

# The Model of Stochastic Optimization of Production under Uncertainty and Risk

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## Abstract

The article describes possibilities of application of models of stochastic optimization of output. The features of the application of various problems of stochastic optimization. The expediency of application of methods of network planning method, PERT, and in the process create the network model of production. It is shown that in conditions of uncertainty and risk when it is difficult to accurately determine the values of the parameters of the problem the use of stochastic models in the planning process of production allows to achieve higher profitability. The authors propose a stochastic model of optimal output and practical recommendations for its application in the activities of industrial enterprises is concluded that to improve the efficiency of planning processes of production it is advantageous to combine the proposed stochastic optimization model with the method of analysis and evaluation of programs.

**Keywords:** stochastic optimization methods, economic-mathematical model, the problem of stochastic optimization, stochastic model, risk, uncertainty, PERT method

## 1. Introduction

In the dynamic external business environment, when the risk becomes a natural and unavoidable factor model stochastic optimization can demonstrate a high efficiency in the process of solving the task of determining the optimal production, inventory levels, investment portfolio. Stochastic formulation of the problem more accurately reflects the economic reality in terms of medium-term planning period. Moreover, the use of stochastic methods significantly exceeds the efficiency of deterministic models in the formation of the optimal production plan, improving financial and economic results of the company's business including profit figures.

It is obvious that the determined characteristics are not suitable for any company in determining the optimal release if you need to increase the planning horizon under conditions of risk and uncertainty. In practice optimization problems with random initial parameters are casual. For example, when planning the activities of industrial enterprises there is always uncertainty associated with the inaccuracy or incompleteness of information on demand, changes in market prices, suppliers and so on.

This is especially true for enterprises with a long production cycle. Random variables include intensity of orders at the enterprises of services industry, number of buyers in shops, number of passengers, etc. Therefore if it is impossible to set the parameter values unambiguously (standards of resources expenses, raw materials stocks, value, etc.) in an applied task, it is referred to as a stochastic one. Methods of task solution with random factors are called the methods of stochastic optimization.

In his studies the author will be guided by rather a detailed analysis of the last achievements and publications which is carried out in the last twenty years (Marti, 2015; Sarker, 2007; Williams, 2013; Song, 2013; Bertocchi, Consigli, & Michael, 2011; Uryasev & Pardalos, 2001; Dzemyda, Saltenis, & Zilinskas, 2002; Zabinsky, 2003; Houmin, George, & Qing, 2006). This source contains the general statements of problems of stochastic optimization, which can serve as the beginning of scientific development.

Common problem here is the absence of concrete practical recommendations what types of probabilistic distributions are to be used in one case or another. Often the sufficient economic justification isn't given to the offered approaches and the mathematics in them is used formally.

The purpose of the article is to develop the model of optimum production in stochastic statement. The author

plans to offer methods of the task realization, having analysed the model on sensitivity to the parameter changes of a production system.

**2. Method**

In mathematical methods of the operations research such direction as stochastic programming is defined. We will pay attention to methods and models of this direction.

Existence of statistical information allows to evaluate selective characteristics of socio-economic systems: empirical function of probabilities distribution, mathematical expectation, dispersion, etc. It is natural that such problems are solved in the conditions of uncertainty and risk. They can be divided into one-stage, two-stage and multi-stage ones (Ali, Khompatraporn, & Zabinsky, 2005).

One-stage problems of stochastic optimization are the ones where the decision is made once and doesn't change any more. Such approach is often implemented to optimally place the production capacities.

In two-stage tasks the initial decision can be modified at the second stage of management of the socio-economic system. For example, the enterprise has no unambiguous information on demand for the new production. Therefore at the first stage production batch allowing to estimate demand is issued. The finishing (second) stage on the basis of information from the first stage forms the modified optimum plan of release.

In two-stage tasks the initial decision can be modified at the second stage of management of social and economic system. For example, the enterprise has no unambiguous information on demand for the new production. Therefore at the first stage a sample quantity is issued allowing to estimate the demand. The finishing (second) stage forms the modified optimum plan of output on the basis of information from the first stage.

Multi-stage problems allow any number of the correction activities. This refers to problems of stockpile management in consecutive periods, optimization of investment projects, operational management of production, technological and other processes.

Problems of stochastic optimization are various in many aspects. For example, if violation of some restrictions leads to serious consequences, it is a task of so-called rigid statement. Restrictions can have probabilistic character. The researcher in stochastic programming might face numerous difficulties regarding the interpretation of probabilistic characteristics, the choice of the efficiency criterion, etc.

