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Obtaining Easily Sums of Powers on Arithmetic Progressions and Properties of Bernoulli Polynomials by Operator Calculus

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Abstract

We show that a sum of powers on an arithmetic progression is the transform of a monomial by a differential operator and that its generating function is simply related to that of the Bernoulli polynomials from which consequently it may be calculated. Besides, we show that it is obtainable also from the sums of powers of integers, i.e. from the Bernoulli numbers which in turn may be calculated by a simple algorithm.

By the way, for didactic purpose, operator calculus is utilized for proving in a concise manner the main properties of the Bernoulli polynomials.

Keywords: Sums of powers on arithmetic progression; Bernoulli numbers; Bernoulli polynomials; Integer valued polynomials

I. Introduction

Sums of powers of integers

$$S_m(n) = 1^m + 2^m + 3^m + ... + (n-1)^m$$

are studied abundantly from the old days but sums on an arithmetic progression

$$S_m(z,n) = z^m + (z+1)^m + (z+2)^m + ... + (z+(n-1))^m$$

are only studied recently for examples by Kannappan & Zhang (2002), Tsao (2008), Bazsóa & Mezőb (2015, Griffiths (2016), etc… Particularly, in an article published on the last decade Chen, Fu and Zhang (2009) showed that sums of powers on arithmetic progression may be deduced from the original Faulhaber theorem or calculated by utilizing the central factorial numbers as in the approach of Knuth (1993). The methods utilized by Faulhaber, Knuth and these authors are respectful but the resulted formulae are not so easy to get and apply, especially for high value of powers.

Searching for another and simpler method for obtaining $S_m(z,n)$, we observe firstly that it is the transform of the monomial z^m by a differential operator built from the derivative operator $D_z \equiv \frac{d}{dz}$

$$S_m(z,n) = (1 + e^{D_x} + e^{2D_x} + ... + e^{(n-1)D_x})z^m .$$

Secondly, from the above property of $S_m(z,n)$ we see that we may study it by the method of operator calculus (Do Tan Si 2016). The most important result we get is obtaining its generating function which has obvious relationship with that of Bernoulli polynomials. This relationship gives rise to the formula for calculating $S_m(z,n)$ from Bernoulli polynomials.

Thirdly, because $S_m(z,n)$ and the Bernoulli polynomials $B_m(z)$ are differential transforms of monomials, we will see that they satisfy the addition formula which in turn leads to formulae allowing the calculations of $S_m(z,n)$ from sums of powers of integers $S_m(n) = S_m(0,n)$ as so as $B_m(z)$ from Bernoulli numbers $B_m = B_m(0)$. The latters may happily be calculated by a simple algorithm based on the relation between $S_m(n)$ and the Bernoulli numbers.

Fourthly for didactic purpose we show that one may obtain the main properties of Bernoulli polynomials simply by utilizing the method of operator calculus.

Finally we propose as applications the calculations of alternate sums of powers of integers and a criteria for a polynomials to have only integer values.

2. Sums of Powers on Arithmetic Progressions

2.1 Preliminary

Consider the sum of powers of elements of an arithmetic progression

$$S_m(a,b,n) = a^m + (a+b)^m + (a+2b)^m + ... + (a+(n-1)b)^m \tag{1}$$

By defining

$$z = \frac{a}{b} \tag{2}$$

we may write

$$S_m(a,b,n) = b^m(z^m + (z+1)^m + (z+2)^m + ... + (z+(n-1))^m)$$

i.e.

$$S_m(a,b,n) = b^m S_m(z,n) \tag{3}$$

where

$$S_m(z,n) = z^m + (z+1)^m + (z+2)^m + ... + (z+(n-1))^m \tag{4}$$

From the definition (4) we have obviously the followed formulae

$$S_m(0,n) = S_m(n) \tag{5}$$

where

$$S_m(n) = 0^m + 1^m + 2^m + ... + (n-1)^m \tag{6}$$

and

$$S_0(z,n) = n \tag{7}$$

$$S_m(z,1) = z^m \tag{8}$$

$$S'_m(z,n) = mS_{m-1}(z,n) \tag{9}$$

$$\int_0^1 S_m(z,n)dz = \frac{1}{m+1}(S_{m+1}(1,n) - S_{m+1}(0,n)) = \frac{1}{m+1} n^{m+1} \tag{10}$$

2.2 Generating Function of $S_m(z,n)$

Utilizing the translation operator

$$e^{aD_z} f(z) = f(z+a)$$

we may write

$$S_m(z,n) = (1 + e^{D_z} + e^{2D_z} .+..+ e^{(n-1)D_z})z^m$$

and, by summing the geometric progression in e^{D_z}, get

$$S_m(z,n) = \frac{e^{nD_z} - 1}{e^{D_z} - 1} z^m \tag{11}$$

By (11) we may say briefly that the operator calculus transforms an arithmetic progression on functions into a geometric progression on operators one always can sum up.

