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The Age or Excess of the $M|G|\infty$ Queue Busy Cycle Mean Value

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Abstract

In this work, approximate and exact results concerning a performance measure of the $M|G|\infty$ system, the age or the excess of the busy cycle, are presented. It will be seen that it is also a measure of the busy period performance. Service distributions for which it is given for a simple expression and others for which this does not happen are considered. For this last case, bounds are deduced. A special emphasis is given to the exponential distribution and to those related with it, useful in reliability theory.

Keywords: age, excess, $M|G|\infty$, busy cycle, busy period

1. Introduction

In a queue system it is usual to call busy period a period that begins when a costumer arrives at the system being it empty, ends when a costumer abandons the system letting it empty, and during it there is always at least one customer being served.

So, in a queue system, there is a sequence of idle periods and busy periods.

Be then the $M|G|\infty$ system initially empty**.** The instants $0, t_1, t_2, \ldots$ at which the system enters in state 0, are a renewal process arrival instants, see (Hokstad, 1979). A cycle is complete whenever a renewal occurs, that is: an entrance at state 0. These cycles are called busy cycles and their length is a random variable designated *Z.*

In (Takács, 1962) it is showed that

$$E[Z] = \frac{e^{\rho}}{\lambda} \text{ and } E[Z^2] = \frac{2e^{2\rho}\int_0^{\infty}\left(e^{-\lambda\int_0^t[1-G(v)]dv} - e^{-\rho}\right)dt}{\lambda} + \frac{2e^{\rho}}{\lambda^2} \tag{1}$$

where λ is the Poisson process arrivals rate, $G(.)$ is the service distribution function, α its service time and $\rho = \lambda\alpha$ the traffic intensity.

Consider now a renewal process which time length between consecutive arrivals is a random variable *Z* and be $A(t)$ the time spent since the last renewal till *t*, or the time spent after *t* till the next renewal. If the renewals represent old devices turning out of order and being replaced, $A(t)$ is the *age* of the device in use at instant *t* or the remaining lifetime of a device in use at instant *t* – *excess* of a device in use at instant *t*, respectively.

Being interested in that device age, or excess, mean value, that is $\lim_{s\to\infty}\frac{\int_0^s A(t)dt}{s}$, it can be computed through, see (Ross, 1983).

$$\lim_{s\to\infty}\frac{\int_0^s A(t)dt}{s} = \frac{E[Z^2]}{2E[Z]} \tag{2}$$

Note that being the $M|G|\infty$ queue a system with no waiting and no losses, it is mandatory to present immediately an available server when every costumer arrives, it will be interesting, for instance, in a given instant of a busy period to have an idea of how much more time it will last. So there will have the notion of for how much time it will be necessary to have the servers available. This time is precisely the busy cycle excess.

The results presented will allow answering to this question in mean value terms.

2. $M|G|\infty$ Queue Busy Cycle Age or Excess Mean Value

Calling the $M|G|\infty$ system busy cycle age, or excess, mean value β_c.

$$\beta_c = \beta + \frac{1}{\lambda} \tag{3}$$

where

$$\beta = \int_0^\infty \left(e^{\rho - \lambda \int_0^t [1-G(v)]dv} - 1\right) dt \tag{4}$$

In (Sathe, 1985) it is showed that $\beta = \frac{1}{2\lambda}\rho^2(\gamma_s^2 + 1)E\left[e^{\lambda U(t)}\right]$ and $1 + 2\rho^{-2}(1+\gamma_s^2)^{-1}\left(e^\rho - 1 - \rho - \frac{\rho^2}{2}\right) \le E\left[e^{\lambda U(t)}\right] \le 2\rho^{-2}(e^\rho - 1 - \rho)$ where γ_s is the service distribution coefficient variation. So

$$\frac{\gamma_s^2}{\lambda}\frac{\rho^2}{2} - \alpha \le \beta_c - E[Z] \le \frac{\gamma_s^2}{\lambda}\sum_{n=2}^{\infty}\frac{\rho^n}{n!} - \alpha \tag{5}$$

The Expressions (3) and (4) show that β_c depends on the whole service distribution structure and so is highly sensible to its form. The bounds given by (5) possess the great advantage of being valid whichever the service distribution is and depend only on ρ, λ and γ_s.

The next proposition, immediate consequence of (5), allows to compare β_c with $E[Z]$, that is insensible to the service distribution, since ρ and γ_s are known.

Proposition 1

$$If\ \gamma_s^2 \le \frac{\rho}{e^\rho - 1 - \rho} \qquad \beta_c \le E[Z]$$
$$If\ \gamma_s^2 \ge \frac{2}{\rho} \qquad \beta_c \ge E[Z]$$

Observation:

$$\frac{2}{\rho} \ge \frac{\rho}{e^\rho - 1 - \rho}$$

But $\lim_{\rho\to 0}\frac{2}{\rho} - \frac{\rho}{e^\rho - 1 - \rho} = \frac{2}{3}$.

