Applied Physics Research; Vol. 8, No. 4; 2016
ISSN 1916-9639 E-ISSN 1916-9647
Published by Canadian Center of Science and Education

The Ground State Electronic Properties of Two Electron Quantum Dot in External Magnetic and Electric Fields

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Received: June 7, 2016 Accepted: June 21, 2016 Online Published: July 29, 2016

doi:10.5539/apr.v8n4p83 URL: http://dx.doi.org/10.5539/apr.v8n4p83

Abstract

The ground state energies of two interacting electrons in two dimensions are studied within the framework of shifted 1/N expansion. The effect of external uniform magnetic and electric have been studied. Energies of the relative part of the Hamiltonian of the system are calculated for both weak and strong field regimes. Our results show a very good agreement with those obtained by other computational methods like asymptotic integration (AIM) and exact diagonalizaion methods.

Keywords: energy eigenvalues, electric field, 1/N expansion, magnetic field, two-electron quantum dot

1. Introduction

Quantum dots QDs are generally refer to nanostructures made from semiconductor materials, in which carries are confined in all spatial dimensions. Modern semiconductor processing techniques allowed the artificial creation of quantum confinement shape of few electrons. Because of the similarity between atoms and quantum dots they are called artificial atoms. However, there are considerable differences in physical characterizations between them. The confining potential of nucleus is singular but that of quantum dots is nonsingular, which makes the parabolic confinement approximation possible. And artificial atoms, typical dimensions range from nanometers to few microns, are much larger than real atoms. Furthermore, there are mainly two energy scales in quantum dots; the Coulomb energy and the confinement potential energy.

The main technological motivation to investigate semiconductor quantum dots is that smaller structures should be faster, dissipate less heat, and quantum mechanical effects are so relevant in such structures that devices with fundamentally new properties can be obtained. Therefore, QDs have many applications in microelectronic and optoelectronic devices such as photodetectors, far-infrared (FIR) laser amplifiers, one electron transistors and high-speed electro-optical modulators (Mendoza, Vazquez, del Castillo-Mussot, & Spector, 2005; Li & Xia, 2007; Zhu & Gu, 1993; Dutta & Das, 1990; Yoffe, 2001; Ahn & Chuang, 1987).

A two electron quantum dot, named as quantum dot helium, is the simplest system that describes the electron-electron interaction, it is also considered an excellent testing ground for various approximation methods used in calculating the energy levels of the system. Several different approaches have been reported in studying such a system. Exact diagonalization method (Wagner, Merkt, & Chaplik 1992; Merkt, Huser, & Wagner 1991), Hartree and Hartree Fock (HF) (Pfannkuche, Gudmundsson, & Maksym, 1993; Pfannkuche, Gerhats, Maksym, & Gudmundsson, 1993; Palacios, Martin-Moreno, Chiappe, Louis, & Tejedor, 1994), the Monte Carlo calculations (Harju, Sverdlov, & Nieminen 1998; Bolton, 1996), and 1/N expansion method (El-Said, 2000).

Electric and magnetic fields influence have been applied to quantum dots using different approaches. Rezaei (Rezaei & Kish, 2013) studied the fields effect on the two dimensional quantum dot using direct matrix diagonalization method. Soylu (Soylu, 2012) investigated the influence of the fields on various quantum states using a asymptotic iteration method. The ground state energies of hydrogenic impurities in cylindrical quantum dots were studies using finite difference method (Wang, Wei, & Yi, 2010), complex absorbing potential (CAP) (Sahoo, Lin, & Ho 2008), and the finite difference methods have been applied to quantum dot under the influence of electric and magnetic fields within the effective mass approximation (Hong, Li-Xue, Xue, Chun-Yuan, & Liun-Jun, 2011).

