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Solution of a Spherically Symmetric Static Problem of General Relativity for an Elastic Solid Sphere

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Abstract

The paper is concerned with the spherically symmetric static problem of the General Relativity Theory (GRT). The classical interior solution of this problem found in 1916 by K. Schwarzschild for a fluid sphere is generalized for a linear elastic isotropic solid sphere. The GRT equations are supplemented with the equation for the stresses which is similar to the compatibility equation of the theory of elasticity and is derived using the principle of minimum complementary energy for an elastic solid. Numerical analysis of the obtained solution is undertaken.

Keywords: general relativity theory, theory of elasticity, spherically symmetric static problem

1. Introduction. Theory of Elasticity Solution

To introduce the proposed approach to GRT problem for elastic solid, consider the problem of the classical theory of elasticity for a sphere whose gravitational field is described by the Newton theory. For a solid sphere with constant density μ and radius R, the Newton gravitational potential ϕ is the solution of the Poisson equation

$$\frac{1}{r^2}\left(r^2\phi'\right)' = 4\pi G\mu \tag{1}$$

Here, r is the radial coordinate $(0 \le r \le R)$, $(\cdots)' = d(\cdots)/dr$ and G is the classical gravitational constant. For the external $(r \ge R)$ space, $\mu = 0$ and Equation (1) has the following well known solution:

$$\phi_e = -\frac{Gm}{r} \tag{2}$$

Where index "*e*" corresponds to the external space and

$$m = \frac{4}{3}\pi\mu R^3 \tag{3}$$

is the mass of a homogeneous solid sphere whose internal space is Euclidean. For the internal $(0 \le r \le R)$ space the solution of Equation (1) which satisfies the regularity condition at the sphere center is

$$\phi_i = \frac{2}{3}\pi\mu G r^2 + C_1 \tag{4}$$

Here, index "*i*" corresponds to the internal space. The integration constant C_1 is determined from the boundary condition on the sphere surface according to which $\phi_e(R) = \phi_i(R)$. Using Equation (2), we can present Equation (4) in the following final form:

$$\phi_i = -\frac{Gm}{2R}\left(3 - \frac{r^2}{R^2}\right) \tag{5}$$

The gravitational body forces which act inside the sphere are

$$f_g = -\mu\phi_i' = -kr\,, \qquad k = \frac{4}{3}\pi G\mu^2 \tag{6}$$

Then, the theory of elasticity equilibrium equation for the sphere element can be presented as

$$r\sigma_r' + 2(\sigma_r - \sigma_\theta) - kr^2 = 0 \tag{7}$$

Here, σ_r and σ_θ are the radial and the circumferential stresses which are accompanied by the corresponding elastic strains following from Hooke's law, i.e.

$$\varepsilon_r = \frac{1}{E}(\sigma_r - 2\nu\sigma_\theta), \qquad \varepsilon_\theta = \frac{1}{E}\left[(1-\nu)\sigma_\theta - \nu\sigma_r\right] \tag{8}$$

in which E and ν are the elastic modulus and the Poisson's ratio of the sphere material. The strains are expressed in terms of the radial displacement u as

$$\varepsilon_r = u', \qquad \varepsilon_\theta = u / r \tag{9}$$

Substituting u from the second of these equations in the first one, we arrive at the following compatibility equation for the strains:

$$r\varepsilon_\theta' + \varepsilon_\theta - \varepsilon_r = 0$$

Using Equations (7), we can write this equation in terms of stresses, i.e.

$$r(1-\nu)\sigma_\theta' - r\nu\sigma_r' + (1-\nu)(\sigma_\theta - \sigma_r) = 0 \tag{10}$$

Thus, we have two equations, Equations (7) and (10), for two unknown stresses. To reduce these equations to one equation with respect to the radial stress, express σ_θ using Equation (7), i.e.

$$\sigma_\theta = \sigma_r + \frac{r}{2}(\sigma_r' - kr) \tag{11}$$

and substitute it in Equation (10) to get

$$r\sigma_r'' + 4\sigma_r' - \frac{3-\nu}{1-\nu}kr = 0 \tag{12}$$

The solution of this equation must satisfy the following boundary conditions:

$$\sigma_r(r=0) = \sigma_\theta(r=0), \qquad \sigma_r(r=R) = 0 \tag{13}$$

Using Equation (11), we can transform the first of these conditions to $\sigma_r'(r=0) = 0$. The final solution for the stresses is

$$\sigma_r = -\frac{(3-\nu)k}{10(1-\nu)}\left(R^2 - r^2\right), \qquad \sigma_\theta = -\frac{(3-\nu)k}{10(1-\nu)}\left(R^2 - \frac{1+\nu}{3-\nu}r^2\right) \tag{14}$$

