case study approach, the research offered an in-depth exploration of students' cognitive strategies, elucidating the intricate methodologies and rationales behind their solutions (Johnson & Onwuegbuzie, 2004). This investigation not only concentrated on the definitive answers provided by the students but also delved deeply into their underlying solution strategies. Such a comprehensive perspective is invaluable, especially in educational contexts.

Understanding both the 'what' (the answer) and the 'how' (the process) through a case study lens can provide educators with actionable insights for pedagogical improvements.

2.1 Participants

This study involved 46 ninth-grade students from a public school as participants; 20 males and 26 females, with ages ranging from 14 to 16 years. The students exhibited diverse academic achievement levels, with their mathematics grades ranging from low to high.

The school featured designated classrooms for each subject, including a specialized mathematics classroom. This setup provided students with easy access to a wide range of hands-on mathematical materials. Additionally, the students hailed from diverse socio-cultural and economic backgrounds. While some lived in the countryside and traveled to the city for their education, others were city residents.

The school was conveniently selected for the study, primarily because the researcher was the mathematics teacher of the participating students, ensuring easy accessibility. At the time of the study, this teacher had five years of experience in mathematics teaching and held a Master of Science degree in mathematics education.

All the students involved were from the same grade, ensuring consistency in their educational experiences. Importantly, although the researcher also acted as the teacher, steps were taken to mitigate any potential biases. To ensure impartiality in gathering information, independent individuals monitored the sessions.

2.2 Procedure

For this study, students were provided with learning kits containing problems related to the similarity of triangles and right-angled triangles. The problems were adapted from various sources, including students' textbooks, activity books, previous national exams, relevant academic papers, and the researcher's experiences. This process involved redefining problem situations to align with contexts familiar to the students. A problems pool, consisting of twenty problems, was established. These problems were categorized into two types: real-world contextual problems and symbolic problems.

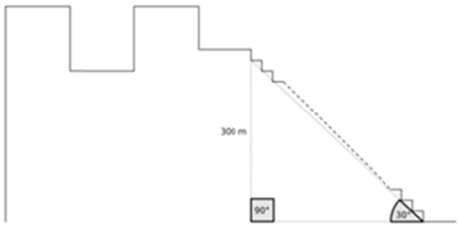
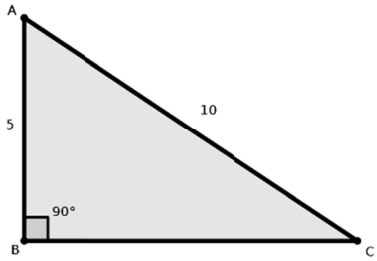
To ensure the relevance and appropriateness of the problems, three doctoral-level mathematics education experts and five experienced mathematics teachers reviewed the problem pool. Their evaluations aimed to assess the problems' credibility, quality, and suitability for the students' comprehension level and the research focus. Based on their feedback and suggestions, six problems were deemed well-structured, aligned with students' comprehension levels, and relevant to the study's objectives. The primary criterion was to identify problems most pertinent to laying the groundwork for trigonometric ratio concepts. Half of these selected problems were real-world contextual, while the other half was symbolic problems.

The initial three problems presented to the students were set in real-world contexts and focused on trigonometric ratios in right-angled triangles. Accompanied by visual representations, these problems aimed to illustrate the problem statements more vividly. The first problem required students to determine the length of a stone ladder used to ascend a hillside castle. The subsequent problem tasked students with calculating the height of a flagstick based on the length of its shadow. In the third problem, students were required to ascertain the distance a plane would cover, given its ascent angle of 37° relative to the ground and its velocity.

The remaining three problems, categorized as symbolic problems, mirrored classical trigonometry questions commonly found in mathematics textbooks or activity books. Most of the time, this kind of problems focus on understanding abstract mathematical concepts independent of real-world contexts. These problems were essentially rephrased versions of the initial three, transformed into symbolic forms with varying lengths or angles (see Table 1).