Moreover, from (11) and the fact that

$$D_z e^{az} = ae^{az}$$

we see that $S_m(z,n)$ has a generating function

$$\sum_{m=0}^{\infty} S_m(z,n)\frac{t^m}{m!} = \frac{e^{nD_z} - 1}{e^{D_z} - 1} e^{zt} = \frac{e^{nt} - 1}{e^t - 1} e^{zt} \tag{12}$$

2.3 Obtaining $S_m(z,n)$ from Bernoulli Polynomials and $S_m(n)$ from Bernoulli Numbers

Eq. (12) may be rewritten as

$$\sum_{m=0}^{\infty} S_m(z,n)\frac{t^m}{m!} = (\frac{nt}{1!} + \frac{n^2t^2}{2!} + ... + \frac{n^k t^k}{k!} + ...)\frac{1}{e^t - 1}e^{zt} \tag{13}$$

Let the Bernoulli polynomials be defined by a generating function (Euler 1738)

$$\frac{t}{e^t - 1}e^{zt} = \sum_{k=0}^{\infty} B_k(z)\frac{t^k}{k!} \tag{14}$$

we may write

$$\sum_{m=0}^{\infty} S_m(z,n)\frac{t^m}{m!} = (n + \frac{n^2 t}{2!} + ... + \frac{n^{m+1}t^m}{(m+1)!} + ...)\sum_{k=0}^{\infty} B_k(z)\frac{t^k}{k!}$$

Identification of coefficients of $\frac{t^m}{m!}$ in both members gives

$$S_m(z,n) = nB_m(z) + ... + \frac{n^k m!}{k!(m-k+1)!}B_{m-k+1}(z) + ... + \frac{n^{m+1}}{(m+1)}B_0(z) \tag{15}$$

i.e. symbolically

$$(m+1)S_m(z,n) = \sum_{k=1}^{m+1}\binom{m+1}{k}B_{m-k+1}(z)n^k := (B(z)+n)^{m+1} - B^{m+1}(z) \tag{16}$$

where undefined notations $B^k(z)$ are to be replaced with $B_k(z)$.

Formula (16) has the merit of giving $\frac{1}{m+1}\binom{m+1}{k}B_{m-k+1}(z)$ as coefficient of n^k in $S_m(z,n)$.

For examples

$$S_0(z,n) := (B(z)+n)^1 - B^1(z) = n$$
$$2S_1(z,n) := (B(z)+n)^2 - B^2(z) = 2B_1(z)n + B_0(z)n^2$$
$$3S_2(z,n) := 3B_2(z)n + 3B_1(z)n^2 + B_0(z)n^3$$

In particular for $n = 1$ we get the known formula (Roman 1984)

$$(m+1)z^m = \sum_{k=1}^{m+1}\binom{m+1}{k}B_{m-k+1}(z) := (B(z)+1)^{m+1} - B^{m+1}(z) \tag{17}$$

which allows the expansion of a polynomial into a set of Bernoulli polynomials.

Another important particular case is that for $z = 0$ we get the well-known formula (Bernoulli 1713)

$$(m+1)S_m(n) = \sum_{k=1}^{m+1}\binom{m+1}{k}B_{m-k+1}n^k := (B+n)^{m+1} - B^{m+1} \tag{18}$$

for calculating sums of powers of integers proven here by operator calculus.

For examples

$$S_0(n) := (B+n)^1 - B^1 = n$$
$$2S_1(n) := (B+n)^2 - B^2 = 2B_1 n + n^2 = -n + n^2$$
$$3S_2(n) := (B+n)^3 - B^3 = 3B_2 n + 3B_1 n^2 + n^3 = \frac{1}{2}n - \frac{3}{2}n^2 + n^3$$

2.4 Obtaining $S_m(z,n)$ from Sums of Powers of Integers $S_m(n)$

We have seen in the studies of Special functions by operator calculus (Si D.T. 2016) that if a polynomial $P_m(z)$ is the transform of a monomial by a differential operator $A(D_z)$

$$P_m(z) = A(D_z)z^m$$

then because

$$D_{z+y}f(z+y) = D_z f(z+y) = D_y f(z+y)$$

it obeys the addition formula

$$P_m(z+y) = A(D_{z+y})(z+y)^m = A(D_y)(z+y)^m = A(D_y)\sum_{k=0}^{m}\binom{m}{k}z^k y^{m-k}$$

$$= \sum_{k=0}^{m}\binom{m}{k}z^k P_{m-k}(y) := (z+P(y))^m \tag{19}$$

From the addition formula we get by putting $y = 0$

$$P_m(z) = \sum_{k=0}^{m}\binom{m}{k}P_{m-k}(0)z^k := (P(0)+z)^m \tag{20}$$

Applying the formula (20) for $S_m(z,n)$ we get the important symbolic formula

$$S_m(z,n) = \sum_{k=0}^{m}\binom{m}{k}S_{m-k}(n)z^k := (S(n)+z)^m \tag{21}$$

allowing us to calculate $S_m(z,n)$ from sums of powers of integers $S_k(n), k \le m$.

