# Novel Approach to Well Tests Data Processing

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## Abstract

A new bounded well function is suggested for processing well tests data. A solution of the basic partial differential equation with physically meaningful initial and boundary conditions is given using its Laplace transform simultaneously with the proof of its unicity. A model for distance-dependence of drawdown is suggested. Results reveal the link between unsteady and steady state of pumping. Related computational problems are discussed. Examples of processing actual data using these results are presented. They illustrate high accuracy of results and a considerable increase of information obtainable from a well test.

**Keywords:** well tests, continuous model of drawdown, heat equation in cylindrical coordinates, unicity and existence of its solution via Laplace transform

## 1. Introduction and Theoretical Background

Pumping tests data are confronted with models of the aquifer to estimate its transmissivity T, storativity  $\sigma$  and other its characteristics. Accuracy and reliability of these estimates depend on the extent of agreement between models and physical facts. Pumping water from a well at constant rate results in a continuous and bounded decrease of the water level. Its adequate model must be a continuous and bounded function of time. It is commonly accepted that the motion of water in a homogeneous, isotropic aquifer of infinite extent is described by the heat equation in cylindrical coordinate system which reads as follows

$$\frac{\partial^2 h[t,R]}{\partial^2 R} + \frac{1}{R} \frac{\partial h[t,R]}{\partial R} = \frac{\sigma}{T} \frac{\partial h[t,R]}{\partial t}, t \ge 0, R \ge 0$$
(1)

To serve as the model of well test equation (1) must be considered together with such initial and boundary conditions which correspond to physical facts and guarantee its unicity among all solutions of the heat equation.

In what follows an attempt is made to follow these basic physical facts and obtain an adequate continuous and (at least piecewise) smooth function of time  $t \ge 0$ , which corresponds to all obtained data of the test (Sec.1.1) and find a unique solution of (1). (Sec.1.3). A model of distance-dependence of drawdown is suggested in Sec.1.2. These results lead to computational and other practical problems which will be discussed in Sec.2. Here also results of processing actual data sets are presented.

## 1.1 Time Dependence of Drawdown

The time-dependent values of drawdown s(t) mirror the actual state of the aquifer outside of the well, in particular violation of its homogenity in a close vicinity of the bore. It means that s(t) should be described by different continuously connected models in diverse time-spans.

The initial phases of drawdown were addressed by Gregor and and Pastuszek (2018), here only results are shortly ansummarized. The function s(t) is obtained step-by-step as follows:

The drawdown in the first 1-4 seconds reflects the transient behavior of the pump, recorded data have no relation to the aquifer and can be neglected.

During the next 10 -30 seconds say for  $t_0 < t < t_1$ , there is practically no measurable inflow to the bore. Therefore for a constant pumping rate the drawdown grows linearly e.g. as

$$s(t) = a + bt. \tag{2}$$

After this "well-bore storage" a transition phase follows, lasting commonly 1-3 minutes, say for  $t_1 < t < t_2$ .

$$s(t) = z + (a + bt_1 - z)exp\left[\frac{2k(t_{1-t})}{r}\right]$$
, where.  $z = 2\pi rk$  (3)

Evidently, in (3) there is  $s(t_1) = a + bt_1$ , which guarantees continuity of the drawdown model for all  $t \in [t_0, t_2]$ , while k [m/sec] can be interpreted as hydraulic conductivity, r stands for the radius of the bore.

For  $t > t_2$  the Theis's solution of (1) called the well function, or its Jacob's approximate are commonly used. The well function is one of an infinite set of solutions for (1). It assumes arbitrary large values for large enough values of t and therefore does not correspond to basic facts on drawdown.

The following model, the alternative well function (awf for short), is proposed

$$awf(t,R) = \lambda - \frac{Q}{4\pi T} \int_{z_1}^{z_2} \frac{\exp(-x)}{x} dx$$
 where  $z_1 = \frac{R^2 \sigma}{4T(t+d)}, z_2 = \frac{R^2 \sigma}{4Tt}$  (4)

Here  $\lambda$  denotes the (positive) difference between the initial and steady-state level of drawdown which is the only directly measurable parameter of aquifer. Meaning of the new parameter *d* will be explained below.

