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A Comparison of Information Criterion for Choosing Copula Models

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Abstract

The object of this paper is to analyse the ability of the Information Criterion in selecting the best copula model. For this study, we carry out a simulation exercise considering five one-parameter copula families: Normal, Student-t with ν degree freedom, Clayton, Gumbel and Frank. For each family copulas, three degrees of dependence and three size samples. The Information Criterion included in the comparison are AIC, BIC, HQIC, SIC. The results obtained are as follow; (i) we find that for a high dependence level (0.9) the reliability of the Information Criterion (IC) is quite good, but it reduces with the dependence level; (ii) the performance of the IC not only depends on the dependence degree but the size sample. In the case of considering negative dependence the reliability of the IC does not depend on the dependence degree but the size sample. As the size sample reduce the performed of the IC reduce. To last, in a comparison among the IC considered, we find that the BIC criterion is the most reliable follow by SIC. AIC and HQIC reaps similar results.

Keywords: Copula, dependence, information criterion, Kendal tau

1. Introduction

A copula is the joint distribution function of random vectors with standard uniform marginal distributions[1]. Copula analysis appropriately describes the dependence structure between random variables so that it has been applied with success in different fields: actuarial science, (Frees & Valdez, 1998; Otani & Imai, 2013) public health and medical (Winkelmann, 2012), hydrology (Genest, Favre, Béliveau & Jacques, 2007) and finance (Nikoloulopoulos, Joe & Li, 2012; Jondeau & Rockinger, 2006; Embrechts, McNeil & Liu, 2002). In this last field, the copula models have many applications, as for instance in the area of risk aggregation (Chavez-Demoulin, Embrechts & Nešlehová, 2006; Embrechts & Puccetti, 2006; Fantazzini, Dalla-Valle & Giudici, 2008) and the study of the contagion effect (Cambriles & Benito, 2023; Rajwani, & Kumar, 2019 and Hussain & Li, 2018). Besides, copula analysis can also be used for analysing the properties of an asset as a diversifier and risk hedge (Kliber, Marszałek, Musiałkowska & Świerczyńska, 2019; Kang, Mclver, & Arreola, 2019 and Feng, Wang, & Zhang, 2018).

As the copula analysis has growth in popularity the number of papers publish in this field has increased enormously, proposing: (i) new parametric copula model (Nelsen, 2007; McNeil & Nešlehová, 2009) (ii) new methods for parameter copula estimation (Joe, 1997; Tsukahara, 2005) or (iii) some goodness-of fit test for selecting copula model (Dobrić & Schmid, 2007; Genest & Rémillard, 2008; Genest, Quessy & Rémillard, 2006; Breymann Dias, & Embrechts, 2003; Genest & Rivest, 1993).

The problem of estimating copula parameter has been largely studied, however, as point Fang, Madsen and Liu (2014) selecting the functional form for copula is an open question in literature. To select the best copula model, it is very important as not all of them impose the same dependency structure between variables. For example, the Gaussian copula imposes a fixed dependency ratio. Other copulas allow us to model changes in this relationship of dependence. Thus, the election of the copula is a crucial issue.

[1] For a thorough literature review of copula, see Nelsen (2007)

A typical problem that arises when fitting copulas to a set of data is how to decide for the best fitting model. To that, different tools can be used such as: (i) graphic methods; (ii) information criterions and (iii) the goodness of fit tests. The problem associated with the GoF test is that they do not provide a ranking of the estimated models. The essence the GoF test is checking whether the unknown copula actually belongs to the chosen parametric copula family or not. Thus, they are only used to check whether we should reject or fail to reject the chosen copula. As a consequence, we cannot use the GoF test to choose the parametric copula that fit the best the data (Fang et al., 2014). Unlike the GoF test the information criterion can be used to select the best copula from a group of copula family.

Although the existing literature on the performance of the information criteria for the selection of a statistical model is quite extensive, in the particular case of copula models it is very limited or practically non-existent. To cover this gap, we carry out an experimental design with the porpoise of studying how accurate are the information criterion for choosing the true copula when we know that the truce copula exists, and it is among the candidates.

For this study the Information Criterion considered are: Akaike information criterion [AIC] (Akaike, 1974, 1976), Bayesian information criterion [BIC] (Schwarz, 1978), Hannan-Quinn information criterion [HQIC] (Hannan & Quinn, 1979), and Shibata information criterion [SIC] (Shibata, 1976, 1980). To assess the performance of these tools we conduct a simulation study comparing the performance of these criterions.

The remainder of the paper is organized as follow. Section II presents a review of the analysis of copula theory, estimation and information criterion. Section III describes the simulation exercise for assessing the performance of the IC. Section IV presents the results and Section IV offers some concluding remarks.

2. A Brief Literature Reviewed on Copula Analysis in the Area of Finance

In the area of finance, copula analysis has many applications. For instance, it is very useful for risk aggregation. In this area, copula analysis can be used for aggregating risks of different nature, as credit risk, exchange rate risk and market risk (see Embrechts, Puccetti, & Rüschendorf, 2013; SyuhadaI & Hakim, 2020; and Bolancé, Guillén & Padilla, 2015). A European investor may have an asset portfolio composed by USA stocks. This portfolio would be affected by two types of risk: market risk and exchange rate risk. To quantify the global risk of the portfolio, both risks must be taken into account. Traditionally, this risk has been calculated as the sum of both, the Value at Risk (VaR) of the portfolio[2] and the VaR of the exchange rate[3]. By doing this, it is assumed that both sources of risk are perfectly correlated. If this assumption fails, the global risk of the portfolio would be overestimated. Copula models, let us to estimate the aggregate risk of a portfolio without assuming a perfect correlation between markets which is pretty usual in practice.

Copula analysis is also applied in the area of operational risk. Operational risk represents the risk of losses arising from the materialization of a wide variety of events, including fraud, thief, computer hacking, loss of key staff members, lawsuits, loss of information, terrorism, vandalism and natural disasters. The risk management group (RMG) of the Basel Committee and industry representatives have agreed on a standardized definition of operational risk (BCBS, 2004): i.e. “the risk arising from inadequate or failed internal processes, people and systems or from external events” (see Benito & López-Martín, 2018).

An important issue in this area has to do with the aggregation of operational losses across business units and event types to obtain a global Operational Value at risk (OpVaR) measure[4]. A simplistic approach involves the estimation of OpVaR measures for each business line and type of risk independently and then adding them up to produce the aggregate measure of bank risk.