It is significant to note that the researcher did not modify the students' regular lessons; the study was conducted after the teacher had covered the relevant topic. Students were instructed to solve the problems within a lesson hour (45 minute), providing explanations for each step of their solutions as clearly and comprehensively as possible. This duration was recommended by experts and experienced teachers, who deemed it sufficient for an average student to tackle all six problems.

Table 1. Some sample problems

Problem Type	Problem Situation
Real-world Contextual Form	Problem 1. Ayşe wishes to visit an ancient castle with an entrance located 300 m above the ground. There's a ladder leading to the entrance that forms a 30° angle with the ground. How many meters must Ayşe Walk along the ladder to reach the castle's entrance? 
Symbolic Form	Problem 4. In an ABC triangle m(B) = 90°, if |AB| = 5 cm and |AC| = 10 cm than what is measure of angle m(C)? 

2.3 Data Collection and Analysis

The primary data for this research were obtained from the students' work on the given problems. Additionally, the researcher observed the students during their problem-solving sessions, and notes from these observations were utilized to gain a deeper understanding of the classroom environment. To highlight differences in students' approaches to contextual versus symbolic problems and to extract the strategies employed in the solving process, their problem sheets were analyzed according to themes and categories developed during preliminary analysis (refer to Table 2).

Table 2. Themes and categories

Themes	Categories
Engagement with the problem	Identifying Given Information Using Geometric Visualizations Using Mathematical Expressions
Strategies to solve the problem	Using Proportional Relations Applying Pythagorean Theorem Utilizing Trigonometric Ratios Formulating Equations

Utilizing qualitative data analysis methods, which facilitate systematic data interpretation as outlined by Strauss and Corbin (1990), the data were examined. The process adhered to the methodology suggested by Miles and Huberman (1994), leading to the aggregation and primary coding of all collected data by the researcher and an expert in mathematics education. Once the entire data set was coded, the resultant codes were organized and reviewed. The consistency among these codes, termed interrater reliability (IRR) (Hallgren, 2012), was assessed, yielding a score of 0.83. This score corresponds to the "high agreement" characterization by Landis and Koch (1977).

Utilizing the constant comparative analysis approach (Strauss & Corbin, 1990), similar data codes were categorized and then presented under appropriate categories. The collaborative efforts of the researcher and the

expert resulted in data interpretations consistent with the phases of the chosen research approach, anchored in the derived codes.

The findings were utilized to provide insights into students' work during problem solving. The strategies employed by students in their responses were identified. These strategies offer deeper insights into the students' approaches to learning and problem-solving, painting a comprehensive picture of their academic strengths and areas in need of improvement.

3. Results

In this study, 46 students were assigned the task of solving six problems related to trigonometric ratios. The initial three problems were presented in real-world contexts, while the remaining three were symbolic form. A comprehensive analysis of the students' responses and methodologies is provided in the subsequent sections.

3.1 Problem Solving Processes of Real-World Contextual Problems

In general, students employed four distinct strategies to comprehend the contextual problems. These strategies included: 'annotating given information on the figures,' 'highlighting pertinent details within the problem,' 'identifying and articulating necessary information,' and 'sketching figures using basic geometric shapes such as triangles and rectangles.'

Analyses revealed that some students combined all these techniques to achieve a clearer understanding of the problem statements. For example, one student highlighted the provided details, transcribed the required information symbolically, and then created their own diagram. In this diagram, they annotated the given information instead of referencing the provided figures (as illustrated in Figure 4).

Students employed various solving strategies for these contextual problems, which can be categorized into three main groups: 'utilizing proportional relations,' 'applying the Pythagorean theorem,' and 'correlating side lengths with angles in a right-angled triangle.'

First, several students endeavored to identify proportional relationships inherent in the problems. For instance, when addressing the second problem concerning the height of a flagstick inferred from its shadow, many students based their calculations on the direct proportionality between the height and shadow length (as illustrated in Figure 5).