The shifted 1/N-expansion, proposed by Sukhatme and Imbo (Sukhatme & Imbo 1983; Imbo, Pagnamenta, & Sukhatme, 1984), is a powerful tool to solve Schrödinger equation for spherical symmetric potentials (Dutta, Mukherji, & Varshni, 1986; Roy & Roychoudhury, 1987; Roychoudhury, & Varshni, 1988; Chatterjee, 1990), and also it was extended successfully to relativistic potentials (Roychoudhury & Varshni 1989; Mustafa & Sever, 1991). The method is simple, and it gives accurate results of energy eigenvalues calculations of the system without dealing with robust numerical calculations or trail wave functions. The shifted 1/N-expansion method has already been used to study various systems, such as two-dimensional magnetoexcitons (Quiroga, Camacho, & Gonzalez, 1995), shallow donor impurities (El-Said, 1994), two-electron spherical quantum dot (Pino & Villalba, 2001), two interacting electrons in two dimensional quantum dot with the presence of magnetic field (El- Said, 2000; Gomez & Romero, 2009). And recently, we have used the method to calculate energies and binding energies for quantum dot with Gaussian potential confinement (Al-Hayek & Sandouqa, 2015), the results show a very good agreement with other computational methods like asymptotic integration method (AIM) and exact diagonalization method.

In this approach, the calculations are carried out for states with arbitrary quantum numbers (the principal and magnetic quantum numbers n and m, respectively) using forth-order perturbation theory in the shifted expansion parameter $1/\bar{k}$, where $\bar{k} = N + 2m - a$. N is the number of spatial dimensions and (a) is a suitable shift parameter which will be discussed later.

The rest of the this work is organized as follows. In section 2, we formulate the Hamiltonian of two electron quantum dot under the influence of external magnetic and electric field. In section 3, we present the shifted 1/N expansion method for arbitrary spherical potential V(r), and we apply the method on the potential of the problem. Numerical energy results and discussion are given in section 4. Conclusion is presented in section 5.

2. The Hamiltonian

The Hamiltonian of two electrons confined in a parabolic quantum dot under the effect of external electric and magnetic fields can be separated into center of mass H_{cm} and relative motion H_{rm} as follows:

$$H_{QD} = H_{cm} + H_{rm} \tag{1}$$

$$H_{cm} = \frac{1}{2M_{cm}}\left(\mathbf{P} - \frac{q}{c} A(R)\right)^2 + \frac{1}{2}M_{cm}\omega_0^2 R^2, \tag{2}$$

$$H_{rel} = \frac{1}{2\mu}\left(\mathbf{P} - \frac{q}{c} A(r)\right)^2 + \frac{1}{2}\mu\,\omega_0^2\, r^2 + \frac{e^2}{\varepsilon r} + q\,F.r, \tag{3}$$

A Where $M_{cm} = 2m^*$ is the total mass, and q = 2e is the total charge and ω_0 is the confinement potential. m^* is the effective mass of each electron, ε and c are the dielectric constant of the dot material and the speed of light, respectively. We introduce the relative coordinate of the two electrons $\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$ and momentum $\mathbf{P} = \mathbf{p_1} + \mathbf{p_2}$, with the reduced mass $\mu = m^*/2$.

The magnetic field effect appears in the effective frequency $\Omega = (\omega_0^2 + \omega_c^2/4)^{1/2}$ in the Hamiltonian, where $\omega_c = eB/m^*c$ is the cyclotron frequency. The magnetic field B is assumed to be uniform and perpendicular to the dot plane along the z axis. The vector potential is chosen to be in the symmetric gauge as $\mathbf{A} = (1/2)\,\mathbf{B} \times \mathbf{r}$. The eigenvalue of the angular momentum operator L_z is mħ, = 0, ±1, ±2 ….is the azimuthal quantum number. We assume the term qF.r in Equation (3) to be F r, Equation (3) can be written as:

$$H_{rel} = \frac{p^2}{2\mu} + \frac{1}{2}\mu\,\Omega\, r^2 + \frac{e^2}{\varepsilon r} + \frac{1}{2} L_z\,\omega_c + F\,r. \tag{4}$$

