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Introduction to the Coherent States Approach for Solving Non-linear Physical Problems

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Abstract

We summarize some crucial results from mathematical models that help us understand the relevance of generalized coherent states (GCS), a fundamental method used for the description of non-linear problems in physics. We mainly concentrate our review in the following models: ferromagnetism, nonlinear Schrödinger equation with external potential, nonlinear quantum oscillators, Bose - Einstein condensation (BEC) and the DNA quasi-spin model. Such models and some applications of the wide variety these involve are outlined. As variational trial states, the coherent states (CS) allow the estimation of the ground state energies and properties, yielding results which become exact in a number of nonlinear differential equations for dynamical variable of each model. Mainly we concentrate our review on two types of coherent states, The first one are based on the Heisenberg - Weyl group where it is applied the bosonization effect. The second one is based on the SU(2)/U(1) Lie group. For this case when the Hamiltonian and all physical operators are constructed by the elements of a Lie algebra, the generalized coherent states approach is directly applied without bozonization.

Keywords: solutions of wave equations: bound states, Semi classical theories and applications, Nonlinear evolution equation, Generalized Coherent states

1. Introduction

In 1926, Schrödinger built the states for quantum harmonic oscillator (Schrodinger, 1926). A year after P. A. M. Dirac, using Maxwell's concept of light and the quantum oscillator, found out a description for electromagnetic radiation. Because of the inequivalent description of the same system, in general, the predictions of Dirac's formulation didn't get right with Maxwell's. Moreover, they were describing opposite sides of the behavior of light. Those were studied again in 1963 by Glauber, Klauder and Sudarshan, among others, for studies on quantum optics (Glauber, 1963a, 1963b). The major contribution of Glauber was, based on the pioneer works of Dirac and Schödinger, to find a model relating both formulations; the theoretical structure of Dirac's work, but achieving the same predictions from the classical theory. In this way, the laser beam played a main role, we owe its basic understanding to Roy J. Glauber. He gave the official name of *coherent states* to the lasers quantum states, which fluctuations on its amplitudes and phases, for example, are simultaneously neglected (Glauber, 1963). Since that moment the coherent states has been widely studied as a fundamental tool of mathematical physics. The interest for such states have been extended to several branches of non-linear physics for modeling multiple non-linear systems and also for systems connected with other groups (including super-groups). Coherent states can be determined for quantum groups as well as using various exponential generalizations (deformations) (Borzov & Damaskinsky, 2006; Combescure & Robert, 2012).

Why does it make so advantageous to employ this method? One of the most important features is that the Coherent State system is over-complete. As it's well known the complete orthogonal systems of basis vectors in the complex Hilbert space (let's not forget this is the quantum analogue of the classical phase space, the analog of a phase-space point is a vector lying in the Hilbert space) is a mind-blowing concept of mathematical physics, in quantum physics we can use over-complete and non-orthogonal systems of state vectors for solving some problems. An over-complete system means, in plain words, that if at least one vector exists in the system which can be removed, while the system remains complete. The system contains more states than necessary to decompose an arbitrary state vector. In addition, coherent state provides a natural framework in which to discuss the relation between quantum and classical mechanics, these states seem to be a natural bridge for studying the classical-quantum correspondence. This because these states minimize the Heisenberg uncertainty relations. And what about the label "coherent" in the name? We ought to look back into the pioneer works and the motivations of these studies (mentioned above) to understand this, so in the next section we present a description closely related with these works, so we go forth just a little to clarify this inquiry in words of Jean-Pierre Gazeau (Perelomov, 2010; Jean-Pierre, 2009).

An electromagnetic field operator is said to be "fully coherent" in the Glauber sense if all of its correlation factorize.

By a close inspection the Coherent States system is related with the Heisenberg-Weyl group. Nevertheless, the Heisenberg-Weyl group is not the universal dynamical symmetry group. The Lie groups systems of states exhibit similar properties for the same system. So, these states can be extended to a more general conception and the outcome will be the so-called Generalized Coherent States, which appear naturally in modern problems, providing a powerful tool to undertake them (Perelomov, 2010). But, why do we associate the field coherent states with groups? It occurs that these states have several properties and we can give a group interpretation. Probably the next question coming naturally is about the representations. We have to look back to quantum states (we refer to position representation, momentum representation), we are acquainted with these representations, and also the Fock representation and the Fock-Bargman representation (Konstant, 1970). We could express for instance the Coherent States in these representations without loss of generality.

This is the starting point to understand a generalization of the states we are dealing with and the next concern will be to introduce particular examples of generalized coherent states associated with groups, that is the case of the ones based on Lie groups, very popular and remarkable for the study of quantum systems. Particularly the group SU(2) and SU(1,1), we're well acquainted with these groups, we have already worked with these in elementary quantum mechanics courses, perhaps without noticing, but we summarize briefly anyway. By means of examples such as the well known Heisenberg theory of ferromagnetism, The Ehrenfest theorem, the Hubbard model, the DNA dynamics, among others we show the importance of the role these quantum states play in non linear physics.

This work is divided in three main sections. The first one is devoted to coherent states of the Weyl-Heisenberg group, we construct the coherent states of electromagnetic field, in fact, this is a major task in order to see clearer the meaning of coherence. As it was told before the efforts of Dirac did not fully match with Maxwell's classical theory, but the structure he employed was useful to manage with these states as we can visualize it. By means of having a wider understanding of the importance of these states we present a couple of examples. With the aid of the cubic-quintic non-linear Schödinger equation obtained from the average treatment of of spin quantum states, and the dynamics of Kennard squeezed states, we discuss briefly how the coherent states are closely related with the soliton-like solutions. In the second section we introduce a generalization of the scheme of coherent states and subsequently we analyze two of the generalized coherent states associated to Lie groups. Finally, in the last section we describe ,. four major examples of using GCS approach, these four examples portray stately the role of GCS in description of physical phenomena. In the last section we comment some crucial points regarding this approach.