The function awf has the following properties:

-awf[t, R] is a solution of equation (1),

-*awf* is continuous, non-negative, increasing and bounded for all  $t \ge 0$  and all  $R \ge r$ ,  $0 \le awf(t, R) \le \lambda$ .

-asymptotic expansion for  $t \to \infty$  of awf[t, r] can be given as follows

$$awf(t,r) \approx \lambda - \frac{Q}{4\pi T} \log\left(1 + d/t\right)$$

With actual data (e.g. from Example 1) the maximal deviation of this expansion from values of awf(t, r) for  $t \in [15, 600]$  was less than  $10^{-6}$ . Although (5) does not solve the equation it offers a useful tool to esimate values of T and d.

-Laplace transform of this asymptotics can be found as

$$S_{as}[p] = \frac{\lambda}{p} - \frac{Q}{4\pi T_p} (Q \exp[p/d] \int_{p/d}^{\infty} \frac{\exp\left[-x\right]}{x} dx)$$
(6)

-at t = 0 the awf is an increasing bounded function of the distance from the well axis with  $\lim_{R\to\infty} awf[0, R] = \lambda$ . At R = r it assumes a positive value

$$awf(0, r) = \lambda - \frac{Q}{4\pi T} \prod_{r=0}^{\infty} \frac{r^2 \sigma}{4dT}$$

-It will be shown that together with (2) and (3) the *awf* gives a continuous and accurate model of all values obtained during a complete well test. Values of awf[t, r] for  $0 < t < t_2$  exceed the recorded drawdown. They show a hypothetical drawdown in an aquifer homogeneous for all R > r.

-an important drawback of *awf* is to be coped with. Taking awf[t, r] and awf[0, R] as boundary and initial conditions respectively, the function awf[t, R] is the unique solution of (1) (see Appendix). However, the initial condition is a growing function, which strictly disagrees with reality. It means that for distance-dependence of drawdown another unique solution of (1) should be found or another model should be suggested (see below).

These results are easy to prove. They complete the treatment of data obtained in a pumping test.

Further information on the aquifer could be obtained analysing solutions of equ. (1). Recalling common treatment of PDE, initial and boundary conditions have to be formulated, uniqueness of a corresponding solution must be proved and existence of a solution satisfying these conditions has to be confirmed. A possible way to solve these problems is the Laplace transform approach which will be summarized below.

## 1.2 Distance-dependence of Drawdown

A suitable model of the distance dependence of drawdown can be suggested along with the Darcy law and steps of deriving the model (3). Comparison of the pumping rate with the influx rate to the bore can be generalized to a hypothetical cylinder of radius R, with R denoting the distance from the well bore axis. As a result, a formula is obtained similar to that used as the approximation of drawdown in the previous paragraph, i.e as

$$s[t, R] = \frac{Q}{2k\pi R} (1 - \exp\left[\frac{2k(t_0 - t)}{R}\right]).$$

Here k[m/sec] can be considered as some average value of hydraulic conductivity of the aquifer. Its value can be estimated applying this model with R = r to all data obtained from the well test similarly as in the previous paragraph.

The model allows to visualize a calculated cone of depression even with its changes at various time instants. Among others this model enables a good estimate for the radius of depression when it is correctly defined as the distance from the well bore axis at which the drawdown assumes, say 98 % of the steady-state. With known parameters k, Q this estimate is a simple inequality for R.

## 1.3 The Use of Laplace Transform

Assume the initial condition to be h[0, R]=0 for all R>0 and the boundary conditions let be a suitable continuous and bounded approximation of drawdown for all  $t \ge 0$ . Under such conditions unicity of a solution can be proved (see Appendix 1.)