As indicated before, calculating global OpVaR by adding the losses from each business line and type of risk

[2] Value at Risk (VaR) of a portfolio asset tell us the maximum amount of an investor may loss over a given time horizon a with a given probability. Formally speaking, the VaR of a portfolio at $\alpha\%$ probability is given by the percentile $\alpha\%$ of the probability distribution of the return portfolio.

[3] The VaR of the exchange rate informs us about the minimum price that a currency could reach in a given period and with a given probability. At $\alpha\%$ probability, this measure is given by the percentile $\alpha\%$ of the probability distribution of the exchange appreciation rate.

[4] In the context of the Actuarial approach, operational risk losses are measured by a high quantile of the loss distribution. By similitude with market risk measure —Value at Risk (VaR)—this measure is called Operational Value at Risk (OpVaR).

involves an assumption of perfect dependence among the different losses[5]. Some empirical papers show that the correlation between the severity data from different types of risk and business line is very small in some cases; hence, calculating the OpVaR of the company by adding the OpVaR of different business lines and risk types may overestimate the risk. Applications of copula model in this field can be found in Chavez-Demoulin et al. (2006), Embrechts and Puccetti (2006), Chapelle, Crama, Hubner & Peters, (2005) and Fantazzini et al. (2008).

Copula analysis can also be applied for studding the contagion effect between financial markets. Forbes and Rigobon (2002) define contagion as the significant increase in cross-market correlations between any two markets from pre-crisis period to crisis period. To assess this subject, correlation analysis and Dynamic Conditional Correlation (DCC) has been traditionally used. Both methods allow us to identify dependence and contagion effect when there exists linear relationship between the marginals or series under the study. However, when the relationship between marginals is non-linear these models will not be able to reap correct results. To overcome these limitations a copula analysis has been proposed. Application of copula analysis for analyzing contagion effect can be found in Rajwani and Kumar (2019), Wang and Liu (2011), Horta, Mendes and Vieira (2010), Hussain and Li (2018), Weng, Wei and Huang (2012), Nguyen, Ishac and Henri (2017) and more recently in Cambriles and Benito (2023).

To last, copula analysis can also be used for analyzing the properties of an asset as diversifier and risk hedge. In this field, recently studies use copula models for studying the ability of Bitcoin to act as a diversification asset and hedge against stock assets risk, see for instance, García-Jorcano and Benito (2020), Kliber et al. (2019), Kang et al. (2019), Klein, Thu, & Walther (2018) and Feng et al. (2018) among others. The motivation for these studies is that, as the literature points out, stock markets are exposed to factors that are different from those of Bitcoin. While the traditional stock markets depend on macroeconomic variables, such as the government's fiscal or monetary policy, Bitcoin depends more on speculative factors and supply and demand. This fact opens the possibility that the Bitcoin market is a source of diversification against the risk of the stock markets (García-Jorcano & Benito, 2020). The results obtained in the papers aforementioned, offer positive evidence to this regard.

3. Copula Theory

3.1 Copula Basics

Let $X_1, X_2, \dots, X_d$ be a set of random variables with a marginal distribution function given by $F_i(x_i)$, where $F_i(x_i) = \Pr(X_i \leq x_i)$ for $i = 1, \dots, d$. A copula is a function that joins (or couples) the univariate distribution functions to a multivariate distribution function (F), as denoted by C in the equation:

$$H(x_1, x_2, \dots, x_d) = C(F_1(x_1), F_2(x_2), \dots, F_d(x_d)) \tag{1}$$

Alternatively, a copula can be defined as the multivariate distribution, C, of a vector of random variables with uniformly distributed marginal $U(0,1)$

$$C(u_1, u_2, \dots u_d) = H(F_1^{-1}(u_1), F_2^{-1}(u_2), \dots, F_d^{-1}(u_d)) \tag{2}$$

where the $u_i = F(x_i)$ and F_i^{-1}'s are the quantile functions of the marginals. A copula extracts the dependence structure from the joint distribution, independent of marginal distributions.

The copula density is given by[6]

$$c(u_1, u_2 \dots, u_d) = h(F_1^{-1}(u_1), F_2^{-1}(u_2), \dots, F_d^{-1}(u_d)) \Big| \prod_{i=1} f_i(F_i^{-1}(u_i)) \tag{3}$$

where $f_i(F_i^{-1}(u_i))$ is the probability density function (*pdf*) for the variable $\mathrm{F_i^{-1}(u_i)}$ and $h(.)$ is the join *pdf* for multivariate vector: $F_1^{-1}(u_1), F_2^{-1}(u_2), \dots, F_d^{-1}(u_d)$.

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[5] Let $z = x + y$; it can be shown that if the correlation coefficient between x and y is equal to 1, then $VaR_z(\alpha) = VaR_x(\alpha) + VaR_y(\alpha)$.

[6] $c(u_1, u_2, \dots u_d) = \frac{\partial c(u_1, u_2, \dots u_d)}{\partial u_1 \partial u_2, \dots \partial u_d}$

Table 1 displays the functional forms of the five copulas used in this paper: (i) Gaussian, (ii) Student-t, (iii) Clayton, (iv) Gumbel and (v) Frank. The two first below the elliptical copula family and the rest to the Archimedean copula family.