2. Coherent States and Bosonization

Let us start with the simplest operators used in describing a quantum mechanical system with one degree of freedom. These operators are the canonical coordinate q and momentum p operator that act in the Hilbert space in addition with the identity operator $\hat{I}$ and satisfy the commutation relations:

$$[q, p] = i\hbar\hat{I}, \quad [q, \hat{I}] = [p, \hat{I}] = 0 \tag{1}$$

where $\hat{I}$ being the identity operator, and $\hbar$ Planck´ s constant. Next it is constructed the Heisenberg -Weyl algebra taking in consideration there operators but in a more comfortable form. Indeed, we have instead of operator p, q the new ones

$$a = \frac{q + ip}{\sqrt{2\hbar}}, \quad a^{+} = \frac{q - ip}{\sqrt{2\hbar}} \tag{2}$$

then, the elements of the Heisenberg-Weyl algebra are written as

$$z = (s; x_1, x_2) = x_1e_1 + x_2e_2 + se_3 \tag{3}$$

with

$$e_1 = i(\hbar)^{-1/2}p, \quad e_2 = i(\hbar)^{-1/2}q, \quad e_3 = i\hat{I} \tag{4}$$

When the creation and annihilation operators, identity operator, and numeric conservation operator as generators are inlcuded, the system of harmonic oscillator possess the Heisenberg-Weyl group dynamics H_4. Glauber (1963) introduced the "term coherent states" and utilized the states of quantum oscillator for studying problems of quantum optics. But Schrödinger (1926) constructed these states as wave packets for determining the link between classical and quantum worlds. It's natural, for that, constructing the coherent states by one-to-one correspondence with the geometric space $H(4)/U(1) \otimes U(1)$. Thus the wave packets that were built in the early development of quantum mechanics, does not spread, and its center moves along the classical trajectory.

The construction of the The Heisenberg Weyl group as is well know, is possible by using the exponentiation

$$exp(z) = exp(isI)D(\beta), \quad D(\beta) = exp(\beta a^+ - \bar{\beta}a) \tag{5}$$

where $z = (s; x_1, x_2)$ that also can be represented using the creation and anihilation operators as $z = is\hat{I} + \frac{i}{\hbar}(Pq - Qp) = is\hat{I} + (\beta a^+ - \bar{\beta}a)$ as the element of Heisenberg Weyl algebra.

When the ground state of the quantum oscillator is defined by the condition

$$a|0>= 0 \tag{6}$$

the coherent states $|\beta>$ can be obtained as follows

$$|\beta>= D(\beta)|0> \tag{7}$$

These coherent states can be represented as a decomposition in terms of oscillator stationary states, More details on the properties of these coherent states could be found in (Gilmore, 1974; Perelomov, 2010).

$$|\beta>= exp\left(-\frac{|\beta|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!}}|n> \tag{8}$$

If we calculate the scalar product one can obtain

$$| <\beta|\alpha> |^2 = exp(-|\alpha-\beta|^2) \tag{9}$$

that means the coherent states are not orthogonal to each other and each quantum state $|\psi>$ can be decomposed in terms of coherent states

$$|\psi>= \int_{-\infty}^{+\infty} |\beta><\beta|\psi> d^2\beta \tag{10}$$

Being $d^2\beta$ related to the element of phase volume $\pi^{-1}d^2\beta = 2\pi\hbar^{-1}dpdx$. Using the coherent states one may represent operators of the Hilbert space as a certain class of functions named also as symbols, that determine the operators completely. Any operator $\hat{O}$ can be also decomposed by using the CS system as

$$\hat{O} = \pi^{-2}\int |\alpha> \tilde{O}(\alpha^*,\beta) <\beta|exp(-\frac{|\alpha|^2+|\beta|^2}{2})d^2\alpha d^2\beta \tag{11}$$

with $\tilde{O}(\alpha^*,\beta) =<\alpha|O|\beta>$

Thus, the coherent state could be interpreted as a condition as close as possible to the state of the classical. The quantum oscillator in the sense that the product of uncertainties (dispersion) of the position and momentum in this state receives a minimum possible value within the uncertainty relation. i.e. $\Delta p\Delta x = \frac{\hbar}{2}$.These properties, the their generalizations can be found in the vast literature, for example in (Sudarshan, 1963; Klauder, & Skagerstam, 1985; Zhang & Gilmore, 1990; Klauder, 1993; Ali, Antoine, Gazeau, & Mueller, 1995; Combescure, & Robert, 2012; Perelomov, 2010), and citation therein.

2.1 The Ferromagnet Model of Heisenberg and the Cubic-Quintic Non-Linear Schrödinger Equation

As a first example of using the CS as a tool for nonlinear problem let's consider the Heisenberg ferromagnet model (Makhankov, 1990). The approach will give us the cubic-quintic non-linear Schrödinger equation . Let's consider the Hamiltonian of interaction between the system of spins and the system of phonons as following

$$\hat{H} = \hat{H}_s + \hat{H}_L \qquad and \qquad H_L = T + V$$

$$y \quad \hat{H}_s = -\frac{1}{4}\sum_{t\sigma}\left[J_{ii+\sigma}(\hat{S}^+_i\hat{S}^-_{i+\sigma} + h.c.) - 2\bar{J}_{it+\sigma}\hat{S}^z_i\hat{S}^z_{i+\sigma}\right] - \mu h\sum_t \hat{S}^z_t.$$

Where T and V represent the kinetic and potential energies of the lattice oscillations, respectively. The Hamiltonian is given in terms of the spin operators $\hat{S}$. Besides, the coupling magnitudes $J_{ii+\sigma}\equiv J(|x_{i+\sigma}-x_i|)$ possess the properties of symmetry $J_{ij} = J_{ji}$ and the external magnetic field vector applied to the system is given by $\vec{B} = (0, 0, h)$ being μ the magnetic susceptibility.

