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Fractional Variational Problems of Euler-Lagrange Equations with Holonomic Constrained Systems

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Abstract

In this paper, we examined the fractional Euler-Lagrange equations for Holonomic constrained systems. The Euler-Lagrange equations are derived using the fractional variational problem of Lagrange. In addition, we achieved that the classical results were obtained are agreement when fractional derivatives are replaced with the integer order derivatives. Two physical examples are discussed to demonstrate the formalism.

Keywords: Holonomic Constraints, Euler-Lagrange equation, fractional variational problem

1. Introduction

The study of Holonomic constrained systems are discussed in most references of classical mechanical (Atom, 1990; Goldstein, 1980). These systems describe dynamic systems with constraints depend only on the coordinates. They do not depend on velocity. The canonical formalism of Holonomic systems was treated by Rabei (1999). In this formalism, the author has treated the regular Lagrangian with Holonomic constraints as singular systems. The Lagrange multipliers for these systems are introduced as generalized coordinates. The equations of motions are written as total differential equations, and then the Holonomic systems are quantized using the WKB approximation (Serhan et al., 2009).

The study of fractional derivatives has reached a great status in various branches of science, applied mathematics, physical systems and engineering (Miller & Ross, 1993; Samko et al., 1993; Gorenflo & Mainardi, 1997), therefore the construction of the fractional Euler-Lagrange equations for Holonomic constrained systems of prime importance. Riewe (1996, 1997) has used fractional derivatives to construct a Lagrangian and a Hamiltonian for non-conservative systems. One can obtain the Lagrangian and the Hamiltonian equations of motion for these systems. Recently, a new formalism for investigating the fractional variational problem of Lagrange was discussed by Agrawal's (1999, 2001). In this formalism, the fractional Euler-Lagrange equation was derived. Besides, the generalization of Lagrangian and Hamiltonian fractional mechanics with fractional derivatives were extended and discussed in details in (Agrawal, 2001, 2002; Rabei et al., 2007) , then this formalism has found a wide range of applications (Hilfer, 2000; Rousan et al., 2002).

This paper is organized as follow. In Section 2, the Euler-Lagrange equation for holonomic constraints was briefly reviewed. In Section 3, basic definitions of fractional derivatives were briefly discussed. In Section 4. the fractional variational problem for Holonomic constraints is examined. In Section 5, two illustrative example are examined. The work closes with some concluding remarks in section 6.

2. Euler-Lagrange Equations for Holonomic constraints

In this section, we will review briefly the Euler-Lagrange equations for the Holonomic constraints without fractional derivatives (Rabei, 1999). The Euler-Lagrange equations take the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=\lambda_\mu \frac{\partial f}{\partial q_i}, \qquad i=1,2,3,...n \tag{1}$$

Here the Lagrangian $L=L(q_i,\dot{q}_i,t)$ is regular and the constraint equation with m constraints can be written as

$$f_\mu(q_i,t)=0, \qquad \mu=n+1,n+2,...,n+m \tag{2}$$

Now, the new Lagrangian is constructed by adding the holonomic constraints multiplied by the Lagrange multipliers to the regular Lagrangian, and then the new Lagrangian has the form

$$L'(q_i,\lambda_\mu,\dot{q}_i,t)=L(q_i,\dot{q}_i,t)+\lambda_\mu f_\mu\,. \tag{3}$$

Here λ_μ are the Lagrange multipliers and they are treated as generalized coordinates. Thus, the new Lagrangian is considered as singular with property that the Hessian determinant vanishes (Rabei, 1999).

$$\begin{vmatrix} \dfrac{\partial^2 L'}{\partial \dot{q}_i \partial \dot{q}_j} & \dfrac{\partial^2 L'}{\partial \dot{q}_i \partial \dot{\lambda}_\mu} \\ \dfrac{\partial^2 L'}{\partial \dot{\lambda}_\beta \partial \dot{q}_j} & \dfrac{\partial^2 L'}{\partial \dot{\lambda}_\beta \partial \dot{\lambda}_\mu} \end{vmatrix}. \tag{4}$$

Thus, the extended Euler-Lagrange equations are increased by m equations

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right)-\frac{\partial L'}{\partial q_i}=0\,, \tag{5a}$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\lambda}_\mu}\right)-\frac{\partial L'}{\partial \lambda_\mu}=0\,. \tag{5b}$$

Equation (5a) leads to Equation (1) while Equation (5b) gives the holonomic constraints Equation (2).