By making the substitution

$$\varphi(r) = r^{-1/2}\,\frac{e^{i m \varphi}}{\sqrt{2\pi}}\chi(r)$$

,

we obtain

$$\frac{d^2\chi(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left(E_{rel} - \frac{m\hbar\,\omega_c}{2} - \frac{(m^2 - \frac{1}{4})\hbar^2}{2\mu r^2} - \frac{1}{2}\mu\Omega^2 r^2 - \frac{e^2}{\varepsilon r} - F\,r\right)\chi(r) = 0\ , \tag{5}$$

The main task is to solve the Hamiltonian of the relative motion (Equation (3)). Using the substitution $r = \sqrt{2}\ \mathrm{l}_0\ \mathrm{x}$ Equation (3) becomes

$$\frac{\mathrm{d}^2\chi(x)}{dx^2} + (\varepsilon - x^2 - \frac{\lambda}{x} - \mathrm{Fx} - \frac{(m^2 - \frac{1}{4})}{x^2})\,\chi(x) = 0\ , \tag{6}$$

where $\varepsilon = \dfrac{E}{\frac{1}{2}\hbar\omega_0}$, $\lambda = \dfrac{\sqrt{2}\ \mathrm{l}_0}{a^*} = 2\sqrt{\dfrac{\mathrm{R}^*}{\hbar\omega_0}}$, $\ell_0 = 2\sqrt{\dfrac{\hbar}{m^*\omega_0}}$, $a^* = \dfrac{\hbar^2\ \varepsilon}{\mathrm{m}^* e^2}$

ℓ_0 is the characteristic length in the x-y plane, a^* and $\mathrm{R}^*$are the effective Bohr radius and effective Redberg, respectively. λ represents the ratio of the electron Coulomb interaction to the harmonic confinement, any change in the Coulomb strength or harmonic confinement results in a change in the coupling constant λ.

Equation (6) related to the relative motion of the two electron quantum dot under the influence of both magnetic and electric fields confined by the potential $V(x) = x^2 + \dfrac{\lambda}{x} + \mathrm{F}\,x$.

3. 1/N Expansion Formalism

Following the work of Imbo (Imbo, Pagnamenta, & Sukhatme, 1984) related to shifted 1/N expansion method, we formulating the radial Schrödinger equation for an arbitrary spherical potential V(r) as

$$\left(\frac{-\hbar^2}{2m*}\frac{d^2}{dr^2} + \frac{(k-1)(k-3)\hbar^2}{8m*r^2} + V(r)\right)\psi(r) = E\psi(r)\,, \tag{7}$$

where $k = N + 2m$. N is the number of spatial dimensions.

In terms of the shifted variable $\bar{k} = k - a$ (a is a shifted parameter), we rewrite Equation (3) as:

$$\left(\frac{-\hbar^2}{2m^*}\frac{d^2}{dr^2} + \bar{k}^2\left(\frac{\hbar^2\left[1-(1-a)/\bar{k}\right]\left[1-(3-a)/\bar{k}\right]}{8m^*r^2} + \frac{V(r)}{Q}\right)\right)\psi(r) = E\psi(r) \tag{8}$$

where Q is a constant which rescales the potential (in large $\bar{k}$ limit) and it will be determined below. The energy eigenvalues are given by an expansion in powers of $1/\bar{k}$.

The shifted 1/N expansion method consists in solving equation (4) systematically in terms of the expansion parameter $1/\bar{k}$. The leading contribution to the energy comes from the effective potential

$$V_{eff}(r) = \frac{\hbar^2}{8m^*r^2} + \frac{V(r)}{Q} \tag{9}$$

$V(r)$ is assumed to be well behaved so that $V_{eff}(r)$ has a minimum at $r = r_0$ and there are well-defined bound states. Q is then determined from the following equation

$$4m^*r_0^3V'(r_0) = \hbar^2 Q \tag{10}$$

It is convenient to shift the origin of coordinates to $r = r_0$ by defining

$$x = \frac{\bar{k}^{1/2}}{r_0}(r - r_0)\,. \tag{11}$$

The energy eigenvalues are given by an expansion in powers of $1/\bar{k}$ where $\bar{k} = N + 2m - a$, N being the number of spatial dimensions and the shifted parameter a (see Appendix).