Existence of the unique solution of (1) can be proved by its construction. It can be obtained using the Laplace transform with respect to the variable t (see e.g. Lavrentev (1958), Fodor (1962))

Denoting  $H[p, R] = \mathcal{L} h[t, R]$  where  $\mathcal{L}$  denotes the Laplace transform equ. (1) is rewritten as follows

$$\frac{d^2H(p, R]}{d^2R} + \frac{dH[p, R]}{dR} / R - apH[p, R] = 0, \text{ with } a = \sigma/T.$$
(8)

It is a non-homogeneous ordinary differential equation of Bessel type with the right-hand part consisting of given initial conditions for the function H. It will be discussed below. The solution of its homogeneous part can be expressed in terms of modified Bessel functions. Denoting the Laplace transform of the boundary condition as S[p] this solution can be adjusted to these conditions.

**Theorem 1** Let S[p] be the Laplace transform of boundary condition h(t, r) = s[t] with  $s[t] < \lambda$ , s[t] > 0 such that S[p] is a meromorphic function with only single poles. Let initial conditions be h[0, R] = 0. Then

$$H[p, R] = \frac{I_0[R\sqrt{p\sigma/T}]}{I_0[r\sqrt{p\sigma/T}]} S[p]$$
(9)

is the Laplace transform of the unique solution of equ.(1) satisfying the given initial and boundary conditions.

Substitution of (9) into equation (8) proves that it is its solution. Put R = r to find that H[p, R] satisfies the boundary condition, the inverse Laplace transform below shows fulfilment of initial conditions. Unicity of this solution follows from Theorem 3. (see Appendix.)

As for the boundary condition:

-equation (1) describes the aquifer as homogeneous, none of its unique solution can copy all results of measurements

-the Laplace transform of the continuous and bounded model of drawdown described above is unacceptable for finding the inverse transform of (9)

In what follows a compromise is suggested for the choice of boundary conditions.

Recall that the result (3) has been found by comparison of the pumping rate with the influx rate (Gregor and Pastuszek, 2018) and it is quite well physically motivated. Therefore its simple modification with  $a = t_0$ , b = 0 can be used for the whole time-span  $t > t_0$  yielding an acceptable approximation of drawdown data. The Laplace transform of this modification is given as

$$S[p] = \frac{Q}{2k\pi R} (1/p - \frac{R \exp{[kt_0/R]}}{2k+p R}),$$

with single poles at 0 and -2k/r. With this approximation a unique solution can be constructed.

Application of Theorem 1. with S[p] as above demands to find the inverse transform of (9). It can be found using the expansion theorem. The result consists of two parts: one corresponds to the two poles of S[p], while the other is an infinite series corresponding to all zeros of  $I_0[r\sqrt{pa}]$  with  $a = \sigma/T$ .

**Theorem 2** The inverse Laplace transform h of H[p, R] is

h[t, R] = h1[t, R] + h2[t, R] with

$$h1[t, R] = \frac{Q}{2k\pi r} (1 - \frac{J_0[R z]}{J_0[r z]} \exp[-k (2t - t_0)/r]), \text{ where } z = \sqrt{2a k/r}$$

$$h2[t,R] = \sum_{n=1}^{\infty} \frac{Q(\left(\exp\left[\frac{kt_0}{r}\right] - 1\right)K_n^2 + 2a \, k \, r)}{kK_n \pi r(K_n^2 - 2akr)} \frac{J_0(RK_n/r)}{J_1(K_n)} \, \exp\left[-\frac{K_n t}{ar^2}\right]$$

Here  $K_n$  is the n-th zero of the Bessel function  $J_0$ ,  $t_0$  denotes the start of wellbore storage. It can be seen that for R = r there is h1[t, r] = s[t] and  $h2[t, r] \equiv 0$ , i.e. h[t, R] satisfies the given boundary conditions. It is worth mentioning that the exponent  $\frac{k_n t}{ar^2}$  is > 100 even for n = 1 in practical applications for all t > 1. Therefore the few considered summands in h2[t, R] become negligible. It should be noted that convergence of the series is rather slow, the results with a finite number of summands exhibit rather large oscillations. Although these oscillations tend to zero with  $t \rightarrow \infty$  numerical verification of initial conditions seems problematic. A few hints to the derivation of Theorem 2. are given in Appendix 2.