For examples from (21) and (18)

$$S_0(z,n) := (S(n)+z)^0 = \binom{0}{0}S_0(n)z^0 = n$$

$$S_1(z,n) := (S(n)+z)^1 = S_0(n)z^1 + S_1(n)z^0 = nz + \frac{1}{2}(-n+n^2)$$

$$S_2(z,n) = S_0(n)z^2 + 2S_1(n)z^1 + S_2(n)z^0 = nz^2 + n(n-1)z + (\frac{1}{6}n - \frac{1}{2}n^2 + \frac{1}{3}n^3)$$

2.5 Obtaining $S_m(z,n)$ from Bernoulli Numbers

From the formula (14) we get

$$\frac{D_z}{e^{D_z}-1}e^{zt} = \sum_{k=0}^{\aleph}B_k(z)\frac{t^k}{k!}$$

so that

$$B_m(z) = \frac{D_z}{e^{D_z}-1}z^m \tag{22}$$

From (22) and (20) we get the symbolic formula allowing the calculation of Bernoulli polynomials from Bernoulli numbers (Lucas 1891)

$$B_m(z) = \sum_{k=0}^{m}\binom{m}{k}B_{m-k}(0)z^k := (B+z)^m \tag{23}$$

For examples

$$B_0(z) := (B+z)^0 = \binom{0}{0}B_0 z^0 = 1$$

$$B_1(z) := (B+z)^1 = (B_1+z) = z - \frac{1}{2}$$

$$B_2(z) := (B+z)^2 = (B_2 + 2B_1 z + B_0 z^2) = \frac{1}{6} - z + z^2$$

Combining (21) and (23) we get the symbolic formula

$$(m+1)S_m(z,n) := ((B+z)+n)^{m+1} - (B+z)^{m+1}$$

For example

$$2S_1(z,n) := ((B+z)+n)^2 - (B+z)^{21}$$

$$= 2(B_1+z)n + n^2 = -n + 2nz + n^2$$

As conclusion we see that from (16), (21) we obtain the theorem

"In $S_m(z,n)$, the coefficients of z^k is $\binom{m}{k} S_{m-k}(n)$ for $k = 0,1,...,m$;

the coefficient of n^k is $\frac{1}{m+1}\binom{m+1}{k} B_{m-k+1}(z)$ for $k = 1,2,...,m+1$" (24)

Moreover, as $S_m(n)$ and $B_m(z)$ may be calculated from the Bernoulli numbers, $S_m(z,n)$ would be perfectly known once the Bernoulli numbers from B_0 to B_m known.

The problem is thus reduced to the obtaining of Bernoulli numbers.

2.6 Algorithm for Obtaining the Bernoulli Numbers and Sums of Powers of Integers

Let

$$S_m(n) = \sum_{k=1}^{m+1} c(m,k) n^k \tag{25}$$

we have from (18)

$$c(m,1) = B_m \tag{26}$$

$$c(m,k) = \frac{1}{m+1}\binom{m+1}{k} B_{m-k+1} \tag{27}$$

$$c(m+1,k+1) = \frac{1}{m+2}\binom{m+2}{k+1} B_{m-k+1} = \frac{(m+1)}{(k+1)} c(m,k) \tag{28}$$

and from (6)

$$S_m(1) = 0 = \sum_{k=1}^{m+1} c(m,k)$$

$$c(m,1) = B_m = -\sum_{k=2}^{m+1} c(m,k) \tag{29}$$

From (28), (29) we get the algorithm for calculating the matrix whose elements are the coefficients $c(m,k)$, $m \geq 0$ as described hereafter

¨ Departing from the definition

$$S_0(n) = 0^0 + 1^0 + 2^0 + ... + (n-1)^0 = n$$

we get the first Bernoulli number

$$B_0 = c(0,1) = 1$$

then by (28) all the elements populating the principal diagonal of the coefficients matrix

$$c(0,1) = 1, \quad c(1,2) = \frac{1}{2}, \quad c(2,3) = \frac{1}{3}, \; ... \;, \quad c(m,m+1) = \frac{m}{m+1}\frac{1}{m}$$

Afterward by (29) we get the second Bernoulli number

$$B_1 = c(1,1) = -c(1,2) = -\frac{1}{2}$$

then by (28) all the elements of the first parallel with respect to the diagonal

$$c(1,1)\,, c(2,2)\,, c(3,3)\,,..., c(m,m) = \frac{m!}{m!} c(1,1) = -\frac{1}{2}$$

Afterward by (29) we get the third Bernoulli number

$$B_2 = c(2,1) = -c(2,2) - c(2,3) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

which by (28) gives all the elements of the second parallel with respect to the diagonal

$$c(2,1)\,, c(3,2)\,, \ldots, \quad c(m,m-1) = \frac{m}{m-1}\frac{m-1}{m-2} ... \frac{3}{2} B_2 = \frac{m}{2} B_2$$