We want to go from the quantum representation of Hamiltonian specified by the spin operators $\hat{S}$ to the semi-classical representation, of indexes (classical φ): the indexes are the complex conjugate canonical variables of the classical system phase space. To archive it we should have the Hamiltonian in terms of the creation $a^\dagger$ and annihilation operators a (known as Bose operators) i.e. to bosonize these spin operators, we use the Holstein-Primakov transformation. It's convenient to remember that the base state is given by the choice of the axis oz as the quantization axis.;

$$\hat{S}^+_j = \sqrt{2s - \hat{n}_j}a_j, \quad \hat{S}^-_j = a^+_j\sqrt{2s - \hat{n}_j}, \quad \hat{S}^z_j = s - \hat{n}_j, \quad \hat{n}_j = a^+_j a_j \tag{12}$$

Being s the value of spin. Once we bosonize the Hamiltonian it's possible to employ the coherent states of the Weyl group based on the creation and annihilation operators (8). where $|n >= \sqrt{n!}(a^\dagger)^n|0>$. Averaging the Hamiltonian on study by the coherent states, it's obtained the classical version

$$H = H_0 - \sum_i J_{n+1}[s(\bar{\varphi}_i\varphi_{i+1} + \bar{\varphi}_{i+1}\varphi_i) - \rho s(|\varphi_i|^2 + |\varphi_{i+1}|^2)] -$$

$$-\sum_i \rho J_{n+1}\left[|\varphi_i|^2|\varphi_{i+1}|^2 - \frac{1}{2\varphi}\bar{\varphi}_{i+1}\varphi_i(|\varphi_i|^2| + \varphi_{i+1}|^2)\right] - \mu h\sum_i|\varphi_i|^2$$

where $\rho = \bar{J}/J > 0$. The kinetic energy of the lattice oscillations is defined as $T = \frac{m}{2}\sum_i \dot{x}_i^2$ and the potential energy is V with the anharmonic terms $V = \frac{mu_o^2}{2a_o^2}\sum_i(x_{i+1} - x_i - a_o)^2 + \frac{V_m}{3!}\sum_i(x_{i+1} - x_i - a_o)^3$ and u_o is the speed of sound. With the purpose of separating the anharmonic effects from the generated by the coupling constant we consider a non-linear approach

$$J_{u+\sigma}\cong J_0 - J_1(X_{i+\sigma} - X_i - a_0) + J_2(X_{i+\sigma} - X_i - a_o)^2$$

and

$$J_k = -\partial^k J/(\partial X_i)^k|_{x_i = x_{i-1} - a_0} \qquad k = 1, 2$$

To obtain the continuous version of the model we make

$$\varphi_j = \varphi(ja_0)\equiv\varphi(\chi)$$

where a_0 is the distance between two near points to the lattice and j is the position at the lattice. In the approach of wavelength we have for the net coordinates the expansion

$$x_{i\pm1} = x\pm x_\chi a_0 + \frac{1}{2}x_{\chi\chi}a_o^2\pm\frac{1}{6}x_{\chi\chi\chi}a_o^3 + \frac{1}{4!}x_{\chi\chi\chi\chi}a_o^4 + ...$$

The lattice dynamics will be studied on the "space-time" (χ, t). We neglect terms bigger than $\varphi_{\chi\chi}$ and supposing that φ has the same order such as $x_{\chi\chi}$ then the field generated by the Hamiltonian after several algebraic manipulations acquire the form

$$\begin{aligned} i\varphi_t &= -\alpha\varphi_{\chi\chi} - \bar{\mu}\varphi + gx\chi\varphi + c_1(x\chi)^2 - \lambda|\varphi|^2\varphi \\ mx_{tt} &= cx_{\chi\chi} + c_1(x\chi|\varphi|^2) - g(|\varphi|^2)\chi \end{aligned} \tag{13}$$

with

$$\alpha = \frac{1}{2}J_0 s a_0^2, \quad \tilde{\mu} = s(J_0 - J_1)(1-\rho) - b\mu, \quad g = -J_1 s(\rho - 1)a_0$$

$$c_1 = 4sJ_2(\rho - 1), \quad \lambda = 2J_o(\rho - 1), \quad c = u_o^2, \quad c_1 = 4sJ_2(\rho - 1)$$

on the quasistationary limit $|mx_{tt}| << |cx_{tt}|$ of the system of equations (13) we obtain

$$i\varphi_t = -\alpha\varphi_{\chi\chi} - \tilde{\mu}\varphi - p\frac{|\varphi|^2\varphi}{1 + b|\varphi|^2} \tag{14}$$

with $b = \frac{c_1}{\sqrt{c}}$ y $p = 4[\frac{J_1 s^2(1-\rho)^2}{c} + \frac{1}{2}sJ_o]$ and the terms of superior order were neglected. The last equation is known as Nonlinear Schrödinger equation with nonlinearity of saturation. If we take into account the nonlinearity less than $0(b|\varphi|^2)$ in the latter term of the equation is obtained the conventional non relativistic equation of the φ^6 fields theory.

$$i\varphi_t + \varphi_{\chi\chi} - \mu\varphi + (|\varphi|^2 - \beta|\varphi|^4)\varphi = 0$$

known as the cubic-quintic non-linear Schrödinger equation.

2.2 Dynamics of Kennard Squeezed States and Squeezions

Another example that emphasizes the importance of application of coherent states is deployed below (for more details consult Belyaeva, Ramirez-Medina, & Serkin, 2012). On this work it's employed a generalization of the Ehrenfest theorem, which consists on the calculation the exchange rate of the average coordinate and momentum values of wave packets. Thus the Gaussian wave was taken as a prototype of a coherent state. It could be also studied the linear and non-linear dynamics of the Gaussian wave packets with variation of parameters (amplitude, width and phase) and the external potential of the harmonic oscillator type.