Having in mind to obtain the solution of the problem under study within the framework of GTR, we should take into account the GRT equations are formulated in the Riemannian space in which the displacement u, as well as the strain-displacement equations, Equations (9), do not exist. Thus, we cannot derive the compatibility equation, Equation (10) using the traditional approach. However, theory of elasticity provides another way to obtain this equation not attracting Equations (9). As known, the compatibility equation formulated in stresses follows from the principle of minimum of the complementary energy under the condition that the stresses satisfy the equilibrium equations. The elastic energy of the solid sphere is

$$U = 4\pi\int_0^R wr^2 dr \tag{15}$$

where

$$w = \frac{1}{2}(\sigma_r\varepsilon_r + 2\sigma_\theta\varepsilon_\theta)$$

is the elastic potential. Substituting the strains from Hooke's law, Equations (8), in Equation (15), expressing σ_θ in terms of σ_r with the aid of Equation (11) and thus satisfying the equilibrium equation, we can reduce the complementary energy to the following functional:

$$U = \frac{2\pi}{E}\int_0^R F(r, \sigma_r, \sigma_r')dr \tag{16}$$

where

$$F=(1-2\nu)(3\sigma_r^2+2r\sigma_r\sigma_r'-2kr^2\sigma_r)r^2+\frac{1-\nu}{2}\left[(\sigma_r')^2-2kr\sigma_r'+k^2\right]r^4 \tag{17}$$

The Euler equation which provides the minimum value of the complementary energy is

$$\frac{\partial F}{\partial \sigma_r}-\frac{d}{dr}\left(\frac{\partial F}{\partial \sigma_r'}\right)=0 \tag{18}$$

Substituting F from Equation (17), we arrive at the compatibility equation, Equation (12).

In conclusion, transform the obtained results introducing the following dimensionless parameters:

$$\bar{\sigma}=\frac{\sigma}{\mu c^2}, \qquad \bar{r}=\frac{r}{R}, \qquad \bar{r}_g=\frac{r_g}{R} \tag{19}$$

Here, c is the velocity of light and

$$r_g=\frac{2mG}{c^2} \tag{20}$$

is the so-called gravitational radius. Using Equations (3) and (6) for m and k and applying Equations (19), we can present the compatibility equation, Equation (12) in the following form:

$$\bar{r}\frac{d^2\bar{\sigma}_r}{d\bar{r}^2}+4\frac{d\bar{\sigma}_r}{d\bar{r}}=\frac{(3-\nu)\bar{r}_g\bar{r}}{2(1-\nu)} \tag{21}$$

The normalized stresses become

$$\bar{\sigma}_r=-\frac{(3-\nu)\bar{r}_g}{20(1-\nu)}(1-\bar{r}^2), \qquad \bar{\sigma}_\theta=-\frac{(3-\nu)r_g}{20(1-\nu)}\left(1-\frac{1+\nu}{3-\nu}\bar{r}^2\right) \tag{22}$$

2. Spherically Symmetric Static Problem of General Relativity for an Elastic Sphere

For a spherically symmetric static problem, the line element is traditionally taken in the following form corresponding to the classical Schwarzchild solution:

$$ds^2=g_{11}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)-g_{44}c^2dt^2 \tag{23}$$

Here r,θ,ϕ and t are space spherical and time coordinates, g_{ij} are the metric coefficients that depend on the radial coordinate r only. For the spherically symmetric static problem and the line element in Equation (23), the conservation equation which is analogous to the equilibrium equation, Equation (7) of the theory of elasticity, is (Synge, 1960)

$$\sigma_r'-\frac{2}{r}(\sigma_\theta-\sigma_r)+\frac{g_{44}'}{2g_{44}}(\sigma_r-\mu c^2)=0 \tag{24}$$