These two theorems expand results given e.g. in Lavrentev and Sabat (1958), Fodor (1962), Cole et. all (2011). On the other end they show the limits of application of equation (1) to well tests data processing discussed below.

#### 2. Computation Issues and Examples

Let first the time dependence of drawdown be discussed. To find values of parameters Q, k, T,  $\sigma$ , and others in (2), (3), (4), (5) means to solve a nonlinear inverse problem for data obtained in a well test. Minimizing the mean square error of data versus model proved to be an effective way to avoid subjective errors. The procedure starts with delimiting the subsets of data used for each part of the model, i.e. fix the values  $t_i$ , i = 0, 1, 2. The first two easily follow from the plot of data (see Fig. 4), while  $t_2$  follows similarly from their semi-log or log-log plot of data. In dubious cases, the trial-error method leads to acceptable results. A crude estimate of the "new" parameter d [sec] (and its physical meaning) can be given as  $d = t_3 - t_2$ , where  $t_3$  is the instant when drawdown first reaches the level near the steady-state.

The accuracy and reliability of parameter estimates from well tests data is often discussed and often neglected. Graphical comparison of data with their models seems rather insufficient. Numerical experiments show e.g. that changes of estimated storativity  $\sigma$  in the interval (0.75  $\sigma$ , 1.25  $\sigma$ ) yields no observable changes in the plots whereas e.g. a 5% change of transmissivity is on a plot easily detected. Numerical differences or relative deviation of models compared to data are much more reliable. In the examples below mean values of absolute deviations (MVD[m]) and mean values of relative deviations (MRD[%]) or graphs of these deviations will be given. Compared to known accuracy of measurements they give a realistic gauge for the reliability of obtained solutions of the inverse problems.

Yet another way can be suggested: Taking for a while the increase of drawdown to be a random variable, the complete model of drawdown can be considered as its cumulative distribution function. Hence standard testing of hypotheses (e.g. Kolmogorov-Smirnov, Pearson a.o.) can be used and adequately interpreted.

Numerical treatment of the obtained solution of (1) via Laplace transform is cumbersome even with rather advanced software. Moreover, results of computation tend to be acceptable for large values of t which is not the most interesting region. Due to its proven unicity it has to be concluded that hope to obtain usefull information on situation off the bore using equ.(1) seems to be fading away. To cope with these facts Cesaro summation of the series in Theorem 2. or asymptotics of Bessel functions for  $p \rightarrow \infty$ , i.e for small values of t, (see Fodor Gy. 1962) could perhaps yield acceptable results. Details of these approaches are out of the scope of this paper.

Taking into account these remarks examples of processing actual data sets can be presented.



Figure 1. A Complete Model of Data at B1a with (2), (3), (4) (black), Theis Model for the Last Part of data(red)(left) and a Complete Graph of *awf* (dashed) with Data



Figure 2. Comparison of Data B1a(blue), Theis Model (red) and (4) (black)

## 2.1 Example

Data of two phases of a well test were analysed. They were obtained on a complete well bore in confined isotropic quaternary aquifer located in a neighborhood of the town Bela Crkva (Serbia). The well, say B1b, was pumped for 97 seconds, and regenerated. The two data sets, B1b and B1a (a for after) lead to following results:

B1b: with declared Q = 0.014, r = 0.149, and  $t_0 = 2$ ,  $t_1 = 14$  it was obtained for the model (2) a = 0.23, b = 0.253355, which casts some doubts on the declared value of Q. Changing this value to  $Q_s = 0.0177$  and following the request of continuity of the derivative, parameter k of (3) was k = 0.001667. With these values deviation of the model exhibits MAD = 0.051 m, MRD = 0.5691%

B1a: With declared Q = 0.0102,  $t_0 = 7$ ,  $t_1 = 18$ ,  $t_2 = 170$  and  $t \le 400$  the parameter estimates were as follows:

 $k = 0.001898, \lambda = 7.0573, T = 0.001374, \sigma = 0.00682,$ 

and a calculated  $Q_s = 0.01156$ .