3. Basic Definitions of Fractional Derivatives

Now, we will give the basic definitions of a fractional derivatives include the left and right RL fractional derivatives (Agrawal, 2001, 2002) and their properties. The left Riemann-Liouville fractional derivatives is defined as

$${}_aD_x^\alpha f(x)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int_a^x (x-\tau)^{n-\alpha-1}f(\tau)d\tau\,. \tag{6}$$

and the right Riemann-Liouville fractional derivatives has the form

$${}_xD_b^\alpha f(x)=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n\int_x^b (\tau-x)^{n-\alpha-1}f(\tau)d\tau\,. \tag{7}$$

where $n\in N$, $n-1\le\alpha<n$ and Γ represents the Euler's gamma function. If α is an integer, these derivatives are defined as follows

$${}_aD_x^\alpha f(x)=\left(\frac{d}{dx}\right)^\alpha f(x)\,, \qquad {}_xD_b^\alpha f(x)=\left(-\frac{d}{dx}\right)^\alpha f(x)\,. \qquad \alpha=1,2,...... \tag{8}$$

The fractional operator ${}_aD_x^\alpha$ can be written as (Igor et al., 2002)

$${}_aD_x^\alpha=\frac{d^n}{dx^n}\,{}_aD_x^{\alpha-n}\quad . \tag{9a}$$

and has the following properties

$$
{}_aD_x^\alpha = \begin{cases} \dfrac{d^\alpha}{dx^\alpha} \ldots\ldots\ldots\ldots\ldots \mathrm{Re}(\alpha) > 0 \\ 1 \ldots\ldots\ldots\ldots\ldots\ldots \mathrm{Re}(\alpha) = 0 \\ \int_a^x (d\tau)^{-\alpha} \ldots\ldots\ldots \mathrm{Re}(\alpha) < 0 \end{cases} . \tag{9b}
$$

Theorem: Let f and g are two continuous functions on [a, b]. Then, for all $x \in [a,b]$, the following properties hold:

(i) For $m > 0$, ${}_aD_x^m[f(x)+g(x)] = {}_aD_x^m f(x) + {}_aD_x^m g(x)$;

(ii) For $m \geq n \geq 0$, ${}_aD_x^m({}_aD_x^{-n} f(x)) = {}_aD_x^{m-n} f(x)$;

(iii) For $m > 0$, ${}_aD_x^m({}_aD_x^{-m} f(x)) = f(x)$;

(iv) For $m > 0$, $\int_a^b ({}_aD_x^m f(x)) g(x) dx = \int_a^b f(x)({}_xD_b^m g(x)) dx.$

4. The Fractional Variational Problem for Holonomic Constraints

The Lagrangian formulation depending on the fractional derivatives for Holonomic constraints is given by the form:

$$
L'(q_i, \lambda_\mu, {}_aD_t^\alpha q, {}_tD_b^\beta q, t) = L(q_i, {}_aD_t^\alpha q_i, {}_tD_b^\beta q_i, t) + \lambda_\mu f_\mu \quad . \quad 0 < \alpha,\ \beta < 1. \tag{10}
$$

be a function with continuous partial derivatives with respect to all its arguments. All functions $q(t)$ have continuous LRLFD of order α and RRLFD of order β for $a \leq t \leq b$, and satisfy the boundary conditions

$$
q(a) = q_a, \quad q(b) = q_b \quad . \tag{11}
$$

We now examine the extrema of the functional

$$
S[q] = \int L'(q, \lambda_\mu, {}_aD_t^\alpha q, {}_tD_b^\beta q, t) dt . \tag{12}
$$

Where $0 < \alpha$, $\beta \leq 1$ and α, $\beta \in R^+$, when $\alpha = \beta = 1$, the above problem reduces to the simplest variational problem.