For any value of the radial quantum number n_r ($n_r = n - m - 1$) and for any value of m the energy $E(n,m)$ is given by an expansion in powers of $1/\bar{k}$

$$E(n,m) = E_0 + E_1 + E_2 + E_3 + \cdots \tag{12}$$

where E_0, E_1, E_2, E_3, …. are given in appendix.

The shift parameter is fixed from the requirement that the term E_1 vanishes. Therefore,

$$a = 2 - 2\left(2n_r + 1\right)\frac{m^*\omega}{\hbar}, \tag{13}$$

ω is the anharmonic frequency parameter given in (Appendix, Equation A7). For any specific choice of n, m and N, the constant Q should be such as to make Eqs. (7) and (8) identical. This means

$$\bar{k} = \sqrt{Q}, \tag{14}$$

using Eqs. (10), (Equation A4), and (13), an explicit equation for determining the root r_0,

$$N + 2m - 2 + \left(2n_r + 1\right)\left[3 + \frac{r_0 V^{(2)}\left(r_0\right)}{V^{(1)}\left(r_0\right)}\right]^{1/2} = \left(\frac{4m^* r_0^3 V^{(1)}\left(r_0\right)}{\hbar^2}\right)^{1/2}. \tag{15}$$

Having determined r_0, all the energy eigenvalues can be computed.

By substituting the applied potential, $V\left(r\right)$, which describes the applied electric and magnetic fileds in Eqs. (10), (15), (Equation A4), we get

$$Q = 2\,F\,r_0^3 + 4\,r_0^4 - 2\,r_0\,\lambda, \tag{16}$$

$$2 - 2\,m - N + \sqrt{2}\,(F\,r_0^3 + 2\,r_0^4 - r_0\,\lambda)^{1/2} = \left(2n+1\right)\left(3 + \frac{2\,(\lambda + r_0^3)}{-\lambda + F\,r_0^2 + 2\,r_0^3}\right)^{1/2}, \tag{17}$$

and

$$\omega = \frac{1}{2}\left(3 + \frac{2\,(\lambda + r_0^3)}{-\lambda + F\,r_0^2 + 2\,r_0^3}\right)^{1/2}. \tag{18}$$

3. Numerical Results Discussions

The calculated energies of two interacting electrons in a two-dimensional quantum dot using 1/N expansion are presented in Tables 1-3. Table 1 lists energy calculations for two interacting electrons in two dimensional quantum dot under the influence of magnetic field only (λ = 1, F = 0). Energies are obtained for ground state with different values of magnetic quantum number m. The energy eigenvalues increase as the magnitude of the orbital quantum number m increases. For comparison of energy accuracy of our results, we list the energies obtained by exact method (García-Castelán, Choe, & Lee, 1998) and asymptotic iteration method AIM (Soylu, 2012). Our results show a very good agreement.

Table 2 displays energy eigenvalues using magnetic field only with higher strength (λ = 10 and F = 0). The effect of enhancing the interaction term, λ, on energies is clearly seen by comparing the results in Table 2 with Table 1 for the same state. This increase in energy is due to increase of the Coulomb interaction energy between the two electrons comparable to the parabolic confinement one. The results show also a very good accuracy with diagonalization and AIM methods (García-Castelán, Choe, & Lee, 1998; Soylu, 2012).

Table 3 shows the calculated energy eigenvalues for the two-electron quantum dot system under the effect of both electric and magnetic field for λ = 1 and different electric field strengths (F = 0, F =10 and F = 100). The dependence of energies on the electric field strength is seen. As the electric field strength increases, the calculated energy eigenvalues increases. Our calculated energy results are compared with those obtained using AIM method. The data shows a very good accuracy of the shifted 1/N expansion method (Soylu, 2012).