We start with the nonlinear Schrodinger equation but now influenced also by an external potential

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} - R|\Psi|^2\Psi + V(x)\Psi \tag{15}$$

where for the general case we consider that $N_0 = \int_{-\infty}^{\infty}|\Psi|^2 dx$ as a constant, not necessary equal to one. Expressions for conserved quantities are obtained following the method of Ehrenfest, as the Hamiltonian of the system, associated with its average value of the energy $< E >= H_{01}$:

$$H_{01} = \int_{-\infty}^{\infty}\left[\frac{1}{2}\left|\frac{\partial\Psi}{\partial x}\right|^2 - \frac{1}{2}R|\Psi|^4 + V|\Psi|^2\right]dx. \tag{16}$$

The space-time evolution of the first momentum, associated with the path of a non-linear wave packet center of mass is described as following

$$iN_0\frac{d < x >}{dt} = \int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}dx. \tag{17}$$

likewise using the same method we obtain for the second momentum

$$iN_0\frac{d^2 < x >}{dt^2} = 4H_{01} - 2\int_{-\infty}^{\infty}\left[2V + x\frac{\partial V}{\partial x}\right]|\Psi|^2 dx \tag{18}$$

Considering the wave packet of the form

$$\psi = \frac{1}{\sqrt{a(t)\sqrt{\pi}}}exp\left\{-\frac{1}{2a^2(t)}x^2 - i\frac{da/dt}{2a(t)}x^2\right\} \tag{19}$$

With $N_0 = 1$ and the parameter $a(t)$ time dependent. As an external potential it's used the harmonic oscillator potential $V(x) = \frac{1}{2}\omega^2 x^2$. By standard procedure one can obtain the following equation of motion

$$\frac{d^2a}{dt^2} - \frac{1}{a^3} + \omega^2 a = 0 \tag{20}$$

The non-linear differential equation (20) is known as Ermakov Equation. The solution for this equation was introduced by Pinney in 1950 and describes a Kennard state, it means, a squeezed state or a localized wave, the position, and the impulse, width and amplitude from which oscillates periodically.

Kennard discovered the coherent states of an harmonic oscillator in 1927, in turn, this work yield as a result one more class of oscillating wave packets for arbitrary initial widths and localizations of wave functions not limited simply for the shifted base state. These solutions are known as squeezed states. This was a turning point to derivative the time dependent uncertainty relation of Heisenberg for wave oscillation packets states. One of the multiple reasons that make of this work something remarkable is that the Kennard states of the wave packets remain Gaussian any moment, as long as its width and amplitude oscillate. This means, if the initial state has a width bigger than the width of the eigenvalue of the potential, the density of probability represents the periodical compression and the Gaussian wave function can be strongly localized in space. This property of the function implies the name of squeezed states.

The special case for a squeezed state, when $(a/a_0^2 = \omega^2 a_0^2)$, or $\omega = 1/a^2$ correspond to a wave packets that represent the eigenstates of the shifted harmonic oscillator Hamiltonian, for which the uncertainty relation of Heisenberg is minimized. These states for definition are the coherent states.

The coherent states behave like non-linear solitary waves, solutions of the non-linear Schrödinger equation with $V(x)$ and $R = 1$. The interaction of the squeezed packets, as well as the solitons, depend of the initial phases. Two packets with the same initial phases form a maximum in the crashing point, but two packets with opposite initial phases form a minimum on this point.

3. Generalized Coherent States

Quickly it was generalized the scheme depicted above and it were built the *Generalized coherent states* (GCS)with similar properties to the ones of coherent states of the harmonic oscillator. The generalized coherent states for arbitrary quantum systems have been developed by many methods based on different physical and mathematical considerations (Klauder & Sudarshan, 1968; Konstant, 1970; Gilmore, 1974; Zhang & Gilmore, 1990; Perelomov, 2010). The basis of this development for Lie groups, specially accomplished by Perelomov (2010), was to connect closely the coherent states with the group dynamics of each physical problem.

The GCS are particularly useful because of its well-defined algebraic and topological properties (Makhankov, 1990£ Agüero, 1991; Zhang, F., & Gilmore, Perelomov, 2010). For the analysis of collective excitations (magnons) on ground states, one has to go from the quantum level to the classical one, and that transition ought to be made with extremely care. Commonly it's handy to use trial functions (that is, some basis) for averaging the quantum Hamiltonian. The choice of coherent states is because of the similarity with the classical states. The collective excitations in many cases are no more than a some sort of soliton like structures. The theory of solitons constitutes a method for studying great variety of non-linear phenomena in fields theory and major applications in optics, gravitations, fluids, particles, superconductivity, biophysics, hydrodynamics, ferromagnetism, etc. (Makhankov, 1990). The concern is formulating a consistent procedure in reducing statistical quantum models to classical field models connected among them. Because of the complexity depicted in the systems of many particles, the coherent states play a main role. Thus, a problem formulated in a quantum mechanics' language with some group dynamics would be analyzed with the help of coherent states bonded to that group and turn it in a quasi-classical system.

On the other hand, it's natural using the generalized coherent states with spin operators, for which it's constructed the generalized coherent states with the $SU(2)$ group spin operators. Such states, for the arbitrary values of spin j, correspond to points of the space $SU(2j+1)/SU(2j) \otimes U(1)$.

To define and analyze some properties of the generalized coherent states we'll make use of the following strategy. Let G be a Lie Group, g is an element of this group and T its unreduced unitary representation acting on the Hilbert space H. Let's denote $|\Psi>$ to a vector in that space, $<\Phi|\Psi>$ to the scalar multiplication and to the projection operator over Ψ as $|\Psi><\Psi|$. We fix some vector Ψ_0 H. We consider a series of vector $\{|\Psi(g)>\}$, such as $|\Psi(g)>= T(g)|\Psi_0>$ and g traverse all the group G. The vectors that defined the same state (the ones that differentiate on the phase) will be grouped in equivalence classes ($|\Psi(g_1)> \sim |\Psi(g_2)>$). This is possible if $|\Psi(g_1)>= exp(i\alpha)|\Psi(g_2)>$, therefore $T(g_2^{-1} \times g_1)|\Psi(g_0)>= exp(i\alpha)|\Psi(g_1)>$.

Let be $K = \{k\}$ a series of elements of the G group that satisfies $T(k)|\Psi(g_0)>= exp(i\alpha(k))|\Psi(g_0)>$. This set of the stationary subgroup of the vector $|\Psi_0>$. From what we propound above, it's easy to see that the vectors $|\Psi>$ being grouped on the left adjoint class $g_{1t}\epsilon g_1 K$ it will be differentiated one from the other just for the phase. This means, they belong to the same class. From this, we conclude that different vectors (states) correspond to the elements g belonging to the space factor $M = G/K$. In this case, for describing the set of different states is enough to take an element of each class. From the geometric point of view the group g_m is treated as a space of fiber-bundle with the basis $M = G/K$ and the cape K. Then, to the elements g_m will correspond some section of the fiber-bundle (Konstant, 1970). Therefore, the generalized coherent states will be defined by the latter one as Perelomov (2010):

The system of coherent states, that we denote as $(T|\psi_0>)$ (T is the representation of the G group acting on the space H, and $|\psi_0>$ is the fixed vector of this space), is the set of the states $\{|\psi_g>\}$ satisfying $|\psi_g>= T(g_m)|\psi_0>$ where $g_m \epsilon G/K$ and K is a stationary subgroup of the referent group $|\psi_g>$ determined by the point $x = x(g)$ of space G/K corresponding to the element $g : |\psi_g>= e^{\alpha}|x>$, $\psi_0 = |0>$.