The necessary conditions for the extremum of the action (12), one can define a family of functions

$$
q(t) = q^*(t) + \epsilon \eta(t) \quad , \tag{13}
$$

where $q^*(t)$ is the desired real function that satisfy the extremum of the action (12), $\epsilon \in R$ is a constant, and the function η defined in $[a,b]$ satisfy the boundary conditions

$$
\eta(a) = \eta(b) = 0 , \tag{14}
$$

Let us define a set of linear operators as follows

$$
{}_aD_t^\alpha q(t) = {}_aD_t^\alpha q^*(t) + \epsilon\, {}_aD_t^\alpha \eta(t) \quad , \tag{15a}
$$

$$
{}_tD_b^\beta q(t) = {}_tD_b^\beta q^*(t) + \epsilon\, {}_tD_b^\beta \eta(t) \quad , \tag{15b}
$$

Substituting Eqs. (13) and (15) into Equation (12), one can find for each $\eta(t)$

$$
S[\in] = \int_a^b [L(t, q^* + \in \eta, {}_aD_t^\alpha q^* + \in {}_aD_t^\alpha \eta, {}_tD_b^\beta q^* + \in {}_tD_b^\beta \eta) + \lambda_\mu f(q^* + \in \eta)] dt. \tag{16}
$$

is a function of $\in$ only. We can note that $S(\in)$ is extremum at $\in = 0.$

One can differentiate Equation (16) with respect to $\in$; we can obtain the variation of $S[q]$ at $q(t)$ along $\eta(t)$

$$\frac{dS}{d\in}=\int_a^b\left[\frac{\partial L}{\partial q}\eta+\frac{\partial L}{\partial_a D_t^\alpha q}{}_aD_t^\alpha\eta+\frac{\partial L}{\partial_t D_b^\beta}{}_tD_b^\beta\eta+\lambda_\mu\frac{\partial f}{\partial q}\eta\right]dt \tag{17}$$

The fundamental necessary condition for $S[\in]$ to have an extremum is that $dS/d\in$ must be zero.

$$\int_a^b\left[\frac{\partial L}{\partial q}\eta+\frac{\partial L}{\partial_a D_t^\alpha q}{}_aD_t^\alpha\eta+\frac{\partial L}{\partial_t D_b^\beta}{}_tD_b^\beta\eta+\lambda_\mu\frac{\partial f}{\partial q}\eta\right]dt=0\,. \tag{18}$$

For all admissible $\eta(t)$. Integrating the second integral in Equation (18) by parts and using the formula for fractional integration by parts, one can write (Samko et al., 1993; Riewe, 1996)

$$\int_a^b\frac{\partial L}{\partial_a D_t^\alpha q}{}_aD_t^\alpha\eta dt=\int_a^b{}_tD_b^\alpha\left(\frac{\partial L}{\partial_a D_t^\alpha q}\right)\eta dt\,. \tag{19}$$

provided that $\partial L/\partial_a D_t^\alpha q$ or η is zero at $t=a$ and $t=b$. By using Equation (14), this condition is satisfied, and it follow that Equation (19) is valid. Similarly, the third integral in Equation (18) can be written as

$$\int_a^b\frac{\partial L}{\partial_t D_b^\beta q}{}_tD_b^\beta\eta dt=\int_a^b{}_aD_t^\beta\left(\frac{\partial L}{\partial_t D_b^\beta q}\right)\eta dt\,. \tag{20}$$

Substituting Eqs. (19) and (20) into Equation (18), we get

$$\int_a^b\left[\frac{\partial L}{\partial q}+{}_tD_b^\alpha\frac{\partial L}{\partial_a D_t^\alpha q}+{}_aD_t^\beta\frac{\partial L}{\partial_t D_b^\beta}+\lambda_\mu\frac{\partial f}{\partial q}\right]\eta dt=0\,. \tag{21}$$