4. Conclusion

In summary, we have calculated the energy eigevalues of two electron quantum dot in two dimensions under the influence of both magnetic and electric fields within the frame work of the shifted 1/N expansion. The effect of both electric and magnetic fields on the ground state energies of the system have been found for different magnetic quantum number m. The solution of the relative part of the Hamiltonian is made for different degrees of the Coulomb to confinement ratios (λ) and electric field strength (F).

The simplicity and efficiency of using the shifted 1/N expansion is shown in our work. Hence, no need to deal with robust numerical calculations or trial wave functions. Our results have a very good agreement with other different numerical methods. Finally, we think that the shifted 1/N expansion can be applied effectively to other kinds of potential confinements in spherical quantum dots.

Table 1. Calculation results for the ground state energies of two-electron quantum dot with $\lambda = 1$ and $F = 0$. Energies are in $1/2h\omega_0$ unit

|n m>	Present results	Exact	AIM
|0 0>	3.42204	3.4952	3.49652
|0 1>	4.85217	4.8553	4.85534
|0 2>	6.65338	6.6538	6.65384
|0 3>	8.54834	8.5485	8.54845
|0 4>	10.4814	10.4814	10.48140
|0 5>	12.434	12.4340	12.43403
|0 6>	14.3983	14.3983	14.39830
|0 7>	16.3701	16.3701	16.37013
|0 8>	18.3472	18.3472	18.34718
|0 9>	20.328	20.3280	20.32802

Table 2. Calculation results for the ground state energies of two-electron quantum dot with. $\lambda = 10$ and $F = 0$. Energies are in $1/2h\omega_0$ unit

|n m>	Present results	Exact	AIM
|0 0>	10.4382	10.4816	10.48157
|0 1>	10.8339	10.8495	10.84954
|0 2>	11.7859	11.7903	11.79025
|0 3>	13.0706	13.0720	13.07195
|0 4>	14.5541	14.5546	14.5564
|0 5>	16.1626	16.1628	16.16284
|0 6>	17.8541	17.8543	17.85425
|0 7>	19.6037	19.6037	19.60371
|0 8>	21.3954	21.3954	21.39538
|0 9>	23.2188	23.2188	23.21878

Table 3. Calculation results for the ground state energies of two-electron quantum dot with $\lambda = 1$ for different electric filed strength F. Energies are in $1/2h\omega_0$ unit

		F = 0		F = 10		F =100	
|n	m>	Present results	AIM	Present results	AIM	Present results	AIM
|0	0>	2.2702	2.3195	9.62726	9.7255	36.9759	37.1878
|0	1>	2.82468	2.8278	12.5221	12.5245	52.7579	52.7602
|0	2>	3.64314	3.6436	15.733	15.7331	67.9271	67.9269
|0	3>	4.54309	4.5432	18.8051	18.8051	81.8352	81.8350
|0	4>	5.47819	5.4782	21.7253	21.7253	94.7532	94.7530
|0	5>	6.43189	6.4319	24.5153	24.5153	106.898	106.8980
|0	6>	7.39678	7.3967	27.1964	27.1963	118.422	118.4217

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Appendix

Substituting Equation (7) in Equation (4) and performing a series expansion in powers of x about x = 0 yields (Imbo, Pagnamenta, & Sukhatme, 1984).

$$\left(-\frac{\hbar^2}{2m^*}\frac{d^2}{dx^2}+\frac{\bar{k}\hbar^2}{8m^*}\left[1+\frac{3x^2}{\bar{k}}-\frac{4x^3}{\bar{k}^{3/2}}+\frac{5x^4}{\bar{k}^2}-\cdots\right]-\frac{(2-a)\hbar^2}{4m^*}\left[1-\frac{2x}{\bar{k}^{1/2}}+\frac{3x^2}{\bar{k}}-\cdots\right]\right.$$

$$\left.+\frac{(1-a)(3-a)\hbar^2}{8\bar{k}m^*}\left[1-\frac{2x}{\bar{k}^{1/2}}+\frac{3x^2}{\bar{k}}-\cdots\right]+\frac{r_0^2\bar{k}}{Q}\left[V(r_0)+\frac{V^{(2)}(r_0)r_0^2x^2}{2\bar{k}}+\frac{V^{(3)}(r_0)r_0^3x^3}{\bar{k}^{3/2}}+\cdots\right]\right)\psi(r)=\frac{Er_0^2}{\bar{k}}\psi(r) \tag{A1}$$