Repeatedly, from $B_j = c(j,1)$ we get by (28) all the elements of the j^{th} parallel

$$c(j,1)\,,\ c(j+1,2)\,,\ldots,c(m,m-j+1)=\frac{1}{m+1}\binom{m+1}{j}B_j(z),\quad m\geq j$$

then by (29) the Bernoulli number

$$B_{j+1}=c(j+1,1)=-\sum_{k=2}^{j+2}c(j+1,k)$$

The operations end whenever we reach the elements

$$c(m,2)\,,\quad c(m,1)=B_m\text{''}. \tag{30}$$

Utilizing this algorithm, from nothing we obtain for example the first seven Bernoulli numbers

$$\left|\begin{array}{lllllll} B_0=1 & & & & & & \\ B_1=\frac{-1}{2} & \frac{1}{2} & & & & & \\ B_2=\frac{1}{6} & B_1 & \frac{1}{3} & & & & \\ B_3=0 & \frac{3B_2}{2} & B_1 & \frac{1}{4} & & & \\ B_4=\frac{-1}{30} & 0 & 2B_2 & B_1 & \frac{1}{5} & & \\ B_5=0 & \frac{5B_4}{2} & 0 & \frac{5}{2}B_2 & B_1 & \frac{1}{6} & \\ B_6=\frac{1}{42} & 0 & 5B_4 & 0 & 3B_2 & B_1 & \frac{1}{7} \end{array}\right| \tag{31}$$

and by (18) the sums of powers of integers

$$\begin{vmatrix} 1S_0(n) \\ 2S_1(n) \\ 3S_2(n) \\ 4S_3(n) \\ 5S_4(n) \\ \ldots \\ mS_{m-1}(n) \end{vmatrix}=\begin{vmatrix} 1B_0 & & & & & & \\ 2B_1 & 1B_0 & & & & & \\ 3B_2 & 3B_1 & 1B_0 & & & & \\ 4B_3 & 6B_2 & 4B_1 & 1B_0 & & & \\ 5B_4 & 10B_3 & 10B_2 & 5B_1 & 1B_0 & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ mB_{m-1} & \binom{m}{2}B_{m-2} & \binom{m}{3}B_{m-3} & \binom{m}{4}B_{m-4} & \binom{m}{5}B_{m-5} & \ldots & B_0 \end{vmatrix}\begin{vmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ \ldots \\ n^m \end{vmatrix} \tag{32}$$

We notify that the algorithm leading to (31), (32) are better than that in our previous work (Si D.T. 2017)

3. Study of Bernoulli Polynomials by Operator Calculus

Apart from the formula (16) coming from the differential representation of $S_m(z,n)$

$$S_m(z,1)=z^m=\frac{1}{m+1}\sum_{k=1}^{m+1}\binom{m+1}{k}B_{m-k+1}(z)$$

we may obtain by operator calculus the following properties of the Bernoulli polynomials.

(i) Derivation and integration

From the differential representation (22) of the Bernoulli polynomials we get

$$B'_{m+1}(z)=D_z\frac{D_z}{e^{D_z}-1}z^{m+1}=(m+1)\frac{D_z}{e^{D_z}-1}z^m=(m+1)B_m(z) \tag{33}$$

$$(e^{D_z}-1)B_m(z)=D_z z^m=B_m(z+1)-B_m(z)=mz^{m-1} \tag{34}$$

and by telescopic summation the Bernoulli formula (Bernoulli 1713)

$$B_{m+1}(z+1)-B_{m+1}(0)=(m+1)\sum_{k=0}^{m}z^k \tag{35}$$

Eq. (34) gives rise also to the integration formula

$$\int_t^{t+1}B_m(z)dz=\frac{1}{m+1}(B_{m+1}(t+1)-B_{m+1}(t)=t^m \tag{36}$$

In particular

$$\int_0^1 dzB_m(z)=\delta_{m0} \tag{37}$$

(ii) The addition formula quoted in (19)

$$B_m(z+y)=\sum_{k=0}^{m}\binom{m}{k}B_k(z)y^{m-k}:=(B+z)^m$$

In particular

$$B_m(z)=\sum_{k=0}^{m}\binom{m}{k}B_k(\frac{1}{2})(z-\frac{1}{2})^{m-k}$$

(iii) The relation between $B_m(z)$ and Bernoulli numbers

This is the symbolic formula quoted in (23) and was proven by Lucas (1881)

$$B_m(z)=\sum_{k=0}^{m}\binom{m}{k}B_k(0)z^{m-k}:=(B+z)^m$$

which leads to

$$B_0(z)=1 \tag{38}$$

and

$$B_m:=(B+z-z)^m:=(B(z)-z)^m \tag{39}$$

(iv) The symmetry formula

By replacing z with $(-z)$ and D_z with $(-D_z)$ we have

$$B_k(-z)=\frac{-D_z}{e^{-D_z}-1}(-z)^k=\frac{D_z}{e^{D_z}-1}e^{D_z}(-z)^k=(-)^k\frac{D_z}{e^{D_z}-1}(z+1)^k$$

$$=(-)^kB_k(1+z)$$

i.e. the well-known formula

$$B_k(1-z)=(-)^kB_k(z) \tag{40}$$

which proves that $B_{2k}(z)$ is symmetric, $B_{2k+1}(z)$ is antisymmetric with respect to the vertical $z=\frac{1}{2}$.