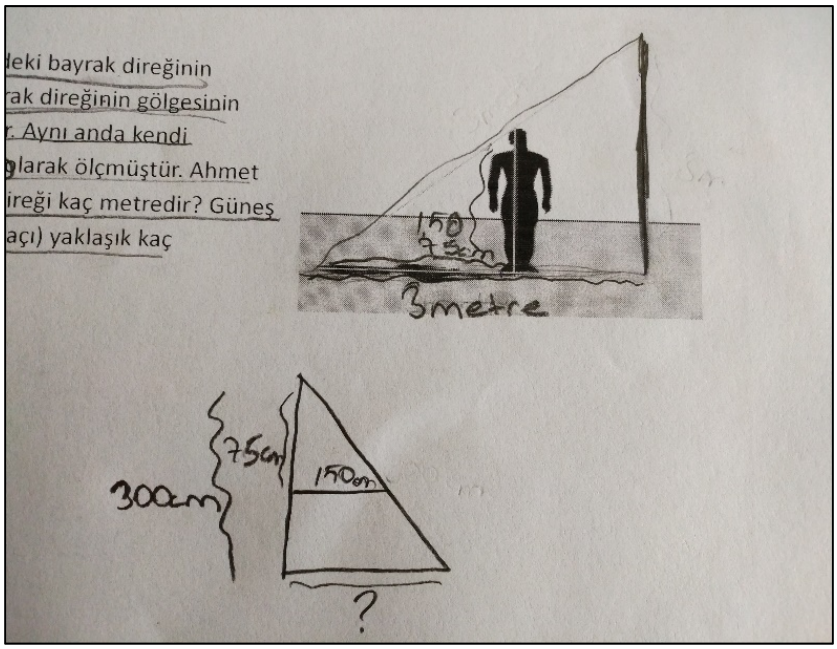


Figure 4. A student redraws own figure to elaborate problem statement

The other strategies, specifically 'applying the Pythagorean theorem' and 'using special triangles or recognizing relationships between side lengths and angles of a right-angled triangle,' were often used interchangeably by

students. Essentially, when a student employed one of these methods for a problem, they tended to use that same method for other similar problems. For example, if a student utilized the Pythagorean theorem for a particular problem, they would favor this theorem in other analogous situations over special triangles or other angle-side relationships within right-angled triangles. Additionally, when students opted to sketch their own diagrams to elucidate problem statements, they were more inclined to apply angle-side relationships.

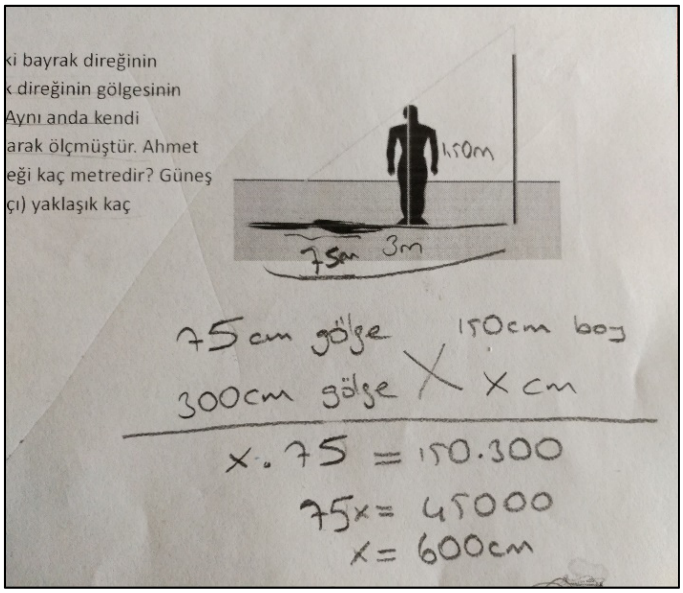


Figure 5. Using direct proportional relations

3.2 Problem Solving Processes of Symbolic Problems

The subsequent three problems were designed to be symbolic terms. Essentially, these problems presented information such as side lengths or angles either within triangle diagrams or through symbolic descriptions. For example, a problem might state, "Given a triangle ABC where angles A and B fall within the range $0° < m(A) < m(B) < 90°$ and $\cos(A) = 0.5$, what is the value of $\cos(B)$?" Broadly speaking, while these problems bore similarities to the initial three, they were framed differently, utilizing symbolic or graphical representations rather than real-world contexts.