The Schrödinger equation for an ℓ-dimensional anharmonic oscillator is

$$\left[\frac{-\hbar^2}{2m^*}\frac{d^2}{dx^2}+\frac{1}{2}m^*\omega^2x^2+\varepsilon_0+P(x)\right]\psi(x)=\lambda\psi(x) \tag{A2}$$

Where $P(x)$ is the perturbation term given by

$$P(x)=g^{1/2}\left(\varepsilon_1x+\varepsilon_3x^3\right)+g\left(\varepsilon_2x^2+\varepsilon_4x^4\right)$$

$$+g^{3/2}\left(\delta_1x+\delta_3x^3+\delta_5x^5\right)+g^2\left(\delta_2x^2+\delta_4x^4+\delta_6x^6\right) \tag{A3}$$

We can compare Eqs. (A1) and (A3) term by term to define all the anharmonic parameters in terms of $\bar{k}$, Q, r_0 and potential derivatives.

Proceeding in a straightforward way we obtain the following identifications:

$$\omega = \left[\frac{3\hbar^2}{4m^{*2}} + \frac{r_0^4 V^{(2)}(r_0)}{m^* Q}\right]^{1/2} = \frac{\hbar}{2m^*}\left[3 + \frac{r_0 V^{(2)}(r_0)}{V^{(1)}(r_0)}\right]^{1/2} \tag{A4}$$

$$g = \frac{1}{\bar{k}}$$

$$\lambda = \frac{E r_0^2}{\bar{k}}$$

$$\varepsilon_0 = \frac{\hbar^2 \bar{k}}{8m^*} - \frac{\hbar^2 (2-a)}{4m^*} + \frac{\hbar^2 (1-a)(3-a)}{8m^* \bar{k}} + \frac{r_0^2 \bar{k} V(r_0)}{Q}, \tag{A5}$$

$$\varepsilon_1 = \frac{\hbar^2}{2m^*}(2-a), \qquad \varepsilon_2 = \frac{-3\hbar^2}{4m^*}(2-a), \qquad \varepsilon_3 = \frac{-\hbar^2}{2m^*} + r_0 \frac{V^{(3)}(r_0)}{6Q}$$

$$\varepsilon_4 = \frac{5\hbar^2}{8m^*} + r_0^6 \frac{V^{(4)}(r_0)}{24Q}, \qquad \delta_1 = \frac{-(1-a)(3-a)\hbar^2}{4m^*}, \qquad \delta_2 = \frac{3(1-a)(3-a)\hbar^2}{8m^*}$$

$$\delta_3 = \frac{(2-a)\hbar^2}{m^*}, \qquad \delta_4 = \frac{-5(2-a)\hbar^2}{4m^*}, \qquad \delta_5 = \frac{-3\hbar^2}{4m^*} + r_0^7 \frac{V^{(5)}(r_0)}{120Q}, \qquad \delta_6 = \frac{7\hbar^2}{8m^*} + r_0^8 \frac{V^{(6)}(r_0)}{720Q}.$$

Where $V^{(n)}(r)$ is the n[th] derivative of $V(r)$ with respect to r.