Table 1. Functional forms for elliptical and archimedean copulas

	Functional form of copula	Density copula
Elliptical copulas		
Gaussian copula	$C(u_1,u_2;\ \rho)=\Phi_2(\Phi^{-1}(u_1),\Phi^{-1}(u_2))$	$c(u_1,u_2;\ \rho)\ =\frac{1}{\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}(x_1^2-2\rho x_1x_2+x_2^2)\right)$
Student-t copula	$C(u_1,u_2;\rho)=\mathrm{T}_{v,}(t_{v_1}^{-1}(u_1),t_{v_2}^{-1}(u_2))$	$c(u_1,u_2;\ \rho)\ =\ \frac{K}{\sqrt{1-\rho^2}}\left[1+\frac{1}{v(1-\rho^2)}(x_1^2-2\rho x_1x_2+x_2^2)\right]^{\frac{v+2}{2}}[(1+v^{-1}x_1^2)(1+v^{-1}x_2^2)]^{\frac{v+2}{2}}$
Archimedean copulas		
Clayton copula	$C(u_1,u_2;\alpha)=(u_1^{-\alpha}+u_2^{-\alpha}-1)^{-1/\alpha}$	$c(u_1,u_2;\ \alpha)=(1+\alpha)(u_1^{-\alpha}+u_2^{-\alpha}-1)^{-\frac{1}{\alpha}-2}\,(u_1u_2)^{-\alpha-1}$
Gumbel copula	$C(u_1,u_2;\delta)=\exp\{-\left((-\ln(u_1))^{\delta}+(-\ln(u_2))^{\delta}\right)^{1/\delta}\}$	$c(u_1,u_2;\delta)=(A+\delta-1)A^{1-\delta}\exp(-A)\,(u_1u_2)^{-1}(-\ln u_1)^{\delta-1}(-\ln u_2)^{\delta-1}$
Frank copula	$C(u_1,u_2;\theta)=-\frac{1}{\theta}\,ln\left(1+\frac{(e^{(-\theta u_1)}-1)(e^{(-\theta u_2)}-1)}{e^{(-\theta)}-1}\right)$	$c(u_1,u_{2;\,\theta})=\frac{\theta[1-\exp(-\theta)]\exp(-\theta u_1u_2)}{\left([1-\exp(-\theta)]-(1-\exp(-\theta u_1))(1-\exp(-\theta u_2)\right)^2}$

Note: Φ_2 is the standard bivariate normal; ρ is simply the linear correlation coefficient; Φ^{-1} is the inverse of the standard univariate Gaussian; T_v is the standard bivariate Student-t with v degrees freedom; $t_{v_i}^{-1}$ is the inverse of the standard univariate Student-t with v_i degree of freedom. In copula Clayton the dependence is measure by α parameter. δ denotes de dependence parameter in copula Gumbel and θ is the dependence parameter in copula Frank. In addition, $K=\Gamma\left(\frac{v}{2}\right)\Gamma\left(\frac{v+1}{2}\right)^{-2}\Gamma\left(\frac{v}{2}+1\right);\ A=\left[(-\ln u_1)^{\delta}(-\ln u_2)^{\delta}\right]^{1/\delta}$

3.2 Estimation

In the experimental design we use the inference in margins (IFM) estimation method for estimating copula parameter[7]. This method is known as a two-step maximum likelihood method. In the first step, the marginal distributions are fitted by any fitting method of univariate probability distribution for each random variable. In the second step, the parameter of the copula model is estimated by maximizing the likelihood function $L(\theta)$

$$\theta^*=\arg max\ L(\theta)$$

$$\theta^*=\arg max\sum_{i=1}^{n} ln\, c\left(u_{1,i},\dots,u_{d,i};\theta\right) \tag{4}$$

where c_θ is the density function of the parametric copula $C_\theta\,\epsilon\, C_0$ and $u_{\mathrm{j}}=F_j\left(x_j\right)$. For practical reasons, the parameter θ is estimated by maximizing

[7] Joe (1997) demonstrates that under standard regularity conditions, this two-stage estimation is consistent, and the parameter estimated are asymptotically efficient.

$$\theta^* = \arg max \sum_{i=1}^{n} ln\, c\left(\hat{u}_{1,i}, \dots, \hat{u}_{d,i}; \theta\right) \quad (5)$$

the symbol $\sum$ of equations 4 and 5 does not look good

where $\hat{u}_i = \{\hat{u}_{i,1}, \dots, \hat{u}_{i,d}\}$ are the pseudo-observations[8] which can be calculated as follow:

$$\hat{u}_{i,j} = \left(\frac{n}{n+1}\right) u_{i,j} \quad (6)$$

As for copula selection, we are only interested in the fit of the copula, we do not wish to introduce any distributional assumption for the margins so that we use the empirical margins to transform the data set into the observed copula. Thus, $u_{i,j}$ will be calculated using the empirical distribution of the X_j for $j = 1, \dots d$.

3.3 Information Criterion

Four different, popularly known, methods are used for our purpose. The one most popular is Akaike's (1974) information criterion. The Akaike Information Criterion (AIC) has commonly been used and significantly known method in the model selection for decades in a wide variety of fields for analyzing actual data. AIC is calculated as follow:

$$AIC = -2 \cdot \log L(\theta) + 2k$$

where $\log L(\theta)$ the log likelihood and k is the number of parameters. AIC has a strong theoretical underpinning, based Kullback-Leibler information[9]. In AIC, the estimation is divided into two sets and the compromise takes place between the log maximized likelihood i.e., $-2\log L(\theta)$ (the lack of fit component) and k (the penalty component) which increases with the number of parameters, and it is used to prevent overfitting. Penalizing likelihood in the model is an attempt to select a more parsimonious model.

The advantages of this criterion are: (i) does not require the assumption that one of the candidate models is the "true" or "correct" model; (ii) all the models are treated symmetrically, unlike hypothesis testing; (iii) can be used to compare nested as well as non-nested models; (iv) can also be used to compare models based on different families of probability distributions.

By other hand, existing studies showed that AIC is not consistent and hence does not lead to the choice of the correct model, with high probability, in large samples. Shibata (1976) showed through empirical evidence that AIC has the tendency to choose models which are over-parameterized. Various modifications have been produced to overcome this lack of consistency. Schwarz (1978) developed a consistent criterion for models defined in terms of their posterior probability (Bayesian approach) which is given as:

$$BIC = -2 \cdot \log L(\theta) + k \cdot log(n)$$

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[8] The scaling factor $(n/(n+1))$ is only introduced to avoid potential problems with C_{θ_n} blowing up at the boundary of $[0,1]^d$.

[9] For a density f that represents the true model and a density g of the model that is going to prove, the relative entropy of Kullback-Leibler can be written as:

$$D_{Kl} = \sum_{x \epsilon X} f(x) \log \frac{f(x)}{g(x)} = E_i[\log(f(x) - \log g(x)]$$

Assuming that the true model is unknown but constant the previous expression is simplified:

$$D_{Kl} = C - E_f[\log g(x)]$$

where C is a constant. The idea is to find the model g to minimize the loss with respect to the true f. By minimizing the expected value of the information loss given by D_{Kl}, the Akaike criterion expression is derived $-2\log L(\theta) + 2k$.

where n is the sample size. Some characteristics of BIC are: (i) consistent unlike AIC; (ii) like AIC, the models need not be nested to use BIC; and (iii) AIC penalizes free parameters less strongly than does the BIC.