According to students' solutions for the symbolic problems, six distinct approaches to understanding these problems emerged. Four of these approaches paralleled the strategies previously described for contextual problems. However, two specific approaches emerged exclusively in the context of these symbolic problems. These can be categorized as 'itemizing the provided information' and 'rephrasing the problem statement for clearer comprehension.'

It was observed that certain students incorporated these approaches alongside other strategies to achieve a clearer understanding of the problem. For instance, in the absence of a provided diagram, students would create their own and annotate it with the given information, while simultaneously rephrasing this information in a descriptive manner (as depicted in Figure 6). Another notable observation was that students generally refrained from redrawing figures if they were presented with basic geometric shapes in the problem.

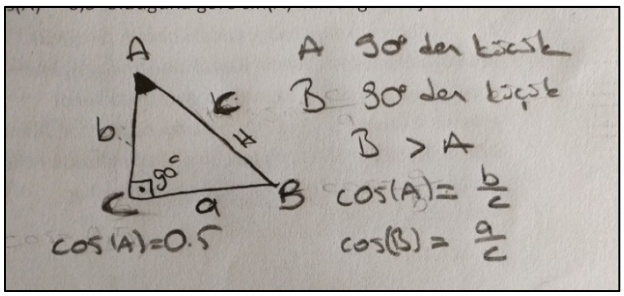


Figure 6. Student draw figure and wrote given information own words

For problems of the symbolic type, students often employed two strategies similar to those used for contextual problems: the 'application of the Pythagorean theorem' and the use of 'special triangles or other angle-side relationships within right-angled triangles'. Additionally, two unique strategies emerged for this type of problem: 'formulating an equation' and 'indicating proportional relationships on diagrams'. Interestingly, while students utilized 'trigonometric ratios such as sine and cosine' in addressing problems presented in symbolic format, this approach was not observed in their solutions for contextual problems, even though the mathematical content of the problems was comparable.

The other strategy, which involved 'indicating proportional relationships on diagrams,' was a nuanced version of the proportional relationship strategy observed in contextual problems. The distinction between these strategies lies in their application: in contextual problems, students typically documented given proportional information and matched it with ratios through calculated processes. Conversely, for problems in the symbolic form, students added lines to diagrams to visually represent and identify proportional relationships between different sections of the provided diagrams (as illustrated in Figure 7).

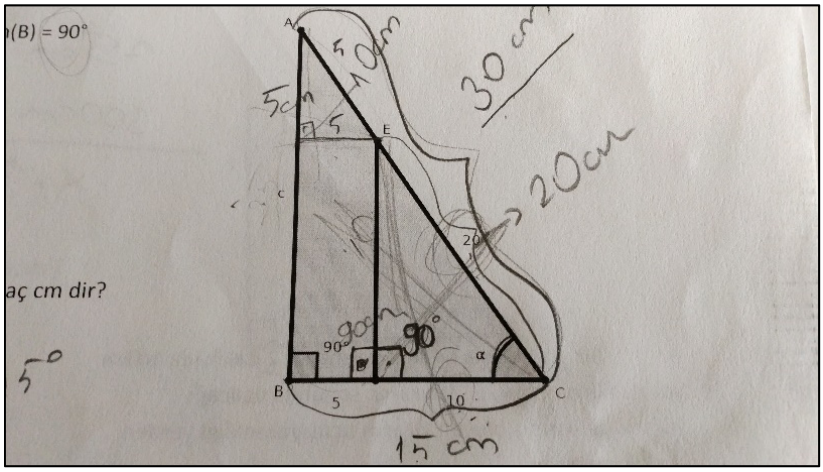


Figure 7. Student described proportional relations on figures

In summary, based on the students' responses, distinct understanding approaches and problem-solving strategies emerged, varying depending on whether the problems were presented in a contextual form or in a symbolic form. The students' solutions provided insights into the different approaches they adopted to comprehend the problem statements, as well as the strategies they employed to solve the problems. The various approaches and strategies can be categorized as detailed in Table 3.