In particular

$$B_k(1)=(-)^kB_k(0)=(-)^kB_k \tag{41}$$

(v) The multiplication formula

By replacing z with $\frac{z}{2}$ and D_z with $2D_z$ we have

$$B_k(\frac{z}{2})=\frac{2D_z}{e^{2D_z}-1}(\frac{z}{2})^k=\frac{2D_z}{(e^{D_z}+1)(e^{D_z}-1)}(\frac{z}{2})^k$$

$$(e^{D_z}+1)B_k(\frac{z}{2})=\frac{2D_z}{(e^{D_z}-1)}(\frac{z}{2})^k=2^{-k+1}B_k(z)$$

i.e.

$$B_k(\frac{z+1}{2})+B_k(\frac{z}{2})=2^{-k+1}B_k(z) \tag{42}$$

In particular

$$B_k(\frac{1}{2}) = (2^{-k+1} - 1)B_k$$

More generally by replacing z with $\frac{z}{n}$ so that D_z is replaced with nD_z we have the formula

$$B_k(\frac{z}{n}) = \frac{nD_z}{e^{nD_z} - 1}(\frac{z}{n})^k = \frac{nD_z}{(e^{D_z} - 1)(1 + e^{D_z} + e^{2D_z} + ... + e^{(n-1)D_z})}(\frac{z}{n})^k \tag{43}$$

which leads to

$$(1 + e^{D_z} + e^{2D_z} + ... + e^{(n-1)D_z})B_k(\frac{z}{n}) = n^{1-k}\frac{D_z}{e^{D_z} - 1}z^k$$

i.e. the formula generalizing (42)

$$\sum_{j=0}^{n-1} B_k(\frac{z+j}{n}) = n^{1-k}B_k(z) \tag{44}$$

For example

$$B_k(\frac{z}{3}) + B_k(\frac{z+1}{3}) + B_k(\frac{z+2}{3}) = 3^{1-k}B_k(z)$$

Now replacing z with nz in (44) we get the multiplication formula given by Raabe (1851) and proven otherwise than here above by Zagier (2005)

$$B_k(nz) = n^{k-1}\sum_{j=0}^{n-1} B_k(z + \frac{j}{n}) \tag{45}$$

A generalization of the multiplication formula of Raabe is possible by remarking that (45) is a relation between polynomials valuable for an infinity of values of n so that it is valuable also if n should be replaced with a complex number.

For example

$$B_1(nz) = (B_1(z) + ... + B_1(z + \frac{n-1}{n})) = (z - \frac{1}{2}) + ... + (z + \frac{n-1}{n} - \frac{1}{2})$$
$$= nz + \frac{n(n-1)}{2}\frac{1}{n} - n\frac{1}{2}) = nz - \frac{1}{2}$$

so that

$$B_1(\omega z) = \omega z - \frac{1}{2}$$

This assumption is to be compared with the result given by Kouba (2016) in a lecture notes about Bernoulli polynomials and their applications.

(vi) The second proof for the theorem $B_{2k+1} = B_{2k+1}(0) = 0, k > 0$

The first proof for the theorem $B_{2k+1} = 0, k > 0$ is seemingly proven by Bardell L.J. leaning on the fact that the function

$$\frac{t}{e^t - 1} + \frac{t}{2} = \frac{t}{2}\coth\frac{t}{2}$$

is odd.

Now, in formula (42) giving for z the values $z = 1$ then $z = 0$ and making a subtraction we get the formula

$$B_k(1) - B_k(0) = 2^{-k+1}(B_k(1) - B_k(0)) \tag{46}$$

which shows that

$$B_k(1) - B_k(0) = 0 \quad if\ k \neq 1 \tag{47}$$

i.e. by (41)

$$(-)^k B_k - B_k = 0 \quad \forall k \neq 1 \tag{48}$$

We may conclude then that

$$B_{2k+1} = 0 \quad k \neq 0 \tag{49}$$

As for the value of B_1 it is given by the algorithm (30), says

$$B_1 = -\frac{1}{2}.$$

This proof was given by Kouba (2016) on the same basis.

(vii) The integration of products of Bernoulli polynomials

Consider $m \geq 1, n \geq 1$ we have successively by utilizing (41), (49), (37)

$$\begin{aligned}
\int_0^1 B_m(z)B_n(z)dz &= \frac{1}{n+1}\int_0^1 B_m(z)B'_{n+1}(z)dz \\
&= \frac{1}{n+1}(B_m(1)B_{n+1}(1) - B_m(0)B_{n+1}(0) - \int_0^1 B'_m(z)B_{n+1}(z)) \\
&= -\frac{1}{n+1}\int_0^1 B'_m(z)B_{n+1}(z)) = -\frac{m}{n+1}\int_0^1 B_{m-1}(z)B_{n+1}(z)) \\
&= ... = (-)^m \frac{m!n!}{(n+m)!}\int_0^1 B_1(z)B'_{n+m}(z) \\
&= (-)^m \frac{m!n!}{(n+m)!}((B_1(1)B_{n+m}(1) - B_1 B_{n+m}) - \int_0^1 B'_1(z)B_{n+m}(z)) \\
&= (-)^m \frac{m!n!}{(n+m)!}(-2B_1)B_{n+m} = (-)^m \frac{m!n!}{(n+m)!}B_{n+m}
\end{aligned} \tag{50}$$