We'll now undertake the study of the generalized coherent states based on the most popular Lie groups

3.1 SU(2) Group

The group $G = SU(2)$ is fundamental in many cases of physics, therefore it's necessary to analyze the generalized coherent states constructed based on this group.

The algebra of this group is defined by the generators $S^{\pm} = S^x \pm S^y$, $S^0 = S^z$ with the commutations $[S^z, S^{\pm}] = \pm S^{\pm}$, $[S^-, S^+] = -2S^z$ and the Casimir operator $\hat{C}_1 = S^{z^2} + \frac{1}{2}(S^+S^- + S^-S^+) = S^{z^2} + S^{x^2} + S^{y^2}$ that commutes with S^z. From the Schur's Lemma this operator is proportional to the unitary operator $\hat{C}_1 = j(j+1)\hat{I}$. This implies that the representation of the $SU(2)$ group is characterized by the number j. To the operator of the unreduced representation $T(g)$ of this group can be parametrize as:

$$T(g) = e^{\alpha S_+ - \bar{\alpha} S_- + i\lambda S_z}, \qquad |\alpha| \geq \pi/2, \qquad \lambda \quad \epsilon \quad \Re$$

If now we choose an eigenvector of the operator S^z as a referring vector $|\psi_0>= |0>$, the action of the operator $e^{i\lambda S^z}$ over the vector $|\psi_0>$ carry only the shift of phase for $|\psi_0>$. This means that the elements as $e^{i\lambda S^z}$ produce a stationary subgroup of the vector $|\psi_0>$ that coincides with the $U(1)$ group. Then, the generalized coherent states in our case will be defined on the homogeneous space $SU(2)/U(1)$, over the sphere S^2, it means, over the projective complex space $\mathbf{CP} = S^2$. Now the system of generalized coherent states can be written as:

$$|\psi>= T(g)|\psi_0>= e^{\alpha S^+ - \bar{\alpha} S^-}|0>= (1 + |\psi|^2)^{-j} e^{\alpha S^+}|0>;$$

with

$$\hat{S^{\pm}} = \hat{S^x} + i\hat{S^y}, \quad \psi = \frac{\alpha}{|\alpha|} Tan|\alpha|, \quad |0>= |j, -j>$$

where α and ψ are complex numbers and j define the unitary representation of the $SU(2)$ group. As can be seen, the series of coherent states or trial functions have the symmetry of the sphere.

The vector $|0>= |j, -j>$ with less weight for the $SU(2)$ group, satisfies the condition

$$\Delta\hat{C}_2 = inf\{\Delta\hat{C}_2\} = j$$

The result is linked intuitively with the situation when the vector with less weight is the "vacuum" for the descending operator $S^-|\psi_0>= 0$. A particular case of this Heisenberg-Weyl group. For other quantum cases when the Hamiltonian is given in terms of the Bose operators, then the more natural basis of trial functions is the one formed with these coherent states. In this case the annihilation operator satisfies $a^-|0>= 0$. When the study involves to infinite degrees of freedom, the construction of the generalized coherent states is very similar to the situation depicted above.

For $j = 1$ the generalized coherent states will be the ones whose live on the $SU(2)/U(1)$ space and can be parametrized by the complex function ψ

$$|\psi> = \frac{1}{1+|\psi|^2}\{|0> + \sqrt{2}\psi|1> + \psi^2|2>\}$$

Being $|i>$ $(i = 0, 1, 2)$ pure spin quantum states (*down,middle and up states*, as usual). The components of the classical spin vector will be:

$$\vec{S} = (S^x, S^y, S^z) = <\Psi|\hat{S}|\Psi>$$

and the quadrupole momentum Q^{ij} for any value of j will be:

$$S^+ = \bar{S}^- = 2j\frac{\bar{\psi}}{1+|\psi|^2}$$

$$S^z = -j\frac{1-|\psi|^2}{1+|\psi|^2}$$

$$Q^{zz} = \frac{j^2(1-|\psi|^2) + 2j|\psi|^2}{(1+|\psi|^2)^2}$$

But as the spin operators commute in different places of cell, then the correlation of S^iS^j will have the form

$$<\psi|\hat{S^i_m}\hat{S^j_{m+1}}|\psi> = <\psi|\hat{S^i_m}|\psi><\psi|\hat{S^j_{m+1}}|\psi>$$

where $|\psi> = |\psi>_m |\psi>_{m+1}$

3.2 SU(1,1) Group

The algebra of this group is defined for the commutation relations $[K^z, K^\pm] = \pm K^\pm$, $[K^-, K^+] = 2K^z$ and, in addition, the Casimir operator

$$\hat{C}_2 = (K^z)^2 - \frac{1}{2}(K^+K^- + K^-K^+) = (K^z)^2 - (K^x)^2 - (K^y)^2$$

commute with all the operators K. Again, according to the Schur's Lemma for the irreducible representations, this Casimir operator is a multiple of the unity $\hat{C}_2 = k(k-1)\hat{I}$.

Let's see the following parametrization of the group $SU(1,1)$, $T_2(g) = exp(\alpha K^+ - \bar{\alpha}K^- + i\lambda K^z)$. Now if we choose to $|\psi_0>$ as an eigenvector of the operator K^2, then the action of the operator $e^{i\lambda K^z}$ over the vector $|\psi_0>$ simply involves to a shift of phase, it means, the elements $b = e^{i\lambda K^z}$ form the stationary group of the vector $|\psi_0>$.