Hannan-Quinn (1979) proposed another consistent criterion based on law of iterated logarithm, which has the quality for penalty function to decrease as fast as possible for a strongly consistent estimator, as the sample size increases. The criterion is given by:

$$HQIC = -2 \cdot \log L(\theta) + 2 \cdot k \cdot log(log(n))$$

Similar to AIC, the HQC introduces a penalty term for the number of parameters in the model, but the penalty is larger than one in the AIC. HQC, like BIC, but unlike AIC, is not asymptotically efficient.

To last information criterion used is the proposed by Shibata [SIC] (Shibata, 1976, 1980). This criterion is defined as follows

$$SIC = -2 \cdot \log L(\theta) + log(n + 2 \cdot k)$$

where n is the number of data, and k is the number of parameters used in the model Thus, one should select the model that yields the smallest value of AIC, BIC, HQIC, and SIC, because this model is estimated to be “closest" to the unknown truth, among the candidate models considered

4. Experimental Design

A large-scale Monte Carlo experiment was conducted to compare the ability of the information criterions (IC) in order to choose the true copula from a series of given candidate families where the true copula is included.

In this experimental design, five one-parameter copula families were considered: Normal, Student-t with v degree freedom, Clayton, Gumbel and Frank. They are abbreviated as N, t_v, C, G and F, respectively, in the forthcoming tables. For each family copulas, three degrees of dependence, were considered both positive and negative $(\tau = 0.9, 0.5, 0.1)$ and three size samples $(n = 1000, 500, 100)$. In addition, for the elliptic copulas and the Frank copula, three negative degrees of dependence have been taken into account (τ = −0.9, −0.5, −0.1). To last, to curtail the computational effort, the comparison was limited to the bivariate case.

For each copula family and fixed value of τ, 1.000 bivariate random samples of size n were simulated. These simulations were used to study the performance of the information criterion. In particular, for the 1000 bivariate samples, we calculate the percentage of times that the candidate copulas are pointed as the best in fitting data for each criterion. The information criterion considered in this study are AIC, BIC, HQIC and SIC.

The procedure for studying the performance of the Information Criterion is as follow:

Step 1: Set $\mathrm{N} = 1000$ and repeat the following steps for every $\mathrm{k} \in \{1,2, \ldots, \mathrm{N}\}$

1.1 For a given copula family considered (N, t_v, C, G and F) we generate a bivariate random sample $U^k = \{U_{1,i}^k\ U_{2,i}^k\}$, $i = 1,2, \ldots, n$ where n is the size sample.

1.2 Assuming that the marginal follows a student-t distribution with v_1 and v_2 degrees freedom respectively we get a random sample of $X^k = \{X_{1,i}^k\ X_{2,i}^k\}$, $i = 1,2, \ldots, n$.

1.3 For each simulated sample X^k we calculate the pseud-observations $\widehat{U}^k = \{\widehat{U}_{1,i}^k\ \widehat{U}_{2,i}^k\}$, fit the candidate copulas (N, t_v, C, G and F) and calculate the *Information Criterion (IC)*. After, we find out for the copula that provide the lowest information criteria $IC = \{AIC, BIC, QHIC, SIC\}$.

Step 2: We calculate the percentage of times that each $IC = \{AIC, BIC, QHIC, SIC\}$ point to candidate copulas as the best in fitting data.

In this simulation, one of the interest is to check whether the true copula used to generate the random sample gives the highest proportions of least *IC*.

5. Results

In this section we report the results obtained in the experimental design. First, we analyse the performance of the information criterion in fitting copula simulated data when we consider positive dependence τ (0.9, 0.5 and 0.1) and three sample size n (1000, 500, 100). Then, we report the results for the case of negative dependence.

5.1 Performance of the Information Criterion with Positive Dependence

For different dependence degree τ (0.9, 0.5 and 0.1) and three sample size n (1000, 500, 100), Table 2 reports the percentage of times that each information criterion (IC) points the candidate copulas as the best in fitting data, knowing that the best copula is that that reap the least IC. As an example, Table 2 shows that when the true copula is the Normal copula with a dependence level of 0.9 and a sample size of 1,000, the AIC criterion point to the Normal copula as the best in fitting data the 89,6% of the times. In this table we show in bold the cases in which the candidate copula is the true copula.

For tree of the copula family considered, the Normal, Student-t and Frank, it is observed that, for a fixed sample size, the accurate of the information criterion is high for a medium and high level of dependence. However, the accurate of these criterions is reduced as the level of dependency decreases. Figure 1 (Panel (a)) illustrates this result for the AIC information criterion. For example, when the true copula is the Frank copula with a dependence level of 0.1 and n = 1000, the AIC criterion points this copula as the best in fitting data the 47.7% of the times and the 100% of the times for dependence levels of 0.5 and 0.9. When the true copula is the Student-t(ν), the effectiveness of this criterion is: 75.9%, 85.0% and 99.9% for dependence level of 0.1, 0.5, and 0.9 respectively. In the case of Gumbel and Clayton, the accurate of the information criterion is high for all dependence levels[10].

Moreover, we notice that the smaller the sample size, the lowest is the accurate of the information criterion. Panel (b) of Figure 1 illustrate these results for AIC information criterion. For example, when the true copula is the Normal copula with a dependence level of 0.9 and $n = 1000$, $n = 500$ and $n = 100$ the AIC criterion points this copula as the best in fitting data the 89.9%, 84,4% and 50.3% respectively. When the true copula is the Student-t(v), the effectiveness of this criterion for a dependence level of 0.5 is: 85.0%, 64.0% and 52.9% for a sample size of 1000, 500, and 100 respectively. In the case of Gumbel and Clayton, the accurate of the information criterion is high for all dependence levels[11].

Table 3, which is a sub table of Table 2, reports the percentage of times that each information criterion (IC) points the candidate copulas as the best in fitting data, when the candidate copula is true. In this table we shade in grey the IC that reap best results, it is to say, the best accurate.

We observed that the BIC criterion reaps better results in 34 of the 42 cases analysed, followed by the SIC criterion with 20, and 13 both, the AIC criterion and the HQIC criterion. Although the SIC criterion performs the best after BIC, it shows a somewhat erratic behaviour, which is reflected in a greater variance of the percentages offered by the information criteria, when the SIC criterion is included in the comparison.

[10] Note that Clayton copula for $\tau = 0.1$ could not be estimated for any sample size.