In the determined case the researcher isolates the optimum decision from a set of admissible decisions which comply with all restrictions of a task. By stochastic optimization it is possible to consider to be the admissible decision the one which does not comply in an insignificant measure with some of restrictions. Such statement complicates a problem and therefore the system of penalties for violation of restrictions is implemented.

**3. Results**

Stochastic models may contain casual coefficients in criterion function and casual components in the system of task restrictions. We will consider the most widespread statements of problems of stochastic programming.

We will assume that coefficients of the criterion function  $C_j$  ( $j = 1, n$ ) are random variables, i.e.  $C_j(\omega)$ . Other parameters of the model are considered to be determined.

We will discuss, whether such an approach will be reasonable. For this purpose we will consider a classical problem of optimum production.

Let the enterprise start producing  $n$  new items of the products  $P_1, P_2, \dots, P_n$ .  $M$  resources  $S_1, S_2, \dots, S_m$  are used for the production by which we will mean raw materials, materials, components, etc.  $B_i$  resources ( $i = 1, m$ ) are stocked in the warehouse.  $A_{ij}$  units the resource  $S_i$  are spent for the production of  $P_j$ . All the listed parameters are determined.

Table 1. Problem of optimum production in stochastic statement

Resources	Resource consumption for a production unit				Resource supplies
	$P_1$	$P_2$	...	$P_n$	
$S_1$	$a_{11}$	$a_{12}$	...	$a_{1n}$	$b_1$
$S_2$	$a_{21}$	$a_{22}$	...	$a_{2n}$	$b_2$
...	...	...	...	...	...
$S_m$	$a_{m1}$	$a_{m2}$	...	$a_{mn}$	$b_m$
Price of a production unit	$c_1(\omega)$	$c_2(\omega)$	...	$c_n(\omega)$	
Quantity of the output	$x_1$	$x_2$	...	$x_n$	

Since products arrive to consumers for the first time, their market price is supposed to be random. We will designate these prices through  $C_j(\omega)$  ( $j = 1, n$ ). The data referring to such a problem of linear programming is placed in Table 1.

Let  $Z(\omega)$  (mon. unit) be the sales revenue. The stochastic problem definition can be reduced to the determined case if we take the mathematical expectation from the criterion function. The mathematical model of a problem of linear programming composed according to table 1 will look as follows:

$$M[Z(\omega)] = \sum_{j=1}^n M[C_j(\omega)] X_j \rightarrow \max,$$

$$\sum_{j=1}^n a_{ij} x_j \leq b_j \quad (i = 1, m),$$

$$x_j \geq 0 \quad (j = 1, n).$$

Adequacy of model will considerably depend on the choice of probabilities distribution of random variables  $C_j(\omega)$  ( $j = 1, n$ ). Since it is about prices for products, these are the continuous random variables possessing the values from the corresponding intervals.

The author suggests using the random variables which are evenly distributed on a segment  $[\alpha; \beta]$ . According to the task the ends of the segment can be only positive numbers.

To determine such random variables, it is necessary to define the distribution segments. It can be done using statistical methods. Having let out a sample quantity of products  $P_j$ , we will determine by selection the smallest  $\alpha_j$  and the largest  $\beta_j$  prices. Having done the same thing with other goods, we will find out that random variables  $C_j(\omega)$  are distributed evenly on segments  $[\alpha_j; \beta_j]$  ( $j = 1, n$ ).

The mathematical expectation of the evenly distributed random variable  $C_j(\omega)$  is calculated as  $[\alpha_j; \beta_j]$ . Having defined  $Z' \stackrel{def}{=} M[Z(\omega)]$ , we will come to such a criterion function:

$$Z' = \sum_{j=1}^n \frac{\alpha_j + \beta_j}{2} x_j \rightarrow \max$$

Such approach allows to apply methods of linear optimization in the determined form. Along with the initial task

$$\left. \begin{aligned} &x_j \geq 0 \quad (j = 1, n), \\ &a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \\ &a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2 \\ &\dots\dots\dots \\ &a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m \\ &Z = c_1x_1 + c_2x_2 + \dots + c_nx_n \rightarrow \max. \end{aligned} \right\}$$

we will consider a dual task

$$\left. \begin{aligned} &y_i \geq 0 \quad (j = 1, n), \\ &a_{11}y_1 + a_{12}y_2 + \dots + y_{m1}x_m \geq c_1 \\ &a_{21}y_1 + a_{22}y_2 + \dots + a_{m2}y_m \geq c_2 \\ &\dots\dots\dots \\ &a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \geq c_n \\ &F = b_1y_1 + b_2y_2 + \dots + b_my_m \rightarrow \min. \end{aligned} \right\}$$

According to the first duality theorem if one of problems of a dual couple has the optimum solution, and other task has the optimum solution, and extreme values of criterion functions coincide:

$$\sum_{j=1}^n c_j x_j^* = \sum_{i=1}^m b_i y_i^*$$

If criterion function of one of tasks isn't limited, the area of feasible solutions of other task is empty.