The calculated MAD and MRD were MAD =0.0104 m MRD =0.3211 %. (see Fig.1.)

The use of the Theis well function as a model for  $170 \le t \le 400$  resulted in  $T_T = 0.00284$ ,  $\sigma_T = 1.0510^{-7}$  with its graph shown in Fig 2.

The right-hand part of Fig 1. shows how the solution awf[t, R] of equation (1), obtained under the assumption of homogenity of the aquifer differs from measured data.

Data from B1a exhibit an anomaly between 500-th and 650-th seconds perhaps due to a temporary dysfunction of the pump. Applying the *awf* model on the time-span 700-1500 seconds, estimates of T,  $\sigma$  showed no significant changes.

Solution of equ.(1) via Laplace transform is presented in Fig.3. Application of Theorem 2. to data from B1a with parameters as above allows to obtain a calculated cone of depression and contour plot. Here the boundary conditions were chosen as approximation of the dashed black line in Fig.1. obtained as the *awf* model for all t > 0. Both graphs are in cartesian coordinates, the cone of depression and the contour plot shows the pressure head in the cutomary way, tics for axes are in meters at time t = 400 sec.



Figure 3. Cone of Depression and Contourplot at B1a

#### 2.2 Example

Here results of processing data of a well test (on a complete well bore in confined isotropic, sandstone aquifer) located near the village Repin in central Bohemia are given. Drawdown of a well named S20 was recorded in 1 sec intervals for 881 seconds. The (inner) radius of the bore was 0.225 = m, the declared pumping rate  $Q = 0.0086m^3/sec$ . Successive applications of models (2), (3, (4), (5) gave values of  $t_0$ ,  $t_1$ ,  $t_2$  as 3, 8, 105 sec, respectively. Minimization of the mean square error between data and models gave the following values of requested parameters:

$$a = -0.2032, b = 0.1131, k = 0.005454, \lambda = 2.3265, T = 0.546216, \sigma = 1.8920910^{-14}$$

In a similar manner parameters of the Theis model have been obtained as  $T_T = 0.0071518$ ,  $\sigma_T = 0.00030$ . Since in a plot of data together with graphs of these results the deviations cannot be distinguished, in Fig.4 the numerical values of these deviations for two separate time segments are presented. It shows that these deviations are significantly less than actual errors of measurements. The complete model with above given values of parameters was tested using Kolmogorov-Smirnov's test with a result "do not reject" at the significance level 0.01.

## 2.3 Example

Data of a well test obtained after regeneration of a bore located in the vicinity of the village Sebuzin in northern Bohemia were processed to illustrate the application of the more simple model (5) compared to (4). The results are presented in Fig.5. showing their high accuracy. (Parts of the model have different colors.) Obtained values of parameters for (2), (3) and (5), with r = 0.1625 m were as follows:

*k* =0.00046686, *T* =0.00018, *d* = 1338.

The calculated and declared pumping rates were  $Q_v = 0.0027$ , Q = 0.00435, respectively. Fig.4 intentionally includes parts of graphs that do not belong to the model to show how time instants  $t_0 = 3$ ,  $t_1 = 18$ ,  $t_2 = 600$  were obtained. Note also that (5) yields no information on storativity.



Figure 4. Deviations of Data S20 and Their Complete Model



Figure 5. Data for S20 and Their Complete Model

Distance-dependence based on the suggestion in paragraph 1.2. gave for the well S20 results displayed in Fig. 6. It shows the estimate of the radius of the cone of depression as about 17 m for the steady-state.