[11] Note that Clayton copula for $\tau = 0.1$ could not be estimated for any sample size.

Table 2. Proportion of the least IC obtained from 1,000 repetitions

T	True Copula	Kendall's τ	Proportion of the least AIC					Proportion of the least BIC					Proportion of the least HQIC					Proportion of the least SIC				
			N	t	C	G	F	N	t	C	G	F	N	t	C	G	F	N	t	C	G	F
1000	N	0.1	**65.9**	4.1	--	8.8	21.2	**68.9**	0.3	--	8.8	22	**68.1**	1.4	--	8.8	21.7	**0**	82.3	--	4.8	12.9
		0.5	**90.3**	9.5	0.2	0	0	**99.2**	0.6	0.2	0	0	**97.2**	2.6	0.2	0	0	**0**	99.8	0.2	0	0
		0.9	**89.6**	10.4	0	0	0	**99.0**	1.0	0	0	0	**95.2**	4.8	0	0	0	**23.4**	76.6	0	0	0
	t	0.1	7.6	**75.9**	0	16.2	0.3	8.1	**74**	0	17.6	0.3	7.8	**75.2**	0	16.7	0.3	7.3	**77.3**	0	15.1	0.3
		0.5	4.7	**85.0**	0	10.1	0.2	4.9	**84.7**	0	10.2	0.2	4.8	**84.8**	0	10.2	0.2	4.6	**85.1**	0	10.1	0.2
		0.9	0.1	**99.9**	0	0	0	0.2	**99.8**	0	0	0	0.1	**99.9**	0	0	0	0.1	**99.9**	0	0	0
	C	0.1	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--
		0.5	0	0	**100**	0	0	0	0	**100**	0	0	0	0	**100**	0	0	0	0	**100**	0	0
		0.9	0	0	**100**	0	0	0	0	**100**	0	0	0	0	**100**	0	0	0	0	**100**	0	0
	G	0.1	0	0.3	--	**99.7**	0	0.1	0.1	--	**99.8**	0	0	0.3	--	**99.7**	0	0	0.5	--	**99.5**	0
		0.5	0	0.1	0	**99.9**	0	0	0	0	**100**	0	0	0.1	0	**99.9**	0	0	0.2	0	**99.8**	0
		0.9	0	0.1	0	**99.9**	0	0	0.1	0	**99.9**	0	0	0.1	0	**99.9**	0	0	0.1	0	**99.9**	0
	F	0.1	12.7	37.5	--	2.1	**47.7**	15.2	25.6	--	2.9	**56.3**	14	33.4	--	2.3	**50.3**	11.4	44.8	--	1.6	**42.2**
		0.5	0	0	0	0	**100**	0	0	0	0	**100**	0	0	0	0	**100**	0	0	0	0	**100**
		0.9	0	0	--	0	**100**	0	0	--	0	**100**	0	0	--	0	**100**	0	0	--	0	**100**
500	N	0.1	**55.0**	4	--	16.7	24.3	**56.9**	0.8	--	17.6	24.7	**56.5**	1.8	--	17	24.3	**37.5**	28.3	--	13.4	20.8
		0.5	**90.4**	8.6	0	0.8	0.2	**98.2**	0.7	0	0.9	0.2	**95.8**	3.2	0	0.8	0.2	**50.6**	48.5	0	0.7	0.2
		0.9	**84.4**	15.5	0	0.1	0	**98.6**	1.3	0	0.1	0	**94.2**	5.7	0	0.1	0	**51.9**	48	0	0.1	0
	t	0.1	0	**96.1**	--	3.9	0	0.5	**88.1**	--	10.7	0.7	0	**93.8**	--	5.7	0.5	0	**97.6**	--	2.4	0
		0.5	13.2	**64.4**	0.1	21.6	0.7	14.3	**62.1**	0.1	22.8	0.7	13.7	**63.7**	0.1	21.8	0.7	12.5	**66.1**	0.1	20.7	0.6
		0.9	0.3	**99.5**	0	0.2	0	1.2	**98.3**	0	0.5	0	0.4	**99.4**	0	0.2	0	0.1	**99.7**	0	0.2	0
	C	0.1	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--	--
		0.5	0	0	**100**	0	0	0	0	**100**	0	0	0	0	**100**	0	0	0	0	**100**	0	0
		0.9	0	--	**100**	0	0	0	--	**100**	0	0	0	--	**100**	0	0	0	--	**100**	0	0
	G	0.1	1.2	1.3	--	**96.8**	0.7	1.4	0.1	--	**97.5**	1	1.3	0.6	--	**97.3**	0.8	0.6	2.5	--	**96.3**	0.6
		0.5	0.2	1.2	0	**98.6**	0	0.3	0.5	0	**99.2**	0	0.3	1	0	**98.7**	0	0	1.9	0	**98.1**	0
		0.9	0.5	0	0	**99.5**	0	0.5	0	0	**99.5**	0	0.5	0	0	**99.5**	0	0.5	0	0	**99.5**	0
	F	0.1	15.5	36.3	--	7.1	**41.1**	20	20.1	--	8.7	**51.2**	17.4	29.3	--	7.8	**45.4**	13.4	46.3	--	5.6	**34.7**
		0.5	4.6	0.2	0	0.1	**95.1**	4.6	0	0	0.1	**95.3**	4.6	0.1	0	0.1	**95.2**	4.5	0.7	0	0.1	**94.7**
		0.9	0	0	--	0	**100**	0	0	--	0	**100**	0	0	--	0	**100**	0	0	--	0	**100**
100	N	0.1	**35.3**	5.9	--	24.7	34.1	**36.9**	0.6	--	26.9	35.6	**36.5**	2.5	--	26	35	**34.9**	7.2	--	24.2	33.7
		0.5	**66.7**	7.3	3.2	14.8	8	**71.8**	0.1	3.3	16.5	8.3	**71.2**	1.3	3.2	16	8.3	**64.7**	9.5	3.1	14.7	8
		0.9	**50.3**	39.5	--	7.7	2.5	**55,4**	25.2	--	14.8	4.6	**53.0**	33.1	--	10.7	3.2	**49,9**	40.1	--	7.6	2.4
	t	0.1	10.5	**49.8**	--	30.5	9.2	16.2	**28.3**	--	41.7	13.9	12.7	**40.5**	--	34.8	11.9	3.7	**78.9**	--	14	3.4
		0.5	11.1	**52.9**	--	26.4	9.6	18.4	**23.4**	--	42.7	15.5	14.3	**39.9**	--	33	12.8	4.1	**81.6**	--	10.8	3.5
		0.9	0.1	**99.7**	0	0.2	0	0.1	**99.7**	0	0.2	0	0.1	**99.7**	0	0.2	0	0.1	**99.7**	0	0.2	0
	C	0.1	--	--	--	--	--	--	--	--	--	--	2	--	--	--	--	--	--	--	--	--
		0.5	1.5	3.7	**93.6**	0	1.2	1.8	2.8	**94.1**	0	1.3	1.8	3	**93.9**	0	1.3	1	5.3	**92.6**	0	1.1
		0.9	0.3	--	**96.9**	0	2.8	0.3	--	**96.9**	0	2.8	0.3	--	**96.9**	0	2.8	0.3	--	**96.9**	0	2.8
	G	0.1	10.9	10.4	--	**76.4**	2.3	11	8.8	--	**77.5**	2.7	10.9	10.2	--	**76.6**	2.3	10.9	10.4	--	**76.4**	2.3
		0.5	8.86	13.29	0.10	**75.73**	2.01	10.47	8.96	0.10	**78.35**	2.11	9.67	10.78	0.10	**77.34**	2.11	5.84	21.55	0.10	**70.59**	1.91
		0.9	0.00	0.00	99.89	**0.11**	0.00	0.00	0.00	99.89	**0.11**	0.00	0.00	0.00	99.89	**0.11**	0.00	0.00	0.00	99.89	**0.11**	0.00
	F	0.1	41.1	4.7	--	17.7	**36.4**	43.1	1.2	--	18.3	**37.3**	42.5	3	--	17.8	**36.6**	26.5	24.9	--	15.6	**32.9**
		0.5	26.9	28.3	1	3.3	**40.5**	29.1	23.5	1.1	3.5	**42.8**	28	26.5	1	3.4	**41.1**	26.1	31.9	0.8	3.2	**38.0**
		0.9	2.5	0.6	--	0.6	**96.3**	2.7	0.2	--	0.7	**96.4**	2.6	0.4	--	0.7	**96.3**	1.2	2.3	--	0.5	**96.0**