3. Values of β_c for Some Service Distributions

As it was emphasised in section 2, β_c depends on the whole service time distribution. Then the values of β_c for some particular service time distributions, obtained after (3), are presented:

Exponential

$$\beta_c^M = \frac{1}{\lambda} + \alpha\sum_{n=1}^{\infty}\frac{\rho^n}{nn!}$$

Constant

$$\beta_c^D = E[Z] - \alpha$$

$\boldsymbol{G(t) = \frac{e^{-\rho}}{e^{-\rho} + (1-e^{-\rho})e^{-\lambda t}}, t \ge 0}$, see (Ferreira & Andrade, 2012)

$$\beta_c = E[Z]$$

$\boldsymbol{G(t) = 1 - \frac{1}{1 + e^{-\rho}\left(e^{\frac{\lambda}{1-e^{-\rho}}t} - 1\right)}, t \ge 0}$, see (Ferreira & Andrade, 2012)

$$\beta_c = \frac{e^\rho + e^{-\rho} - 1}{\lambda}$$

Power function with parameter *c* ($\boldsymbol{\alpha = \frac{c}{c+1}}$)

$$\beta_c^P = \frac{1}{\lambda} + \sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{\rho^{n-k}}{(n-k)!}\sum_{j=0}^{k}\frac{(-1)^j}{(c+1)^j j!\,(k-j)!\,(k+jc+1)}$$

Uniform in $[\mathbf{0}, \mathbf{1}]$ (*c*=1)

$$\beta_c^P = \frac{1}{\lambda} + \sum_{n=1}^{\infty}\sum_{k=0}^{n}\frac{\rho^{n-k}}{2^{n-k}(n-k)!}\sum_{j=0}^{k}\frac{(-1)^j}{2^j j!\,(k-j)!\,(k+j+1)}$$

But note that the result for the constant service distribution may be derived easily from (5) making $\gamma_s = 0$.

For any service time distribution, after (5),

$$\beta_c \ge E[Z] - \alpha + \frac{\rho^2}{2\lambda}\gamma_s^2 \tag{6}$$

So, for the $M|G|\infty$ system, fixed α and λ, the least value of β_c happens in the case of constant service time.

The values of β_c are then computed for some values of α and λ and presented in Tables 1 and 2.

Table 1. β_c values for $\lambda = 1$ and various values of α, for some service time distributions

Service Time Distribution	$\alpha = .5$	$\alpha = 1$	$\alpha = 5$	$\alpha = 10$	$\alpha = 50$
Exponential	1.2850757	2.3178568	186.93907	24755.984	5.2920661×10^{21}
Constant	1.1487213	1.7182818	143.41316	22016.466	5.1847055×10^{21}
$\boldsymbol{G(t) = \frac{e^{-\rho}}{e^{-\rho} + (1 - e^{-\rho})e^{-\lambda t}},\ t \geq 0}$	1.6487213	2.7182818	148.41316	22026.466	5.1847055×10^{21}
$\boldsymbol{G(t) = 1 - \frac{1}{1 + e^{-\rho}\left(e^{\frac{\lambda}{1-e^{-\rho}}t} - 1\right)},\ t \geq 0}$	1.2552519	2.0861613	147.41990	22025.466	5.1847055×10^{21}

Table 2. β_c values for $\alpha = 0.5$ and various values of λ, for some service time distributions

Service Time Distribution	$\lambda = 2$	$\lambda = 10$	$\lambda = 20$	$\lambda = 100$
Exponential	1.1589511	19.099311	1244.7304	5.9392749×10^{19}
Constant	0.8591409	14.341316	1100.8233	5.1847055×10^{19}
$\boldsymbol{G(t) = \frac{e^{-\rho}}{e^{-\rho} + (1 - e^{-\rho})e^{-\lambda t}},\ t \geq 0}$	1.3591409	14.841316	1101.3233	5.1847055×10^{19}
$\boldsymbol{G(t) = 1 - \frac{1}{1 + e^{-\rho}\left(e^{\frac{\lambda}{1-e^{-\rho}}t} - 1\right)},\ t \geq 0}$	1.0430806	14.741990	1101.2733	5.1847055×10^{19}
Power function with parameter c ($\alpha = \frac{c}{c+1}$)	1.9626517	17.272158	168.2805	5.2381918×10^{19}

For the service time distributions exponential and power function the β_c values were obtained through (3) by numerical methods.

The values in Tables 1 and 2 evidence the dependence of β_c from the service time distribution structure, although for high values of ρ that dependence vanishes expressively.

After (5) and the expression showed in this section for β_c^M it is possible to obtain lower and upper bounds for this parameter, since the busy period mean value is $E[B]=\frac{e^{\rho}-1}{\lambda}$ for any service time distribution.