Table 3. Students' approaches and strategies for problems

Problem Type	Understanding Approaches	Solving Strategies
Real-world Contextual Form	writing given information on figures	using proportional relations
	underlining given information describing needed information	applying Pythagorean theorem
	redrawing figures by using basic geometric figures	relationship for side lengths and angles of right-angled triangle
Symbolic Form	writing given information on figures	applying Pythagorean theorem
	underlining given information describing needed information	relationship for side lengths and angles of right-angled triangle
	redrawing figures by using basic geometric figures	writing an equation
	listing given information rewriting problem statement	marking proportional relations on figures

Results indicate that students employed similar or related approaches for understanding and solving both types of problems. For problems presented in a symbolic form, students typically listed the given information and rephrased the problem statement in their own words. In cases where problems were articulated using trigonometric equations or mathematical symbols, students tended to list and explain the given information using their own phrasing. Additionally, some students opted to redraw figures to make them more closely resemble familiar geometric shapes. Notably, students typically applied trigonometric ratios when addressing problems presented in symbolic formats. Finally, the strategy of using proportional relations manifested in two distinct ways, depending on the type of problem: in a conventional ratio form for contextual problems, and as a representation of length or angle ratios on figures for symbolic problems.

In the present study, a detailed analysis of the approaches of 46 students to problems concerning trigonometric ratios was conducted. These problems were presented in two distinct formats: real-world contextual and symbolic form. Various strategies were observed in students' problem-solving processes. For real-world contextual problems, students typically annotated the provided figures, highlighted critical details, identified necessary information, and sketched their own representations. For symbolic problems, students often listed and rephrased given information, and certain strategies, such as the application of trigonometric ratios, predominantly emerged in this context. Additionally, the strategy of utilizing proportional relationships displayed variations based on the nature of the problem. Overall, this research provides profound insights into students' cognitive strategies and emphasizes the importance of understanding both the definitive answers and the methodologies employed by students in their problem-solving endeavors.

4. Discussion

Analyses of the students' performance and work revealed that many students have difficulty interpreting their solutions in trigonometry as functions of angles. They often become confused when expressing trigonometric ratios. Interestingly, if they interpret their results using ratios within the geometric context of right-angled triangles, they tend to correct their mistakes. Students successfully interpret their solutions when they determine the trigonometric values using the ratio of side lengths. Additionally, students interpret their results more successfully in real-world contextual problems than in symbolic ones. Therefore, if trigonometry is introduced with definitions of trigonometric relationships, explaining their meaning and applications before delving into angles, many problems related to trigonometric relationships could be resolved. Consequently, real-world contexts can be employed to make trigonometry more appealing to students, especially at lower levels. Once they gain a basic conceptual understanding of trigonometric relationships and ratios, they will be better prepared to benefit from the advantages of the unit circle method.

The NCTM Standards (NCTM, 2000, p. 98) argue that trigonometry instruction should shift away from emphasizing the "memorization of isolated facts and procedures and proficiency with paper-and-pencil tests" and move towards "programs that emphasize conceptual understanding, multiple representations and connections, mathematical modeling, and problem-solving." However, if students cannot interpret their work correctly when first introduced to trigonometry, they tend to memorize ratios and procedures without gaining a meaningful understanding of the subject (Kendal & Stacey, 1996; 1997). Previous research on students' understanding of trigonometry has revealed that they generally memorize symbolic forms of trigonometric relations and struggle with verbal and non-routine problems, including those involving real-life situations. Due

to this reliance on memorization, they often adopt algebraic approaches using symbolic formulas of trigonometry rather than geometric ones. Furthermore, using visual models and ratio relationships can provide students with a deeper understanding of trigonometric relationships (Orhun, 2004; Weber, 2005).

In this study, students employed various approaches for understanding and strategies for solving both real-world contextual and symbolic problems. The classification for the trigonometric problem-solving procedure is twofold: understanding, which includes reading, clarifying, and comprehending the given problem phases, and solving, which involves planning, employing strategies to solve problems, and verifying the correctness of the answers. These phases of solving trigonometric problems resemble Allen's (2017) classification of strategies for understanding, simplifying, and solving problems.