The formation of a dual task enables us to carry out the economic-mathematical analysis of the initial problem of linear optimization.

We will consider the initial problem of optimum production in a matrix-vector form:

$$\begin{aligned} Z &= \vec{C} * \vec{X} \rightarrow \max, \\ \vec{A}X &\leq \vec{B} \\ \vec{X} &\geq 0 \end{aligned}$$

Here A is a matrix from coefficients at unknowns in the limitation system;  $\vec{B}$  is a vector of resources supplies;  $\vec{C}$  is a vector of the product prices;  $\vec{X}$  is the required plan of production which will maximize the revenue from realization of Z.

Let  $\vec{Y} = (y_1, y_2, \dots, y_m)$  be the solution to the dual task which has to minimize costs of resources F. Having introduced the transposed matrix  $A^T$ , we will write down the dual task in a matrix-vector form:

$$\begin{aligned} F &= \vec{B} * \vec{Y} \rightarrow \min, \\ A^T \vec{Y} &\geq \vec{C}, \\ \vec{Y} &\geq 0 \end{aligned}$$

According to the second duality theorem, if some variable  $x_j^*$  ( $j=1, n$ ) of the optimal solution of the original problem is positive, then the j-th restriction of the dual problem of its optimal solution becomes strict equality. If the optimal solution of the original problem turns an i -th ( $i = 1, m$ ) restriction in strict equality, the optimal solution of the dual problem variable  $y_i^*$  is bigger than 0.

Thus dual estimates are an indication of the scarcity of resources and products. Value  $y_i^*$  is called the dual assessment or shadow price of the i-th resource. If  $y_i^* > 0$ , then the resource is scarce and by the implementation of an optimal plan  $\vec{X}^*$  is completely consumed. I.e. the i-th constraint addressyы the original problem in strict equality. Acquisition of an additional unit of this resource will result in an increase in revenue from the sale of the value of Z by the value of  $y_i^*$ . The higher the value of the shadow price is, the more scarce the resources are. For a readily available resource  $y_i^* = 0$ .

Further analysis of solving linear optimization is based on a study of the sensitivity of the optimal plan to changes in the values of the parameters  $C_j, a_{ij}, b_i$  ( $i=1, m; j = 1, n$ ). Therefore, such an analysis is briefly called sensitivity analysis.

Suppose that the initial vector of resource stocks has the form of  $\vec{B} = (b_1, b_2, \dots, b_m)$ . Let us introduce the increment  $\Delta b_i$  and the result of such increments  $b_i + \Delta b_i$  ( $i = 1, m$ ). Denoting with  $\Delta \vec{B} = (\Delta b_1, \Delta b_2, \dots, \Delta b_m)$  the increment vector, we obtain a new vector of resource stocks  $\vec{B} + \Delta \vec{B}$ . Now substitute the  $\vec{B}$  vector in the original and dual problems with the  $\vec{B} + \Delta \vec{B}$ . Thus a symmetrical pair of dual multiparameter problems will be formed.

This article has already mentioned the first duality theorem. Its main assertion is that  $Z_{\max} = F_{\min}$  or in another form  $\vec{C} \cdot \vec{X}^* = \vec{B} \cdot \vec{Y}^*$ , where Y is a vector of unknown dual problem. We denote with  $\vec{X}_{\Delta}^*$  the vector of the optimal solutions multiparameter problem. By duality we obtain that  $\vec{C} \cdot \vec{X}_{\Delta}^* = (\vec{B} + \Delta \vec{B}) \cdot \vec{Y}^*$ .

Consider the increase of the objective function of the original problem:

$$\Delta \vec{Z}_{\max} = \vec{C} \cdot \vec{X}_{\Delta}^* - \vec{C} \cdot \vec{X}^* = (\vec{B} + \Delta \vec{B}) \cdot \vec{Y}^* - \vec{B} \cdot \vec{Y}^* = \Delta \vec{B} \cdot \vec{Y}^*$$

If we change only the i-th constraint, then  $(\Delta \vec{Z}_{\max})_i = \Delta b_i y_i^*$ . From which we derive that:

$$y_i^* = \frac{(\Delta Z_{\max})_i}{\Delta b_i} \quad (i = 1, m).$$

After passing the limit we have:

$$y_i^* = \frac{\partial \max}{\partial b_i} \quad (i = 1, m).$$

Therefore, the dual estimates are an indication of the impact of restrictions on the value of the objective function.