As can be seen, the stationary group $g = e^{i\lambda S^z}$ and the group $b = e^{i\lambda K^z}$ coincide with the $U(1)$ group; therefore, in the two cases analyzed above, the generalized coherent states will be defined on the homogeneous spaces $SU(2)/U(1)$, $SU(1,1)/U(1)$ respectively, it means, over the sphere S^2 on the first case and the pseudosphere $S^{1,1}$ in the second one. The system of generalized coherent states will be easily constituted as

$$|\alpha>_2 = D_2(\alpha)|\psi_0>_2 = e^{\alpha K^+ - \bar{\alpha}K^-}|\psi_0>_2$$

The latter discussion will be distinct for the two cases. Thus, the states $|j,\mu>$ with a spin projection over the axis basis X_3 provide the basis on the irreducible unitary representation of the space $T(g)$ of the $SU(2)$ group. It's very a special and simple system of coherent states formed by means of the referring vector $|\psi>_1 = |j, -j>$, it means $\mu = -j$, such as $S^-|\psi_0> = 0$.

In the second case, the irreducible unitary representation of the $SU(1,1)$ group have the fundamental series, two discrete series $T^{(+)}$, $T^{(-)}$ and an additional one. Therefore, it can be constructed a number of systems of coherent states related with this series. The most important one could be the system related with the discrete series that can be generalized by means of the creation and annihilation operators. It's enough to consider only one of the two series, for instance $T^{(-)}$, because all the results can be easily translated to $T^{(+)}$. Thus, $K_0|k,\mu>=\mu|k,\mu>$. We can choose to the vector $|\psi_0>_2$ to the form

$$|\psi_0>_2=|k,k>$$

and we have $K|\psi_0>_2=0$. Finally the coherent states constructed with the latter vectors have the form

$$|\xi>_2=(1-|\xi|^2)^2 e^{\xi K^+}$$

those will be coherent states of pseudo-spin.

3.3 Properties

The generalized coherent states have important properties analogous to the properties of the Heisenberg-Weyl coherent states. Thus,

1. The operators $T(g)$ transfer a coherent state to another one.

2. The generalized coherent states are complements (more precisely) are over-complete.

3. The generalized coherent states are not orthogonal among itself.

$$<\xi'|\xi>_1=[(1+|\xi'|^2)(1+|\xi|^2)]^{-j}((1+\xi\bar{\xi'})^{2j}$$
$$<\xi'|\xi>_2=[(1-|\xi'|^2)(1-|\xi|^2)]^{k}((1-\xi\bar{\xi'})^{-2k}$$

   where the first one corresponds to the sphere S^2, and the second one to the pseudosphere $S^{1,1}$.

4. In addition, the coherent states minimize the scattering:

$$\Delta C_2=<\psi_g|C_2|\psi_g>-g^{ik}<\psi_g|x_j|\psi_g><\psi_g|x_k|\psi_g>$$

   with $C_2=g^{ik}x_ix_j$ that it's the quadratic Casimir operator, x_i are the generators of the Lie algebra and g^{ik} is the Cartan-Killing metric tensor.

   The analogous relation to the latter one for the special case of the common coherent states based on the Weyl algebra, is the relation of the Heisenberg uncertainty, obtaining $\Delta p\Delta x=\frac{1}{2}$

5. In the limit of large values of j (or k) the generalized coherent states tend to the bosonic states. The demonstration of this is by means of the substitution

$$S^+,\quad (K^+)\rightarrow\sqrt{2j}a^+,\quad \xi\rightarrow\frac{\alpha}{\sqrt{2j}},(j\rightarrow k)$$

   Then, it's assumed that $j,(k)\rightarrow\infty$. Therefore the coherent states of the these Lie groups can be written as

$$|\xi>_1=\lim_{j\to\infty}\left[1+\frac{|\alpha|^2}{2j}\right]^{-j}exp(\alpha a^\dagger)|\psi>_1=exp\left[-\frac{1}{2}|\alpha|^2\right]exp(\alpha a^\dagger)|\psi_0>$$

$$|\xi>_2=\lim_{k\to\infty}\left[1-\frac{|\alpha|^2}{2k}\right]^{k}exp(\alpha a^\dagger)|\psi>_2=exp\left[-\frac{1}{2}|\alpha|^2\right]exp(\alpha a^\dagger)|\psi_0>$$

   For other groups the generalized coherent states are constructed using the fundamental representation

$$|\psi>=\left\{\sum_i^{2j}(\xi\hat{T_i^\dagger}-\bar{\xi}_i\hat{T_i^-})\right\}|0>=\left[1+\sum_i^{2j}|\psi_i|^2\right]\left\{|0>+\sum_i^{2j}\psi_i|i>\right\}\qquad(21)$$

   where $\hat{T_i^\dagger}$ y $\hat{T_i^-}$ are the generators of the $SU(2j+1)$ group in the fundamental representation and

$$\psi_i = \frac{\xi_i}{|\xi|} tan|\xi| \qquad |\xi| = \sqrt{\sum_i^{2j} |\xi_i|^2}$$

$$|0 >= (0, ..., 0, 1)^{2j} \qquad |j >= (0, ..., 0, 1, 0, ..., 0)^{2j}$$

In the case of $\mathbf{CP}^2$, the Lie group to study is $G = SU(3)$, the space will be $G/H = SU(3)/SU(2)\otimes U(1)$ then we have

$$|\psi >= (1 + |\psi_1|^2 + |\psi_2|^2)^{\frac{1}{2}} \{|0 > +\psi_1|1 > +\psi_2|2 >\} \tag{22}$$

Something will be remarked of the latter examples of generalized coherent states defined over the sphere S^2 in the first case and by the pseudosphere $S^{1,1}$ in the second one.

4. Applications of the Generalized Coherent States

4.1 The Anti-Ferromagnetic Case of the Heisenberg Theory

The Heisenberg model provides the foundation for the theoretical study of a wide class of ferromagnetic (and anti-ferromagnetic) phenomena for the quantum level. In the last years the study of non-linear spin excitations in terms of solitary waves in the Heisenberg model of ferromagnetism with different magnetic interactions in the classical and semi-classical limits under the linear approach has attracted a lot of interest. Because of the low temperatures phenomenon of ferromagnetism have a microscopic character, it's possible to give a classical or semiclassical description of its behavior. It should be formulated a procedure for reducing quantum statistical models of classical field connected between themselves. A known example is the one-dimensional model of Hubbard corresponding to the 2-components of the chain of Heisenberg spin with inter-component interaction. Generalizing the model of Hubbard we obtain the multicomponent spin chain, which can be used to describe collective excitations, in addition of the statistical properties on systems that have distinct classes of spin. Because of the complexity that systems of many particles present it is introduced the concept of coherent states, and so forth we deal with the problem of formulating in the language of quantum mechanics to a semiclassical problem.