Furthermore, the observed patterns in student performance underscore the complexities of problem-solving in the context of trigonometry. The emphasis on real-world applications, as highlighted in recent literature, aligns with the findings presented in Table 3. The data suggest that an integrated approach, combining theoretical knowledge with practical applications, is crucial for enhancing student outcomes. Moreover, the varying degrees of success across different problem-solving strategies underscore the importance of adaptive pedagogical methods (Aydoğdu & Ayaz, 2008; Spangenberg, 2021).

Within this problem-solving dichotomy for trigonometric problems, there are both approaches and strategic components. In this study, students adopted various approaches to understand given trigonometric problems. These approaches can be characterized as writing given information on figures, underlining given information, describing needed information, redrawing figures using basic geometric shapes, listing given information, and rewriting the problem statement for clarity.

These approaches align with previous research, such as the understanding and simplifying strategies proposed by Allen (2017) and the data organization and simpler strategy forms suggested by Posamentier and Krulik (2009). After understanding trigonometric problems through these approaches, students employed specific strategies to solve them. These strategies are in line with the problem-solving strategies suggested by Allen (2017), Posamentier and Krulik (2015), and Schoenfeld (2014), which were proposed for general mathematics problem-solving rather than specifically for trigonometry. However, since trigonometry is a subset of mathematics, these strategies can also be applied to trigonometry.

In conclusion, attempting to solve a trigonometric problem presented in either a contextual or symbolic form without a strategy is likely to be ineffective. Identifying which problems can be more effectively solved with the aid of a strategy requires practice. Such practices stem from learners' experiences with problem-solving scenarios during the learning process (Posamentier & Krulik, 2015). The results of this study led to a dichotomous process for solving both symbolic and real-world contextual trigonometry problems. These findings support the existing literature and previous research concerning approaches to understanding and strategies for solving. In light of this dichotomous process, educators can structure their mathematics lessons and assessments to foster a meaningful understanding of trigonometric concepts. Additionally, researchers interested in trigonometric problem-solving processes might consider the approaches and strategies presented in this study to develop more intricate and detailed problem-solving processes. Finally, these findings can be expanded upon with empirical evidence through experimental studies.

5. Conclusion

This study undertook a detailed inquiry into the strategies employed by ninth graders when confronting trigonometric problems. Spanning six problems—three contextual and three symbolic—the examination of 46 students revealed intriguing patterns with significant implications for the pedagogy of trigonometry. Findings related to the accuracy of students' answers indicated that students tend to provide more accurate answers to problems presented in real-world contexts or those requiring the construction of a geometric figure. For instance, 63.1% of students effectively tackled the first problem, which was contextual. However, when problems transitioned into the symbolic realm, a decline in performance was observed. This aligns with the overarching narrative of the research, emphasizing the efficacy of real-world scenarios and geometric figures in facilitating an intuitive understanding of trigonometry.

Furthermore, the strategies identified from the students' responses characterized the intricate interplay of contextual understanding and symbolic representation. While students demonstrated a preference for proportional relations, notably in real-world scenarios, they also exhibited a tendency to use visual aids in symbolic problems. This duality in approach underscores the necessity for a balanced pedagogical methodology that integrates real-world applications with symbolic nuances.

The study underscores the urgency of transitioning from rote memorization to a more holistic conceptual understanding. By incorporating real-world contexts in the initial phases of trigonometry instruction, educators can foster a foundational understanding, thereby paving the way for the introduction of more abstract concepts. In summary, the findings of this study provide insights into the pedagogical strategies employed in solving trigonometry problems. By merging the tangible with the abstract, educators can cultivate a generation of students who do not merely memorize trigonometric formulas but understand and appreciate their inherent logic and beauty. As a foundational step in this academic realm, this study invites further research to delve deeper, potentially uncovering more nuanced strategies and insights, thus enriching the field of trigonometric pedagogy.

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Note

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Authors contributions

All authors have equally contributed to conceptualization, methodology, data curation, investigation, formal analysis, writing the original draft, visualization, validation, and reviewing and editing.

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The authors have no conflicts of interest to declare that are relevant to the content of this article.

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Data sharing statement

No additional data are available.