Panel (a): Fixed a sample size Panel (b): Fixed a dependence level

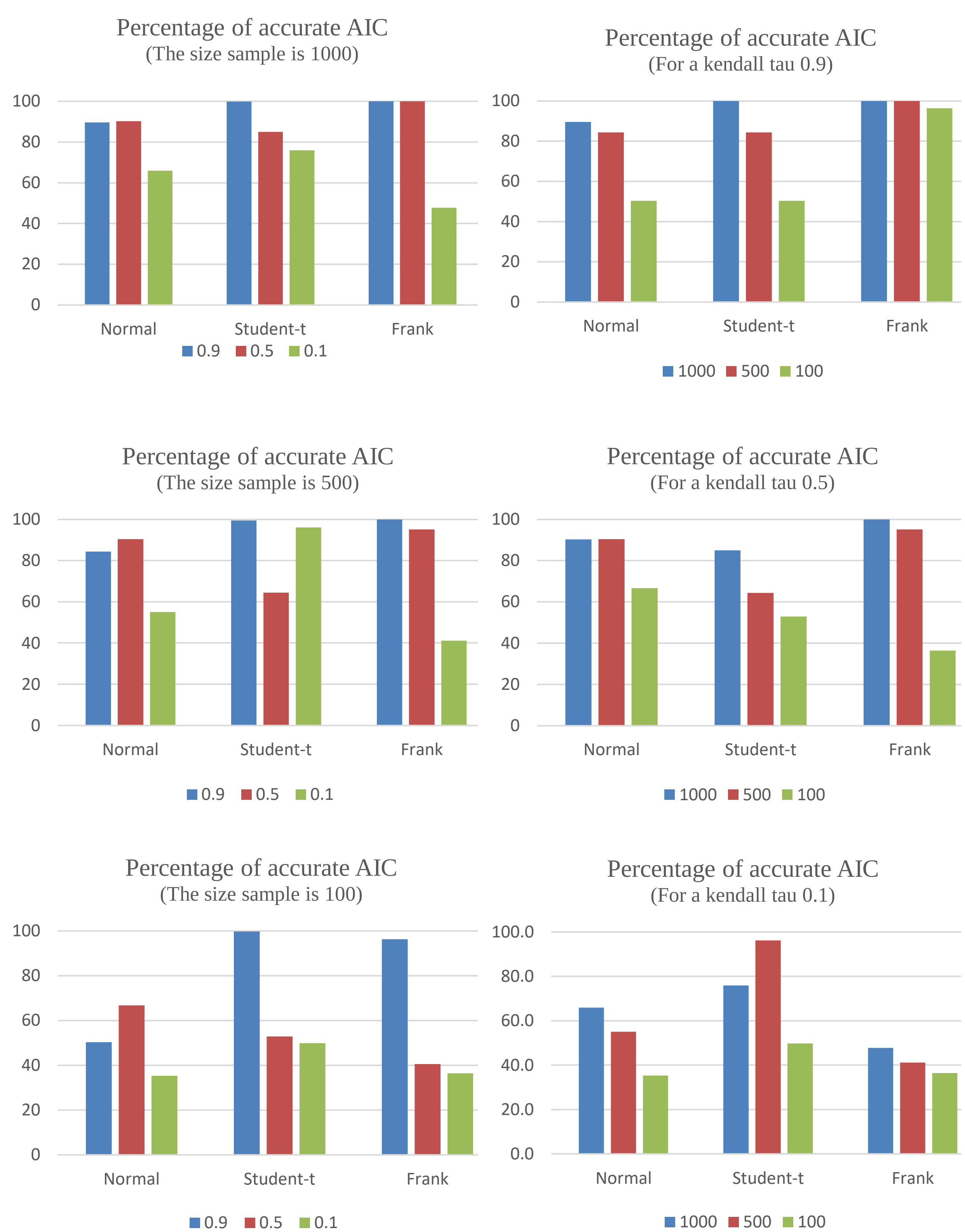

Figure 1. Accurate of AIC information criterion

Table 3. Percentage of times that the true copula reaps the least IC

	True Copula	τ	AIC	BIC	HQIC	SIC
1000	N	0.1	65.9	68.9	68.1	0.0
		0.5	90.3	99.2	97.2	0.0
		0.9	89.6	99.0	95.2	76.6
	t	0.1	75.9	74.0	75.2	77.3
		0.5	85.0	84.7	84.8	85.1
		0.9	99.9	99.8	99.9	99.9
	C	0.1	--	--	--	--
		0.5	100	100	100	100
		0.9	100	100	100	100
	G	0.1	99.7	99.8	99.7	99.5
		0.5	99.9	100	99.9	99.8
		0.9	99.9	99.9	99.9	99.9
	F	0.1	47.7	56.3	50.3	42.2
		0.5	100	100	100	100
		0.9	100	100	100	100
500	N	0.1	55	56.9	56.5	37.5
		0.5	90.4	98.2	95.8	50.6
		0.9	84.4	98.6	94.2	51.9
	t	0.1	96.1	88.1	93.8	97.6
		0.5	64.4	62.1	63.7	66.1
		0.9	99.5	98.3	99.4	99.7
	C	0.1	--	--	--	--
		0.5	100	100	100	100
		0.9	100	100	100	100
	G	0.1	96.8	97.5	97.3	96.3
		0.5	98.6	99.2	98.7	98.1
		0.9	99.5	99.5	99.5	99.5
	F	0.1	41.1	51.2	45.4	34.7
		0.5	95.1	95.3	95.2	94.7
		0.9	100	100	100	100
100	N	0.1	35.3	36.9	36.5	34.9
		0.5	66.7	71.8	71.2	64.7
		0.9	50.3	55.4	53	49.9
	t	0.1	49.8	28.3	40.5	78.9
		0.5	52.9	23.4	39.9	81.6
		0.9	99.7	99.7	99.7	99.7
	C	0.1	--	--	--	--
		0.5	93.6	94.1	93.9	92.6
		0.9	96.9	96.9	96.9	96.9
	G	0.1	76.4	77.5	76.6	76.4
		0.5	75.73	78.35	77.34	70.59
		0.9	0.11	0.11	0.11	0.11
	F	0.1	36.4	37.3	36.6	32.9
		0.5	40.5	42.8	41.1	38.0
		0.9	96.3	96.4	96.3	96.0

This table reports the percentage of times that the *IC* point the true copula as the best in fitting data. For instance, being Normal copula the true copula for a size sample of 1000 and a dependence degree of 0.1 AIC point to the Normal copula as the best in the 65,9% of the sample analysed.

5.2 Performance of the Information Criterion with Negative Dependence

As the Clayton and Gumbel copula do not permit the estimation of negative dependence, this study has been done just only for Normal, Student -*t* and Frank copula.

Table 4 reports -for different dependence degree (τ =-0.9, -0.5 and -0.1) and three sample size (n=1000, 500, 100), the percentage of times that each information criterion (*IC*) points to the three candidate copulas as the best in fitting data.

As the results obtained in the case of positive dependence, we find that the performance of the IC reduces as the size sample reduces. However, in the case the dependence degree does not affect the performance of the IC. Just only in the Normal copula, the accurate of the IC reduce with the dependence degree.