(viii) The Fourier series expansion

Consider the function identical to $B_m(z)$ on the interval $[0,1]$ and $\tilde{B}_m(z)$ the juxtaposition of the successive translates by ± 1 of it on the real axis. $\tilde{B}_m(z)$ is periodic so that we may write

$$\tilde{B}_m(z) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C(m,k)e^{i2\pi kz} \tag{51}$$

Because

$$\int_0^1 e^{-i2\pi kz}e^{i2\pi k'z}dz = \frac{2\sin\pi(k-k')}{2\pi(k-k')}e^{i\pi(k-k')} = \delta(k-k')$$

we have for $k \neq 0$ and $0 \leq z \leq 1$

$$\begin{aligned}
C(m,k) &= \int_0^1 B_m(z)e^{-i2\pi kz}dz = \frac{1}{-i2\pi k}\int_0^1 B_m(z)(e^{-i2\pi kz})'dz \\
&= \frac{1}{i2\pi k}(-(B_m(1) - B_m) + \int_0^1 B'_m(z)e^{-i2\pi kz}dz) \\
&= \frac{1}{i2\pi k}\int_0^1 B'_m(z)e^{-i2\pi kz}dz = \frac{m}{i2\pi k}\int_0^1 B_{m-1}(z)e^{-i2\pi kz}dz \\
&= ... = \frac{m(m-1)...2}{(i2\pi k)^{m-1}}\int_0^1 B_1(z)e^{-i2\pi kz}dz \\
&= -\frac{m!}{(i2\pi k)^m}((B_1(1) - B_1) - \int_0^1 B'_1(z)e^{-i2\pi kz}dz) = -\frac{m!}{(i2\pi k)^m}
\end{aligned} \tag{52}$$

From (52) we get the formula given by Hũrwitz in 1890 (Costabile 2006)

$$B_m(z) = -\frac{m!}{(i2\pi)^m}\sum_{k=1}^{\infty}\frac{1}{k^m}(e^{i2\pi kz} + (-)^m e^{-i2\pi kz}) \quad 0 \leq z \leq 1 \tag{53}$$

(ix) The relation with Riemann Zeta function

From (53) we get

$$B_{2m}(z) = (-)^{m-1} \frac{(2m)!}{(2\pi)^{2m}} \sum_{k=1}^{\infty} \frac{2}{k^{2m}} \cos 2k\pi z$$

and

$$\int_0^1 B_m(z) B_n(z) dz = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C^*(m,k) C(n,k)$$

i.e.

$$(-)^m \frac{m!n!}{(n+m)!} B_{n+m} = (-)^{m+1} \frac{m!n!}{(i2\pi)^{m+n}} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k^{m+n}}$$

$$\zeta(n+m) = \sum_{k=1}^{\infty} \frac{1}{k^{m+n}} = -\frac{(i2\pi)^{m+n}}{(n+m)!2} B_{n+m} \ \ for\,(m+n)\ even \tag{54}$$

(x) The recurrence relation of Bernoulli polynomials

From (19) we see that

$$\frac{D_{x+y}}{e^{D_{x+y}}-1} B_m(z+y) = \frac{D_y}{e^{D_y}-1} B_m(z+y) = \sum_{k=0}^{m} \binom{m}{k} B_k(z) B_{m-k}(y)$$

i.e.

$$\frac{D^2{}_{x+y}}{(e^{D_{x+y}}-1)^2} (z+y)^m = \sum_{k=0}^{m} \binom{m}{k} B_k(z) B_{m-k}(y) \tag{55}$$

Let Y be the Eckaert operator defined by the relation $Yf(y) = yf(y)$. Utilizing the fundamental identity of operator calculus (Do Tan Si 2016) we have

$$D_y Y \equiv Y D_y + 1$$

$$A(D_y) Y \equiv Y A(D_y) + A'(D_y)$$

so that

$$\frac{D_y}{e^{D_y}-1} Y \equiv Y \frac{D_y}{e^{D_y}-1} + \left(\frac{D_y}{(e^{D_y}-1)}\right)' \equiv Y \frac{D_y}{e^{D_y}-1} + \frac{1}{e^{D_y}-1} - \frac{D_y e^{D_y}}{(e^{D_y}-1)^2}$$