Stability interval unit price of products is called the  $i$ -th interval  $[C_i^{\min}; C_i^{\max}]$  with the following properties. If the price of  $C_i \in [C_i^{\min}; C_i^{\max}]$ , and the prices for other types of production are fixed, the optimal production plan  $\vec{X}^*$  remains unchanged.

In order to determine the stability intervals we need to recall the geometric interpretation of linear programming problems. Supporting plans - this corner points formed by the intersection of the hyperplanes system limitations. In finding the extreme value of the objective function hyperplane-isoobjective with the normality vector  $\vec{C} = (c_1, c_2, \dots, c_m)$  moves to the optimal plan  $\vec{X}$  etc.

**4. Discussion**

The organization of a modern and high-tech industry requires a large number of calendar linking of interrelated activities. The compilation and analysis of the schedules is a complex task in dealing with which the method of network planning. These methods make it possible to determine the following. First, what works or operations of the many that make up the project are critical of their effect on the overall duration of the project calendar. Second, how to build the best possible schedule of all activities of the project to support the target date at minimal cost.

Under the network model of organization of production we mean the economic-mathematical model that reflects the full range of activities and events related to the project. We describe a method of analysis and evaluation of programs PERT (Program Evaluation and Review Technique). It was proposed for practical purposes in 1958.

The method of analysis and evaluation of programs differs from the deterministic methods that are calculated for each transaction of its probabilistic characteristics. It is used to control the timing of the project. The PERT method is focused on the analysis of such projects, for which the duration of the implementation of all or some of the work is not possible to determine accurately. First of all, it is about the design and implementation of new productions. In such projects, many of the works are unique. As a result, there is uncertainty in the timing of the project as a whole.

The PERT method assumes that the execution time of each operation is a random variable. It is necessary to define the following three estimates:  $a$  - optimistic time (performance under the most favorable conditions);  $m$  - the most likely time (execution time of normal operation);  $b$  - pessimistic time (performance in adverse conditions).

Numerous studies (Catoni and Picard, 2004; Spall, 2004; Cao, 2007; Shukla and Mishra, 2014; Marti, 2015; Williams, 2013; Song, 2013; Bertocchi, Consigli and Michael, 2011) showed that the execution of the work is well described by a beta probability distribution. The expectation (average) time performance can be estimated from the formula:

$$M_t \approx \frac{a + 4m + b}{6}$$

The estimated variance equals:

$$D_t \approx \frac{b - a}{6}$$

And given the fact that  $b \geq a$ , we obtain an estimate for the standard deviation of the time of the operation:

$$\sigma_t \approx \frac{b - a}{6}$$

Suppose  $T$  is the time the project. If the project requires works to be done, the timing performance of which one can only assume that the time  $T$  is a random variable. The expectation of the time of the project  $M [T]$  is the sum of the expected values of working time  $M_t$ , lying on the critical path. A similar assumption is made with respect to the dispersion and  $D [T]$ .

To determine the critical path of the project critical path method is used. At this stage, the analysis of project execution time work is assumed to be the expected time  $t$ .  $E. M_t$ .

It is expected that the project time  $T$  is the sum of a sufficiently large number of independent, identically distributed random  $t$ -variables. Under these conditions, the central limit theorem of probability theory. This means that the random variable  $T$  has asymptotically normal probability distribution with time parameters  $M [T]$  and  $\sigma [T] = \sqrt{D [T]}$ .

We can set a specific deadline for the project  $T_0$ . Then the probability that the project time  $T$  does not exceed a

specified period  $T_0$ , is approximately calculated as follows:

$$P\{T \leq T_0\} \approx \frac{1}{2} + \Phi\left(\frac{T_0 - M[T]}{\sigma[T]}\right)$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-x^2/2} dx$$

Despite the prospects of using models of stochastic optimization, it is necessary to note that stochastic methods do not guarantee obtaining exact solutions, but they usually allow you to find close enough for the practical use of the solution in a reasonable time. Stochastic nature of most of the methods makes the application of a non-trivial task, since each algorithmic implementation for each class of optimization problems the efficiency, performance, convergence, the impact of the problem and algorithm parameters require thorough research.

## 5. Conclusion

This article offers a fairly wide range of methods for the optimization of socio-economic systems.

As a practical orientation problems the problem of optimal production is addressed. In order to develop production plans the author proposes a specific model of stochastic optimization. This model provides a detailed feasibility study. Methods for solving such problems are described.

It is proposed to use the model not in isolation, but in conjunction with the method of analysis and evaluation of programs. This method allows to design complex manufacturing systems in conditions of uncertainty.

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