Table 5, which is a sub table of Table 4, reports the percentage of times that each information criterion (IC) points the candidate copulas as the best in fitting data, when the candidate copula is true. In this table we shade in grey the IC that reap best results, it is to say, the best accurate.

We observed that the BIC criterion reaps better results in 21 of the 27 cases analysed, followed by the SIC criterion with 8, HQIC with 4 and AIC 2. Thus, again we find that BIC perform the best in selecting the true copula model between the candidates.

Table 4. Proportion of the least IC obtained from 1,000 repetitions

			Proportion of the least AIC			**Proportion of the least BIC**			**Proportion of the least HQIC**			**Proportion of the least SIC**		
			N	t	F	N	t	F	N	t	F	N	t	F
1000	N	-0.1	**73.3**	5.9	20.8	**78.2**	0.1	21.7	**77.4**	1.0	21.6	**42.3**	39.1	18.6
		-0.5	**62.3**	37.6	0.1	**66.7**	33.2	0.1	**64.1**	35.8	0.1	49.3	**50.3**	0.1
		-0.9	**86.8**	0.5	0	**99.5**	4.4	0.0	**95.6**	4.4	0	**53.7**	46.3	0.
	t	-0.1	0	**100**	0	0	**100**	0	0	**100**	0	0	**100**	0
		-0.5	17.9	**82**	0.1	18.6	**81.3**	0.1	18.6	**81.3**	0.1	17	**82.9**	0
		-0.9	0	**100**	0	0	**100**	0	0	**100**	0	0	**100**	0
	F	-0.1	3.9	0.5	**95.6**	4	0	**96**	2.2	3.6	**94.2**	2.2	3.6	**94.2**
		-0.5	1.1	96.3	2.6	3.1	90.2	6.7	1.9	93.7	4.4	0.4	98.7	0.9
		-0.9	6.1	2.3	**91.6**	7.1	0.7	**92.2**	6.8	1	**92.2**	4.1	5.2	**90.7**
500	N	-0.1	**63.3**	7.4	29.3	**68.7**	1	30.3	**67.5**	2.7	29.8	**37.7**	37.3	25.0
		-0.5	**79.1**	20.6	0.3	**86.1**	13.5	0.3	**83.4**	16.2	0.4	48.7	**51.2**	0.1
		-0.9	**74.6**	25.4	0.0	**85.5**	14.5	0	**81.4**	18.6	0	49.3	**50.7**	0
	t	-0.1	0	**99.9**	0.1	1.4	**97.3**	1.3	0.3	**99.5**	0.2	0	**100**	0
		-0.5	17.9	**82**	0.1	19.9	**80**	0.1	18.6	**81.3**	0.1	17	**82.9**	0
		-0.9	0.7	**99.3**	0	1.8	**98.2**	0	0.9	**99.1**	0	0.5	**99.5**	0
	F	-0.1	14.7	7.8	**77.5**	15.6	5.3	**79.1**	15.3	6	**78.7**	8.5	17.8	**73.7**
		-0.5	11.4	2.2	**86.4**	12.2	0.4	**87.4**	12.1	0.8	**87.1**	6.4	10.5	**83.1**
		-0.9	7	1.1	**91.9**	7.4	0.5	**92.1**	7.3	0.7	**92**	4.3	5.3	**90.4**
100	N	-0.1	**52.6**	7.0	40.4	**56.4**	1.0	42.6	**55.0**	3.5	41.5	33.8	**35.0**	31.2
		-0.5	**81.3**	8.5	10.2	**87.4**	1.9	10.7	**84.6**	4.9	10.7	**46.9**	44.2	8.9
		-0.9	**68.0**	26.31	5.7	**82.0**	2.0	6.0	**75.0**	19.3	5.7	**37.7**	**57.4**	4.9
	t	-0.1	27.4	**66.3**	24.1	39.2	**45**	33.6	33.1	**55.7**	29	18.6	**82.3**	8.9
		-0.5	21.9	**70.2**	7.9	38	**49.9**	12.2	28.2	**62.1**	9.7	4.6	**90.2**	0
		-0.9	32.2	5.6	**69.2**	31.9	7	**68.1**	32	6.4	**68.6**	32.5	5.1	3.5
	F	-0.1	20.9	4	**75.1**	22.3	1.1	**76.6**	21.8	2.3	**75.9**	13.3	16.3	**70.4**
		-0.5	21.2	3.6	**75.2**	22.4	1	**76.6**	22	2.1	**75.9**	13.7	15.7	**70.6**
		-0.9	18.4	3.6	**78**	19.1	1.2	**79.7**	18.9	2.1	**79**	11.5	15.3	**73.2**