For the antiferromagnetic case we take into consideration the following Hamiltonian

$$H = +J \sum_n (S_n S_{n+1} + \delta S_n^z S_{n+1}^z) \tag{23}$$

where S_n^x, S_n^y, S_n^z are the spin operators acting on the n-site, and δ is the anisotropy coefficient. Directly applying the generalized coherent states to the previous Hamiltonian we get excitations with negative energies, it will produce instability on the system, in other words, the quantum vacuum over whose we construct excitations will turn unstable (Makhankov, Agüero, & Makhankov, 1996). Apparently, this is the reason for seeking the vacuum in the antiferromagnetic case, a very complex problem. To avoid difficulties excitations will be proved with positive energy, in accordance with Konstant (1970) and Perelomov (2010) we use the following procedure to overwrite the Hamiltonian by means of terms of the $SU(1, 1)$ group operators.

$$K^{\pm} = iS^{\pm} \qquad K^z = S^z$$

to get te representation of pseudo-spin for the antiferromagnet:

$$H = -J \sum_n \left[\frac{1}{2}(K_n^+ K_{n+1}^- + H.C.) - K_n^z K_{n+1}^z (1 + \delta) \right]$$

Applying the scheme of averaging as it has been done before with respect to the quantum hamiltonian and using generalized coherent states $\mathbf{L}^{1,1}$ to get the classical lattices model.

$$H = -J \sum_n \frac{2(\xi_n^* \xi_{n+1} + \xi_n \xi_{n+1}^*) - (1 + \delta)(1 + |\xi_n|^2)(1 + |\xi_{n+1}|^2)}{(1 - |\xi_n|^2)(1 - |\xi_{n-1}|^2)}$$

In the continuous limit we have

$$H = -J \sum_n K^2 + \frac{a_o}{2} J \int dx (K_x K_x + \rho K^z K^z)$$
$$= J K^2 N + \frac{a_o}{2} J \int dx (K_x K_x + \rho K^z K^z)$$

for the representation of the σ-model, or

$$H = const. + 2k^2 a_o J \int dx \frac{|\xi_x|^2 + \rho |\xi|^2}{(1 - |\xi|^2)^2}$$

for the stereographic projection.

Thus, we avoided the problem of excitations with negative energy and instead we dealt with the problems of non-compact groups and manifolds (σ-model representation) or singular expression (stereographic projection). For the antiferromagnetic case in the continuous approach, we applied the procedure of expanding in Taylor series the function $\psi_{n\pm1}$; we suppose that $\lambda_{a_o} << 1$ with $\alpha = a_o n$. Then, for the case $\mathbf{S}^{1,1}$ the equation of motion is

$$i\xi + \Delta\xi + 2\frac{(\nabla\xi)^2\xi^*}{1 - |\xi|^2} + \delta\frac{1 + |\xi|^2}{1 - |\xi|^2} = 0$$

It's considered this way of writing the equation of motion as the stereographic projection of the non-compact Landau-Lifshitz model defined on the hyperboloid $\mathbf{S}^{1,1}$. In particular, for $\delta > 0$, this last equation is equivalent to the repulsive cubic non-linear Schrödinger equation. The σ-model of this equation yields the right quasi-classical description of the Bogolyubov condensation, foretell the coupling between the anti-ferromagnetism and the superfluids.

4.2 The Hubbard Model and the Superconductivity

Recently the interest for super-symmetrical models increased, by one hand those related with the superstring theory and the superconformals and, by the other hand, they're bonded with theories of the atomic nuclei. The supersymmetry in condensate media theories is used in the disordered metals and the superconductivity (Nambu, 1985). The theoretical physicists for the model of Hubbard in the strong correlation electrons domain (Anderson, 1987) has been revitalized due to the discovery of superconductivity. In the atomic representation, the phase space of the model on each place of the crystalline cell is determined by four vectors. Two states have an odd number of particles: $|0>$ determines the vacancy and the state $|2>= \hat{c}^+_\uparrow \hat{c}^+_\downarrow |0>$ is the state of two particles. Two states of particles $|\uparrow>= \hat{c}^+_\uparrow$ with spin-up and $|\downarrow>= \hat{c}^+_\downarrow$ with spin down. Here $c\sigma^+$ y $c\sigma$ (where $\sigma = \uparrow, \downarrow$) are the fermionic creation and annihilation operators that satisfy

$$\{c_\sigma, c^+_{\sigma'}\} = \delta_{\sigma\sigma'}$$
$$\{c_\sigma, c_{\sigma'}\} = \{c^+_\sigma, c^+_{\sigma'}\}$$

and $\hat{c}_\sigma|0>= 0$. The Hubbard operators $\hat{X}^{pq}_i = |i, p>< q, i|$ defined in these states instead i generate a graduated Lie algebra $pl(2/2)$

$$\{\hat{X}^{pq}_i, \hat{X}^{nm}_j\} = \delta_{ij}(\hat{X}^{pm}_i \delta_{qn} - \hat{X}^{nq}_j \delta_{mq})$$
$$[\hat{X}^{pq}_i, \hat{X}^{nm}_j] = \delta_{ij}(\hat{X}^{pm}_i \delta_{qn} - \hat{X}^{nq}_i \delta_{mp})$$