Here, the stresses and the density are expressed in terms of the metric coefficients with aid of Einstein's equations which can be presented as (Synge, 1960)

$$\chi\sigma_r=\frac{1}{r^2}-\frac{1}{rg_{11}}\left(\frac{g_{44}'}{g_{44}}+\frac{1}{r}\right) \tag{25}$$

$$\chi\sigma_\theta=-\frac{1}{2g_{11}}\left[\frac{g_{44}''}{g_{44}}-\frac{1}{2}\left(\frac{g_{44}'}{g_{44}}\right)^2+\frac{1}{r}\left(\frac{g_{44}'}{g_{44}}-\frac{g_{11}'}{g_{11}}\right)-\frac{g_{11}'g_{44}'}{2g_{11}g_{44}}\right] \tag{26}$$

$$\chi\mu c^2=\frac{1}{r^2}-\frac{1}{rg_{11}}\left(\frac{1}{r}-\frac{g_{11}'}{g_{11}}\right) \tag{27}$$

where

$$\chi=8\pi G/c^4 \tag{28}$$

is the GRT gravitational constant. As known, substitution of Equations (25)-(27) in Equation (24) identically satisfies this equation.

Perform some transformations. First, consider Equation (27). The solutions of this equation for the external $(r \geq R, \mu = 0)$ and internal $(0 \leq r \leq R, \mu = const)$ spaces are specified by the well known exterior and interior Schwarzchild solutions which have the form (Synge, 1960)

$$g_{11}^{e} = \frac{1}{1 - r_g / r}, \qquad g_{11}^{i} = \frac{1}{1 - r_g r^2 / R^3} \tag{29}$$

Here, indices "e " and "i" correspond to external and internal spaces, whereas r_g is the gravitational radius specified by Equation (20). Second, introduce function $f(r)$ as

$$\frac{g'_{44}}{g_{44}} = f, \qquad \frac{g''_{44}}{g_{44}} = f' + f^2 \tag{30}$$

Finally, substituting the second of Equations (29) and Equations (30) in Equations (25) and (26), we arrive at

$$\chi\sigma_r = \frac{r_g}{R^3} - \left(\frac{1}{r} - \frac{r_g r}{R^3}\right) f \tag{31}$$

$$\chi\sigma_\theta = \frac{r_g}{R^3} - \frac{1}{2}\left(1 - \frac{r_g r^2}{R^3}\right)\left(f' + \frac{f^2}{2}\right) - \left(\frac{1}{2r} - \frac{r_g r}{R^3}\right) f \tag{32}$$

Following the approach described in Section 1, express f in terms of σ_r using Equation (31)

$$f = \frac{r}{1 - r_g r^2 / R^3}\left(\frac{r_g}{R^3} - \chi\sigma_r\right)$$

and substitute this result in Equation (32), i.e.

$$\sigma_\theta = \frac{1}{4(1 - r_g r^2 / R^3)}\left[2r\left(1 - \frac{r_g r^2}{R^3}\right)\sigma'_r + 4\sigma_r - \chi r^2 \sigma_r^2 - \frac{3 r_g^2 r^2}{R^6 \chi}\right] \tag{33}$$

According to the basic idea of GRT, the stresses specified by the Einstein equations, Equations (31) and (32), identically satisfy the equilibrium equation, Equation (24). Transforming this equation with the aid of Equations (3), (20) and (28) for m, r_g, χ and substituting σ_θ from Equation (33), we can readily prove that the equilibrium equation is satisfied. Thus, to determine σ_r, we can apply the principle of minimum of the complementary energy discussed in Section 1. In the Riemannian space, Equation (15) for the complementary energy is generalized as

$$U = 4\pi \int_0^R w \sqrt{g_{11}}\, r^2 dr$$

Using Equation (32), we can reduce it to the functional in Equation (16). Omitting the explicit expression for the function F which is rather cumbersome, present the Euler equation, Equation (18), which takes the following final form:

$$2\bar{r}(1-\nu)(1-\bar{r}_g \bar{r}^2)^2 \frac{d^2\bar{\sigma}_r}{d\bar{r}^2} + 2(1-\nu)(1-\bar{r}_g\bar{r}^2)(4-3\bar{r}_g\bar{r}^2)\frac{d\bar{\sigma}_r}{d\bar{r}} + \bar{r}_g\bar{r}\left[4(2+\nu) - \bar{r}_g\bar{r}^2(7+5\nu)\right]\bar{\sigma}_r$$

$$+3\bar{r}_g\bar{r}\left[1-7\nu+2\bar{r}_g\bar{r}^2(1+2\nu)\right]\bar{\sigma}_r^2 - 9\bar{r}_g^2\bar{r}^3(1-\nu)\bar{\sigma}_r^3 - \bar{r}_g\bar{r}(3-\nu-2\bar{r}_g\bar{r}^2) = 0 \tag{34}$$