Applying this identity on y^m and utilizing (55) we get successively

$$B_{m+1}(y) = Y B_m(y) + \frac{1}{(e^{D_y}-1)} y^m - e^{D_y} \frac{D_y}{(e^{D_y}-1)^2} y^m$$

$$D_y B_{m+1}(y) = (Y D_y + 1) B_m(y) + B_m(y) - e^{D_y} \frac{D^2{}_y}{(e^{D_y}-1)^2} y^m$$

$$(m-1) B_m(y) = y D_y B_m(y) - e^{D_y} \frac{D^2{}_y}{(e^{D_y}-1)^2} y^m$$

$$(m-1) B_m(y+z) - (y+z) D_y B_m(y+z) = -e^{D_y} \frac{D^2{}_y}{(e^{D_y}-1)^2} (y+z)^m$$

$$(m-1)B_m(y+z)-m(y+z)B_{m-1}(y+z)=-\sum_{k=0}^{m}\binom{m}{k}B_k(y+1)B_{m-k}(z) \tag{56}$$

$$=-\sum_{k=0}^{m}\binom{m}{k}(B_k(y)+ky^{k-1})B_{m-k}(z)$$

$$=-\sum_{k=0}^{m}\binom{m}{k}(B_k(y)B_{m-k}(z)-D_y\sum_{k=0}^{m}\binom{m}{k}y^k B_{m-k}(z)$$

$$=-\sum_{k=0}^{m}\binom{m}{k}B_k(y)B_{m-k}(z)-D_yB_m(y+z)$$

$$(m-1)B_m(y+z)-m(y+z-1)B_{m-1}(y+z)=-\sum_{k=0}^{m}\binom{m}{k}B_k(y)B_{m-k}(z)$$

$$(1-m)B_m(y+z)+m(y+z-1)B_{m-1}(y+z):=(B(y)+B(z))^m \tag{57}$$

Eq. (57) is mentioned without proofs by Weisstein (n.d.).

The formula (56) leads also to

$$mB_m(z)=mzB_{m-1}(z)-\sum_{k=1}^{m}\binom{m}{k}B_k(1)B_{m-k}(z)$$

$$=m(z-\frac{1}{2})B_{m-1}(z)-\sum_{k=2}^{m}\binom{m}{k}(-)^k B_kB_{m-k}(z)$$

$$B_m(z)=B_1(z)B_{m-1}(z)-\frac{1}{m}\sum_{k=2}^{m}\binom{m}{k}B_kB_{m-k}(z) \tag{58}$$

which is seemingly a new recurrence formula for Bernoulli polynomials.

For examples

$$B_2(z)=B_1(z)B_1(z)-\frac{1}{2}B_2B_0(z)=(z-\frac{1}{2})^2-\frac{1}{12}$$

$$B_3(z)=B_1(z)B_2(z)-B_2B_1(z)=(z-\frac{1}{2})(z^2-z+\frac{1}{6}-\frac{1}{6})=z^3-\frac{3}{2}z^2+\frac{1}{2}z$$

From (58) we have the curious formula concerning Bernoulli numbers

$$(1-m)B_m=\sum_{k=0}^{m}\binom{m}{k}(-)^k B_kB_{m-k}:=(B-B)^m \tag{59}$$

which is to be compared with the recurrence formula coming from (27), (29)

$$(-1-m)B_m=\sum_{k=2}^{m+1}\binom{m+1}{k}B_{m-k+1} \tag{60}$$

4. Miscellaneous Applications

4.1 Alternate Sums of Powers on Arithmetic Progressions

Let us consider some simple cases utilizing (3) and (21)

$$\text{(i)}\quad S_2(a,b,n)=b^2S_2(\frac{a}{b},n)=(a^2+\frac{b^2}{6}-ab)n+(ab-\frac{b^2}{2})n^2+\frac{b^2}{3}n^3 \tag{61}$$

$$S_2(1,2,n)=1^2+3^2+...+(2n-1)^2=\frac{4}{3}n^3-\frac{1}{3}n$$

$$S_2(2,2,n)=2^2+4^2+...+(2n)^2=\frac{4}{3}n^3+2n^2+\frac{2}{3}n$$

i.e.

$$1^2-2^2+...+(2n-1)^2-(2n)^2=S_2(1,2,n)-S_2(2,2,n)=-2n^2-n \tag{62}$$

As exercise the readers are proposed to prove that

$$S_2(\cos t, i\sin t, n)-S_2(\sin t, i\cos t, n)=(\frac{7}{6}n-\frac{1}{2}n^2+\frac{1}{3}n^3)\cos 2t$$

(ii) $$S_3(a,b,n)=(\frac{ab^2}{2}-\frac{3a^2b}{2}+a^3)n+\frac{3b}{2}(a^2+\frac{b^2}{6}-ab)n^2$$

$$+b^2(a-\frac{b}{2})n^3+\frac{b^3}{4}n^4 \tag{63}$$

which gives

$$1^3-2^3+3^3-...+(2n-1)^3-(2n)^3=S_3(1,2,n)-S_3(2,2,n)=-4n^3-3n^2 \tag{64}$$

(iii) $$S_4(a,b,n)=\frac{1}{5}b^4n^5+\frac{2ab^3-b^4}{2}n^4+2(\frac{1}{6}b^4+a^2b^2-ab^3)n^3$$

$$-(3a^2b^2-ab^3-2a^3b)n^2-(\frac{1}{30}b^4+2a^3b-a^2b^2-a^4)n \tag{65}$$

$$S_4(1,2,n)=1^4+3^4+...+(2n-1)^4=\frac{16}{5}n^5-\frac{8}{3}n^3+\frac{7}{15}n$$

$$S_4(2,2,n)=2^4+4^4+...+(2n)^4=\frac{16}{5}n^5+8n^4+\frac{16}{3}n^3-\frac{8}{15}n$$

$$1^4-2^4+3^4-4^4+...+(2n-1)^4-(2n)^4=-8n^4-8n^3+n \tag{66}$$

4.2 Sufficient Condition for a Polynomial to Have Only Integer Values

The sums $S_m(k,n)$ with k integer are all equal to integer numbers. Nevertheless they appear often as sums of rational numbers. By this remark we may obtain many forms of sums of rational numbers which result in integers.