If the service time distribution is *NBUE*-New **B**etter than **U**sed in **E**xpectation with mean α, $\int_t^\infty [1 - G(v)]dv \leq \int_t^\infty e^{-\frac{v}{\alpha}}dv$, see (Ross, 1983), and the upper bound obtained for β_c^M is good for β_c^{NBUE}.

If the service time distribution is *NWUE*-New **W**orse than **U**sed in **E**xpectation with mean α, $\int_t^\infty [1 - G(v)]dv \geq \int_t^\infty e^{-\frac{v}{\alpha}}dv$, see (Ross, 1983), and the lower bound obtained for β_c^M is good for β_c^{NWUE}.

If the service time distribution is DFR-**D**ecreasing **F**ailure **R**ate, $1 - G(t) \geq e^{-\frac{t}{\alpha} - \frac{\gamma_s^2}{2} - \frac{1}{2}}$, see (Ross, 1983), and a lower bound for β_c^{DFR} may be obtained following a methodology analogous to the one that allowed to obtain the lower bound for β_c^M. If the service time distribution is IMRL-**I**ncreasing **M**ean **R**esidual **L**ife, $1 - \frac{\int_0^t [1-G(v)]dv}{\alpha} \geq e^{-\frac{2t\alpha}{\mu_2} - \frac{2\alpha}{3\mu_2^2}\mu_3 + 1}$, being μ_r the $G(.)$ r^th^ order moment around the origin, see (Brown, 1981) and (Cox, 1962), and it is possible to find a lower bound for β_c^{IMRL} analogous to the one for β_c^M. For the power function service distribution, as $\gamma_s^2 = [c(c+2)]^{-1}$, a lower bound and an upper bound for β_c^P that, for $c = 1$, are also valid for the uniform in $[0,1]$ service distribution are easily obtained. So

- **$\boldsymbol{\beta_c}$ lower bounds**

  a) M and **NWUE**

$$\mathrm{E}[Z]-\alpha+\frac{\alpha\rho}{2}\left(1+\frac{\rho}{6}\right)$$

  b) DFR

$$\mathrm{E}[Z]-\alpha+\frac{\alpha\rho}{2}\left(2e^{\frac{1+\gamma_s^2}{2}}-1+\rho\frac{3e^{1-\gamma_s^2}-2}{6}\right)$$

  c) IMRL

$$\mathrm{E}[Z]-\alpha+\frac{\lambda}{4}\left(\mu_2 e^{1-\frac{2\alpha}{3\mu_2^2}\mu_3}-2\alpha^2+\rho\frac{\mu_2 e^{2\left(1-\frac{2\alpha}{3\mu_2^2}\mu_3\right)}-4\alpha^2}{6}\right)$$

  d) Power function with parameter *c*

$$\mathrm{E}[Z]+\frac{\rho-2c(c+2)}{2(c+1)(c+2)}$$

  e) Uniform in $[\mathbf{0},\mathbf{1}]$ ($\boldsymbol{c}=\mathbf{1}$)

$$\mathrm{E}[Z]+\frac{\frac{\lambda}{2}-6}{12}$$

- **$\boldsymbol{\beta_c}$ upper bounds**

  a) M and **NBUE**

$$\frac{1}{\lambda}+min\left\{2\left(E[B]-\alpha\right),\frac{\rho}{2}\left(E[B]+\alpha\right)\right\}$$

  b) Power function with parameter *c*

$$\frac{1}{\lambda}+\frac{(c+1)^2}{c(c+2)}E[B]-\frac{c+1}{c+2}$$

  c) Uniform in $[\mathbf{0},\mathbf{1}]$ ($\boldsymbol{c}=\mathbf{1}$)

$$\frac{1}{\lambda}+\frac{4}{3}E[B]-\frac{2}{3}$$

Finally, the ratio of the difference between the upper and the lower bound over the real value, for the exponential and power function service distributions were computed taking $\alpha=0.5\ and\ \lambda=2,10,20,100$, and the results are in Table 3.

Table 3. Ratio of the difference between the upper and the lower bound over the real value- $\alpha=0.5$

Service Time Distribution	$\lambda=2$	$\lambda=10$	$\lambda=20$	$\lambda=100$
Exponential	0.024818024	0.62565866	0.87899084	0.87295261
Power function with parameter *c*; $\boldsymbol{\alpha=\frac{c}{c+1}}$	0.018536302	0.25071787	0.28865152	0.32992972

The best results (that is: the lowest) happen for the power service distribution and for the lowest traffic intensities.

4. Conclusions

It was already emphasized the interest of the $M|G|\infty$ system age or excess of the busy cycle, in the management of that queue, particularly of the availability of the servers.

Then this search was oriented to look for the properties of this parameter. Of course, important are the exact formulae to compute it for the various service time distributions, but some of it result quite complicate, involving infinite sums, making its applicability problematic.

So the importance of the lower and upper bounds that it was possible to compute, mathematically much simpler, namely for service time distributions important in reliability theory such as: Exponential, NBUE, NWUE, DFR and IMRL.

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