To simplify this equation, we use dimensionless parameters in Equations (19) and Equations(3), (21) and (28) for m, r_g and χ. Neglecting the terms with $\bar{r}_g$ in comparison with unity and omitting nonlinear terms, we arrive at equation (21) of the theory of elasticity.

3. Numerical Analysis

For the numerical analysis, we take $\nu = 0$ and reduce Equation (34) to

$$2\bar{r}(1-\bar{r}_g\bar{r}^2)^2\frac{d^2\bar{\sigma}_r}{d\bar{r}^2}+2(1-\bar{r}_g\bar{r}^2)(4-3\bar{r}_g\bar{r}^2)\frac{d\bar{\sigma}_r}{d\bar{r}}+\bar{r}_g\bar{r}(8-7\bar{r}_g\bar{r}^2)\bar{\sigma}_r+3\bar{r}_g\bar{r}(1+2\bar{r}_g\bar{r}^2)\bar{\sigma}_r^2$$

$$-9\bar{r}_g^2\bar{r}^3\bar{\sigma}_r^3=\bar{r}_g\bar{r}(3-2\bar{r}_g\bar{r}^2) \quad (35)$$

This equation is numerically integrated under the boundary conditions that follow from Equations (13), i.e. $\bar{\sigma}_r'(\bar{r}=0)=0$ and $\bar{\sigma}_r(\bar{r}=1)=0$. The circumferential stress is found from Equation (33) which can be transformed to

$$\bar{\sigma}_\theta=\frac{1}{4(1-\bar{r}_g\bar{r}^2)}\left[2\bar{r}(1-\bar{r}_g\bar{r}^2)\frac{d\bar{\sigma}_r}{d\bar{r}}+4\bar{\sigma}_r-3\bar{r}_g\bar{r}^2\bar{\sigma}_r^2-\bar{r}_g\bar{r}^2\right] \quad (36)$$

It is interesting to compare the results that follow from Equations (35) and (36) with the theory of elasticity solution specified by Equations (22) and with the interior Schwarzchild solution obtained for a sphere of perfect incompressible fluid. This solution has the following form (Synge, 1960):

$$\bar{p}=\frac{p}{\mu c^2}=-\frac{\sqrt{1-\bar{r}_g\bar{r}^2}-\sqrt{1-\bar{r}_g}}{\sqrt{1-\bar{r}_g\bar{r}^2}-3\sqrt{1-\bar{r}_g}} \quad (37)$$

and demonstrates a specific behavior. Taking $\bar{r}=0$ in Equation (37), determine the pressure at the sphere center, i.e.

$$\bar{p}_0=-\frac{1-\sqrt{1-\bar{r}_g}}{1-3\sqrt{1-\bar{r}_g}} \quad (38)$$

The denominator of this expression is zero for the sphere with radius $\bar{r}_g=8/9=0.8888$, and the pressure becomes infinitely high at the sphere center. This result is sometimes used to support the existence of the objects referred to as the Black Holes (Thorne, 1994).

The results of the numerical analysis are presented in Figures 1, 2. Figure 1 demonstrates the distributions of the normalized radial (solid lines) and circumferential (dashed lines) stresses over the radial coordinate for $\bar{r}_g=0.2, 0.4, 0.6, 0.8, 0.99$.