In fact, let $P_m(n)$ be a polynomial of order m. If $P_m(n)$ is always equal to an integer then so is the polynomial $P_m^-(n)$ obtained from $P_m(n)$ by omitting evident integers and arranging so that all the coefficients are positive and less than unity. For example the ones taken from (16) are

$$S_1(z,n)=(z-\frac{1}{2})n+\frac{1}{2}n^2 \Rightarrow S_1^-(k,n)=\frac{1}{2}n+\frac{1}{2}n^2$$

$$S_2(z,n)=(z^2-z+\frac{1}{6})n+(z-\frac{1}{2})n^2+\frac{1}{3}n^3 \Rightarrow S_2^-(k,n)=\frac{1}{6}n+\frac{1}{2}n^2+\frac{1}{3}n^3$$

$$4S_3(z,n)=4B_3(z)n+6(z^2-z+\frac{1}{6})n^2+4(z-\frac{1}{2})n^3+n^4$$

$$S_3^-(k,n)=(\frac{1}{2}k+\frac{1}{2}k^2)n+(\frac{1}{4}+\frac{1}{2}k+\frac{1}{2}k^2)n^2+\frac{1}{2}n^3+\frac{1}{4}n^4 \quad \forall k\in Z$$

i.e. because $(\frac{1}{2}k+\frac{1}{2}k^2)$ is an integer

$$S_3^-(k,n)=\frac{1}{4}n^2+\frac{1}{2}n^3+\frac{1}{4}n^4$$

$$S_4^-(k,n)=\frac{29}{30}n+\frac{1}{3}n^3+\frac{1}{2}n^4+\frac{1}{5}n^5 \tag{67}$$

A sufficient condition for a polynomial $Q_m(n)$ to have only integer value is that $Q_m^-(n)$ is a linear combination with integer coefficients of the sums $S_{m-1}^-(k,n)$, $S_{m-2}^-(k,n)$, …, $S_1^-(k,n)$.

For example the class of reduced polynomials

$$Q_3^-(n)=\alpha S_1^-(k,n)+\beta S_2^-(k,n)=\alpha(\frac{1}{2}n+\frac{1}{2}n^2)+\beta(\frac{1}{6}n+\frac{1}{2}n^2+\frac{1}{3}n^3)$$
$$=(\frac{\alpha}{2}+\frac{\beta}{6})n+(\frac{\alpha}{2}+\frac{\beta}{2})n^2+\frac{\beta}{3}n^3 \quad \forall\alpha,\beta\in Z \tag{68}$$

has certainly integer values as so as all the polynomials of 3^{th} order which may be reduced into one of them.

From (67) we see that the polynomial $Q_3(n)=an+bn^2+cn^3$ would have only integer values if

$$a+c=b, 3c\in N, 2b\in N, 6a\in N$$

for example

$$Q_3(n)=\frac{1}{6}n+\frac{3}{2}n^2+\frac{4}{3}n^3.$$

Apart sums of powers on arithmetic progressions, we ignore if there are other forms of polynomials "naturally" valuing integers.

5. Remarks and Conclusion

The main remark about this work is that it steadily utilizes the operator calculus method for obtaining a simple algorithm for calculating the Bernoulli numbers and sums of powers of integers leading to the obtention of Bernoulli polynomials and sums of powers on arithmetic progressions by the theorem " In $S_m(z,n)$, the coefficients of z^k is $\binom{m}{k}S_{m-k}(n)$ for $k=0,1,...,m$; the coefficient of n^k is $\frac{1}{m+1}\binom{m+1}{k}B_{m-k+1}(z)$ for $k=1,2,...,m+1$ ".

The readers who like this work may find in Ref. (Si D.T. 2016) many interesting applications of operator calculus method for examples in the resolutions of differential equations; in obtaining concisely and coherently the properties of Hermite, Laguerre, associated Laguerre, Gegenbauer, Chebyshev polynomials; in obtaining the differential representations and properties of many kinds of transforms from translation, dilatation to partial Fourier, Fourier and Laplace transforms. We would like to insist that operator calculus is more powerful than the Heaviside operational calculus.

The second remark is that although Bernoulli polynomials are largely studied from many centuries ago we dare present herein the proofs of the main properties of them by operator calculus. It is observed that these proofs are concise and made easy to verify by everyone.

Finally some interesting applications are also given for embellishing the work.

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