Since η is arbitrary, it follows that (Samko et al., 1993)

$$\frac{\partial L}{\partial q}+{}_tD_b^\alpha\frac{\partial L}{\partial_a D_t^\alpha q}+{}_aD_t^\beta\frac{\partial L}{\partial_t D_b^\beta q}+\lambda_\mu\frac{\partial f}{\partial q}=0\,. \tag{22}$$

Equation (22) is the formulation of Euler-Lagrange equations for the fractional calculus of variational problem for holonomic constraints.

5. Examples

5.1 As a first example, let us consider a bead of mass m is constrained to move on a frictionless horizontal circular wire of radius R.

The Lagrangian of our problem is given by

$$L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)-mgr\cos\theta\,. \tag{23}$$

Is subject to the Holonomic constraint

$$f=r-R=0\,. \tag{24}$$

The Lagrangian in fractional form can be written as

$$L=\frac{1}{2}m[({}_0D_t^\alpha r)^2+r^2({}_tD_1^\beta\theta)^2]-mgr\cos\theta\,. \tag{25}$$

The Euler-Lagrange equations corresponding to Equation (22) become

$$mr({}_tD_1^\beta\theta)^2-mg\cos\theta+{}_tD_1^\alpha m({}_0D_t^\alpha r)+\lambda_r=0\,, \tag{26a}$$

$$mgr\sin\theta + {}_0D_t^{\beta} mr^2 ({}_tD_1^{\beta}\theta) = 0 . \tag{26b}$$

From Eqs. (26), we can obtain the classical results if α and β are equal to unity. One can get the angular acceleration

$$\ddot{\theta} = \frac{g\sin\theta}{R} . \tag{27}$$

and the force of constraint (Lagrange multiplier) is given by

$$\lambda = mg(3\cos\theta - 2) . \tag{28}$$

5.2 As a second example, let us consider the motion of a disk of mass m and radius R that is rolling down an inclined plane without slipping.

The Lagrangian of our problem is given by

$$L = \frac{1}{2}m\dot{y}^2 + \frac{1}{4}mR^2\dot{\theta}^2 + mgy\sin\phi . \tag{29}$$

is subject to the Holonomic constraint

$$f = y - R\theta = 0 . \tag{30}$$

where ϕ is the angle of the incline plane.

The extended Lagrangian in fractional form can be written as

$$L' = \frac{1}{2}m({}_0D_t^{\alpha} y)^2 + \frac{1}{4}mR^2({}_tD_1^{\beta}\theta)^2 + mgy\sin\phi + \lambda(y - R\theta) . \tag{31}$$

The Euler-Lagrange equations corresponding to Equation (22) become

$$mg\sin\phi + m\,{}_tD_1^{\alpha}({}_0D_t^{\alpha} y) + \lambda = 0 , \tag{32a}$$

$$\frac{mR^2}{2}\,{}_0D_t^{\beta}({}_tD_1^{\beta}\theta) - R\lambda = 0 . \tag{32b}$$

From Eqs. (32), we can obtain the classical solution if α and β are equal to unity. One can get the acceleration and the angular acceleration

$$\ddot{y} = \frac{2}{3}g\sin\phi , \tag{33a}$$

$$\ddot{\theta} = \frac{2g}{3R}\sin\phi . \tag{33b}$$

The force of constraint (Lagrange multiplier) is given by

$$\lambda = -\frac{1}{3}mg\sin\phi . \tag{34}$$

6. Conclusion

This paper is mainly concerned with the calculus of variation for Lagrangian containing fractional derivatives, especially for Holonomic constrained systems. The fractional Euler-Lagrange equations for these systems were derived. The solutions of Euler-Lagrange equations were obtained and the recovery of the classical results was discussed. Two examples were examined.

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