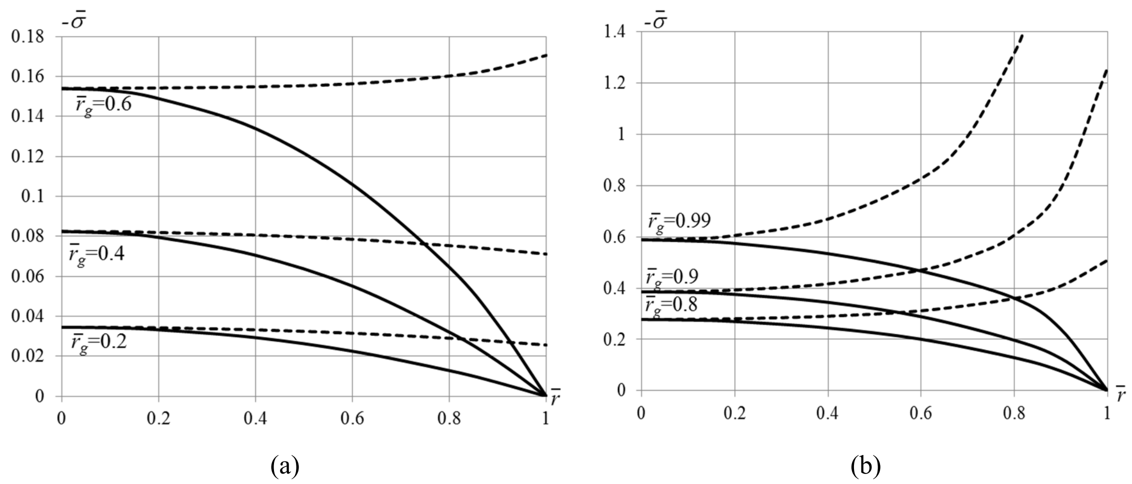

Figure 1. Distributions of the normalized radial (solid lines) and circumferential (dashed lines) stresses over the radial coordinate for (a) $\bar{r}_g=0.2, 0.4, 0.6$ and (b) $\bar{r}_g\, 0.8, 0.9, 0.99$

Figure 2 shows the normalized stresses at the sphere center as functions of the normalized gravitational radius.

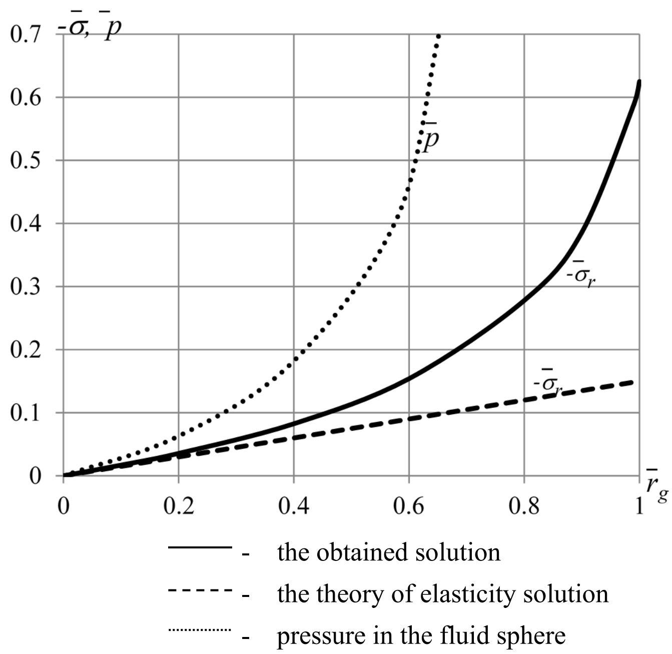

Figure 2. Dependences of the normalized stresses at the sphere center on the radial coordinate

As can be seen, in contrast to the pressure specified by Equation (38) which becomes infinitely high at $\bar{r}_g = 8/9$, the stresses are finite for the sphere with the radius $R = 9/8r_g$. For $\bar{r}_g = 1$, the numerical solution does not converge. However, it does not look like the stresses are singular in this case. It seems that the tolerance of the applied numerical procedure is not as high as should be. The last result $\bar{\sigma} = -0.5886$ is obtained for $\bar{r}_g = 0.99$. Dashed line in Figure 2 corresponds to the theory of elasticity solution specified by Equations (22).

4. Conclusion

The solution of the Schwarzchild spherically symmetric static problem for a fluid sphere is generalized for a linear elastic sphere. The equation for the stresses missing in GRT is derived using the minimum complementary energy principle of the theory of elasticity. In contrast to the singular Schwarzchild solution for the pressure in the fluid, the stresses do not demonstrate singular behavior for the elastic sphere whose radius is equal to the gravitational radius.

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