Numerous other well test data were processed with comparable accuracy results. It has to be admitted that application of the mean square method, let alone of Laplace transform, requires rather sophisticated software (here Wolfram's MATHEMATICA package was used). However, in the 21-th century accuracy and reliability becomes more important than simplicity, numerical calculations are preferable compared to curve fitting or other graphical approaches. Ready-made procedures for minimization in the described approach reduces the computer time for parameter estimates to a few of seconds even together with visualisation of results. For these reasons the paper did not evades rather complicated mathematics. (Data for included examples can be obtained via e-mail in .xlsx format upon request.)



Figure 6. Evolution of Drawdown at Distance R and Contour Plot of Pumping Results

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#### Appendix 1.

**Theorem 3** The equation (1) with boundary and initial conditions

$$h[t, 0] = \varphi[t], \quad \forall t > 0 \tag{10}$$

and

$$h[0, R] = \psi[R] \quad \forall R > 0, \tag{11}$$

where  $\varphi$  and  $\psi$  are given continuous, smooth, non-negative and bounded functions  $\leq \lambda$  has at most one non-constant solution.

#### **Proof:**

Start by assuming that there exist two such solutions of (1), say  $h_1$ ,  $h_2$  with  $h_1 \neq h_2$ . Then  $h = h_1 - h_2$  solves (1) with zero initial and boundary conditions (i.e. with conditions (10), (11),  $\lambda$  replaced by 0 and  $\varphi = \psi \equiv 0$ ).

Noting that h[t, R] is independent of the angle we conclude that the graph of any solution of (1) in cylindrical coordinates is the surface of a solid of revolution with the z-coordinate as its axis. Denoting its formatting curve by f[R] we may conclude that f = h[., R] is continuous and smooth and with f[r]=0, and  $\lim R \to \infty f[R]=0$ . Hence this function f has a maximum  $f[R_{max}]$ , which depends on the variable t. Take a fixed value for t, say  $t_0$ . For the maximum of f it must be  $h_r[t_0, r]=0$  and  $h_r[t_0, r] < 0$  and (1) implies that  $h_t[t_0, r] < 0$ . This means that for  $t_0 - \Delta t$  with  $\Delta t > 0$  the maximal value of f will grow. Therefore for a sequence  $t_0 > t_1 > t_2$ ... with a zero limit the maximal values of f for these values of t cannot have a zero limit as requested by the zero initial conditions and consequently, f[r] cannot have a positive maximal value, which means that h[t, r] cannot be strictly positive. Similarly, it can be concluded, that it cannot be strictly negative, hence it must be identically zero. The two solutions  $h_1$ ,  $h_2$  must be identical, which proves the theorem.

#### Appendix 2.

Since the function (10) is a meromorphic function of p with single poles only, the expansion theorem can be used. For the Laplace transform of F[p, R] = G[p]/H[p] it states that  $f[t, R] = \sum_{n=1}^{\infty} G[p_n]/H[p_n] \exp[p_n t]$ , where  $p_n$  are all zeros of H[p].

The zeros of the denominator include values 0, -2k/r, which yield h1[t, R]. Further an infinite set of negative zeros of  $I_0$  has to be dealt with. For these zeros the consecutive steps are as follows. Since with  $\sigma/T = a$  there is

$$I_0[r\sqrt{p_n a}] = 0 = J_0[ir\sqrt{p_n a}]$$

it is obtained that  $ir\sqrt{p_n a} = k_n$ , where  $k_n$  is the n-th zero of  $J_0$  for n = 1, 2... Therefore

$$\sqrt{p_n a} = -ik_n/r$$
 and  $p_n = -\frac{k_n^2}{r^2 a}$ 

For the derivative of the denominator in (10) at  $p = p_n$  it is

$$\frac{d}{dp}I_0[r\sqrt{pa}]_{p=pn} = \frac{ar I_1[r\sqrt{p_na}]}{2\sqrt{p_na}} = \frac{ar^2 J_1(k_n)}{2k_n}$$

Using  $I_1[iz] = -iJ_1[z]$ ,  $i^2 = -1$ , equ.(10) can be expanded to find the given result.

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