For any value of the radial quantum number n_r ($n_r = n - m - 1$) and for any value of ℓ the energy $E(n_r, m)$ is given by an expansion in powers of $1/\bar{k}$

$$E(n_r, m) = E_0 + E_1 + E_2 + E_3 + \cdots \tag{A6}$$

Where

$$E_0 = \bar{k}^2 \left[\frac{\hbar^2}{8m^* r_0^2} + \frac{V(r_0)}{Q}\right]$$

$$E_1 = \frac{\bar{k}}{r_0^2}\left[\left(n_r + \frac{1}{2}\right)\hbar\omega - (2-a)\frac{\hbar^2}{4m^*}\right]$$

$$E_2 = \frac{1}{r_0^2}\left[\frac{\hbar^2}{8m^*}(1-a)(3-a) + \left\{(1+2n_r)\tilde{\varepsilon}_2 + 3\tilde{\varepsilon}_4\left(1+2n_r+2n_r^2\right)\right\} - \frac{1}{\hbar\omega}\left\{\tilde{\varepsilon}_1 + 6(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_3 + \left(11+30n_r+30n_r^2\right)\tilde{\varepsilon}_3^2\right\}\right]$$

$$E_3 = \frac{1}{\bar{k} r_0^2}\left[\left\{(1+2n_r)\tilde{\delta}_2 + 3\left(1+2n_r+2n_r^2\right)\tilde{\delta}_4 + 5\left(3+8n_r+6n_r^2+4n_r^3\right)\tilde{\delta}_6\right\}\right.$$

$$-\frac{1}{\hbar\omega}\left\{(1+2n_r)\tilde{\varepsilon}_2^2 + 12\left(1+2n_r+2n_r^2\right)\tilde{\varepsilon}_2\varepsilon_4\right.$$

$$+2\left(21+59n_r+51n_r^2+34n_r^3\right)\tilde{\varepsilon}_4^2 + 2\tilde{\varepsilon}_1\tilde{\delta}_1 + 6(1+2n_r)\tilde{\varepsilon}_1\tilde{\delta}_3$$

$$+30\left(1+2n_r+2n_r^2\right)\tilde{\varepsilon}_1\tilde{\delta}_5 + 6(1+2n_r)\tilde{\varepsilon}_3\tilde{\delta}_1$$

$$\left.\left.+2\left(11+30\mathrm{n}_\mathrm{r}+30n_r^2\right)\tilde{\varepsilon}_3\tilde{\delta}_3 + 10\left(13+40n_r+42n_r^2+28n_r^3\right)\tilde{\varepsilon}_3\tilde{\delta}_5\right\}\right]$$

$$+\frac{1}{\hbar^2\omega^2}\left\{4\tilde{\varepsilon}_1^2\tilde{\varepsilon}_2+36\left(1+2n_r\right)\tilde{\varepsilon}_1\tilde{\varepsilon}_2\tilde{\varepsilon}_3+8\left(11+30n_r+30n_r^2\right)\tilde{\varepsilon}_2\tilde{\varepsilon}_3^2\right.$$

$$+24\left(1+2n_r\right)\tilde{\varepsilon}_2^2\tilde{\varepsilon}_4+8\left(31+78n_r+78n_r^2\right)\tilde{\varepsilon}_1\tilde{\varepsilon}_3\tilde{\varepsilon}_4$$

$$\left.+12\left(57+189n_r+225n_r^2+150n_r^3\right)\tilde{\varepsilon}_3^2\tilde{\varepsilon}_4\right\}$$

$$+\frac{1}{\hbar^3\omega^3}\left\{8\tilde{\varepsilon}_1^2\tilde{\varepsilon}_3+108\left(1+2n_r\right)\tilde{\varepsilon}_1^2\tilde{\varepsilon}_3^2+48\left(11+30n_r+30n_r^2\right)\tilde{\varepsilon}_1\tilde{\varepsilon}_3^3\right.$$

$$\left.+30\left(31+109n_r+141n_r^2+94n_r^3\right)\tilde{\varepsilon}_3^4\right\}\Bigg]$$

where

$$\tilde{\varepsilon}_j=\frac{\varepsilon_j}{\left(2m^*\omega/\hbar\right)^{j/2}}$$

$$\tilde{\delta}_j=\frac{\delta_j}{\left(2m^*\omega/\hbar\right)^{j/2}}\text{, j = 1, 2, 3, ….}$$