Table 5. Percentage of times that the true copula reaps the least IC

	True Copula	τ	AIC	BIC	HQIC	SIC
1000	N	-0.1	73.3	78.2	77.4	42.3
		-0.5	62.3	66.7	64.1	49.3
		-0.9	86.8	99.5	95.5	53.7
	t	-0.1	100	100	100	100
		-0.5	82	81.3	81.3	82.9
		-0.9	100	100	100	100
	F	-0.1	95.6	96	94.2	94.2
		-0.5	2.6	6.7	4.4	0.9
		-0.9	91.6	92.2	92.2	90,7
500	N	-0.1	63.3	68.7	67.5	**37.7**
		-0.5	79.1	86.1	83.4	48.7
		-0.9	74.6	85.5	81.4	49.3
	t	-0.1	99.9	97.3	99.5	100
		-0.5	82	80	80.3	100
		-0.9	99.3	98.2	99.1	99.5
	F	-0.1	77.5	79.1	78.7	73.7
		-0.5	86.4	87.4	87.1	83.1
		-0.9	91.9	92.1	92	90.4
100	N	-0.1	52.6	56.4	55.0	33.8
		-0.5	81.3	87.4	84.6	46.9
		-0.9	68.0	82.0	75.0	37.7
	t	-0.1	66.3	45	55.7	82.3
		-0.5	70.2	49.9	62.1	90.2
		-0.9	5.6	7	6.4	5.1
	F	-0.1	75.1	76.6	75.9	70.4
		-0.5	75.2	76.6	75.9	70.6
		-0.9	78	79.7	79	73.2

This table reports the percentage of times that the *IC* point the true copula as the best in fitting data. For instance, being Normal copula the true copula for a size sample of 1000 and a dependence degree of -0.1, AIC point to the Normal copula as the best in the 73.3% of the sample analysed.

6. Conclusions

Copula analysis outperforms traditional linear correlation analysis and the Dynamic Conditional Correlation (DCC) model in describing the dependence structure between random variables. Because of this the use of copula models has become very popular during the last years specially in the financial field.

As the copula analysis has growth in popularity the number of papers publish in this field has increased enormously, proposing new copulas and new methods for parameter copula estimation. A typical problem that arises when fitting copulas to data is how to decide for the best fitting model. Meanwhile the problem associated with the parameter copula estimation has been large study in the literature selecting the functional form for copula is still an open question in literature. To cover this gap, in this paper we assess the ability of the Information Criterion for choosing the best copula model when we know that the true copula exists, and it is among the candidates.

To assess the performance of the Information Criterion we conduct a simulation study. In this study, five one-parameter copula families were considered: Normal, Student-t with v degree freedom, Clayton, Gumbel and Frank. For each family copulas three degrees of dependence, were considered (0.9, 0.5, 0.1) and three size samples (1000,500,100). In addition, for the elliptical copulas and the Frank copula have been taken into account three negative degrees of dependence (−0.9, −0.5, −0.1).

The results obtained are as follow: First, we find that for a high dependence level (0.9) the reliability of the Information Criterion (IC) is quite good. Overall, all the IC point the true copula as the best in fitting data in around the 90% of the cases (size sample equal to 1000). Second, as the dependence decreases the performance of the IC decreases. Thus, for a dependence level of 0.1 and a size sample of 1000 the percentage of success reduce to 50% in many cases. Third, the performance of the IC not only depend on the dependence degree but the size sample. To this respect we find that, as the size sample reduces, the reliability of the IC reduces. Thus, for a size sample 100 and 0.1 dependence level, the success percentage of the IC reduce to 35% in same cases. In the case of considering negative dependence the reliability of the IC does not depend on the dependence degree but the size sample. As the size sample reduce the performed of the IC reduce. To last, in a comparison among the IC considered, we find that the BIC criterion is the most reliable follow by SIC. The AIC and HQIC reaps similar results.

The results aforementioned suggest that to use IC for choosing the best copula model when the dependence degree between the variables is potentially low and the sample size is reduced, may not be appropriate, as the probability of failing in choosing the best copula model is very high. The problem has to do with the fact that not all copula models impose the same dependency structure. For instance, Gaussian copula is symmetric and does not exhibit tail independence; the Student-t copula has symmetric but nonzero tail dependence; Clayton copula has asymmetric tail dependence being positive in lower tail and zero in upper tail; the same as the Gumbel copula but in reverse. Thus, to study dependence using the inappropriate copula could lead us to draw erroneous conclusions.

The same problem may appear when we analyse the properties of an asset as diversifier and hedge risk, where the dependence between variable is potentially low. In these cases, to use goodness of fit copula test may be preferred. As far as we know, this is the first study that highlight the limitations of Information Criterion in choosing copula model.

In the same line of this paper, for future research it would be interesting to evaluate the performance of the goodness of fit copula test and to analyse if -as in the case of the Information Criterion- it depends on the dependence degree of the variables analysed. It would be even more interesting to compare the reliability of the information criteria - which are very easy to implement- with some of the tests developed for the selection of copulas. As the computational cost of analysing the performance of these tests is very high, we have considered it convenient to leave this task for future research.

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