It's used the proper anticommutators for two fermionic operators changing the number of electrons inside of the site of the even number. In the case of having just one bosonic operator (we do change the number of electrons for an odd number) this suffices to use the commutator. In this regime of the strong repulsion ($U \to \infty$) the occupation of double state is neglected and the corresponding superalgebra of operators $\hat{X}^{pq}_i$ is reduced to the $pl(2/1)$ algebra. The Hubbard operators on the site i generate a complete system

$$\sum_{p=0,\uparrow,\downarrow} \hat{X}^{pq}_i = I$$

on such a way that any operator $\hat{A}_i$ can be represented in terms of its own linear combination

$$\hat{A}_i = \sum_{p,q} < p, i|\hat{A}_i|i, q > \hat{X}_i^{p,q}$$

Particularly, for the fermionic operators and the operator of number of particles (Hubbard operators) we have

$$\hat{c}_\uparrow = \hat{X}^{0\uparrow} + \hat{X}^{\downarrow 2}, \quad \hat{c}_\uparrow^+ = \hat{X}^{\uparrow 0} + \hat{X}^{2\downarrow}$$
$$\hat{c}_\downarrow = \hat{X}^{0\downarrow} - \hat{X}^{\uparrow 2}, \quad \hat{c}_\downarrow^+ = \hat{X}^{\downarrow 0} + \hat{X}^{2\uparrow}$$
$$\hat{n}_\uparrow = \hat{c}_\uparrow^+ \hat{c}_\downarrow = \hat{X}^{\uparrow\uparrow} + \hat{X}^{22}, \quad \hat{b}_\downarrow = \hat{c}_\downarrow^+ \hat{c}_\downarrow = \hat{X}^{\downarrow\downarrow} + \hat{X}^{22}$$

The Hubbard Hamiltonian can be written then as

$$H = \sum_{i,j} \sum_{\sigma=\uparrow,\downarrow} t_{ij} \hat{c}_y^+ \hat{c}_{j\alpha} + \sum_{i,j} U_{ij} \hat{n}_{i\uparrow} \hat{n}_{j\uparrow}$$

and consists of two parts: the first one describes the transition of the electron from the place j to the place i of the net, where t_{ij} is the coefficient of transition; the second one describes the repulsion of electrons in the sites i, j (on the original work, Hubbard just considered correlations for just one site, it means $U_{ij} = U\delta_{ij}$, and for the Pauli's exclusion principle just are allowed combinations of spin components with opposite directions).

Then, in terms of Hubbard operators the Hamiltonian will take the following form

$$H = \sum_{i,j} \sum_{p,q,n,m} t_{ij} g_{pq,nm} \hat{X}_i^{pq} \hat{X}_j^{nm} + U_{ij} h_{pq,nm} \hat{X}_i^{pq} \hat{X}_j^{nm}$$

It's clear that the Hubbard Hamiltonian, written in terms of operators $\hat{X}_i^{pq}$, has the same form of a generalized Heisenbeg model

$$H = \sum_{i,j} \sum_{\alpha,\beta} \mathfrak{F}_{ij} g_{\alpha\beta} \hat{S}_i^\alpha \hat{S}_j^\beta$$

with the superalgebra $pl(2, 2)$. In the strong correlation of the electron, when $U >> 1$, the Hamiltonian of the model acquires the form

$$H = \sum_{ij} \{t_{ij}(\hat{X}_i^{\uparrow 0} \hat{X}_j^{\uparrow 0} + \hat{X}_i^{\downarrow 0} \hat{X}_j^{\downarrow 0}) + U_{ij} \hat{X}_i^{\uparrow\uparrow} \hat{X}_j^{\downarrow\downarrow}\}$$

and when $U_{ij} = U\delta_{ij}$, $t_{ij} = t\delta_{ij}$

$$H = t \sum_i (\hat{X}_i^{\uparrow 0} \hat{X}_{i+1}^{0\uparrow} + \hat{X}_i^{\downarrow 0} \hat{X}_{i+1}^{0\downarrow})$$

takes the form of the generalized Heisenberg model with the superalgebra $spl(2/1)$. This fundamental representation is tridimensional and it describes the super-conterparts - holes and excitations of spin. Then, the effective Hamiltonian with strong correlation can be considered as a supergeneralization of magnetic Hamiltonians.

In these cases it results more convenient the use of coherent states based on the Grassman algebra, again averaging

$$|\Psi >= \prod_i |\psi_1, \psi_2 >_i$$

where

$$|\psi_1, \psi_2 >= exp\left\{\frac{1}{2}(|\psi_1|^2 + |\psi_2|^2)\right\} exp(\psi_1 c_\uparrow^+ + \psi_2 c_\downarrow^+)|0 >$$

where $|0>$ is the ferromagnetic vacuum. The model of Hubbard generates in the continuous limit the Grossman odd type classical Hamiltonian $U(2)$ of the non-linear Schrödinger equation. This means that we can consider the classical model $SU(2/1)/S(U(2)\otimes U(1))$ of Heisenberg, being an equivalent norm to the classical limit of the model of Hubbard expressed in terms of generators of the supergroup generators. The fact of the existence of another model of the non-linear Schrödinger equation and its analogous norm to $SU(2/1)/S(L(1,1)\otimes U(1))$ of Heisenberg model implies the existence of a new phase in the quantum initial Hamiltonian. This relative phase to a quantum average of the Bose condensation type is depicted by the constant field $\varphi(x,t)$ in the version of the non-linear Schrödinger equation and for $S^{(0)}$ in the magnetic version, too similar to the classical limit of the high temperature superconductivity Hamiltonian.

$$H = -\sum_{ij} t_{ij}\alpha_i^{+}\alpha_j\left\{S_i^{+}S_j^{-} + (S_i^{z}+\frac{1}{2})(S_j^{z}+\frac{1}{2})\right\}$$

The $OSPU(1,1/1)$ non-compact magnet can be used if the base state of system is antiferromagnetic type, then the version of the non-linear Schrödinger equation is a super-extension of its repulsive version and pretend to describe the antiferromagnetic one. The last models admit generalizations involving $SU(N/M)$ superalgebras and bidimensional models, such as the model of Ishimori (Makhankov, & Pashev, 1992).

4.3 Generalized Coherent States and the Representation of Bose Einstein Condensation

In the next example we provide the brief explanation of how the generalized coherent states should be used as a quantum state of the many body problem in the Bose - Einstein condensation. We will follow the good and explicative work done by Chernyak et al. The main idea in his work was to show that the quantum many body state of Bose-Einstein condensates(BEC) consistent with the time-dependent Hartree-Fock-Bogoliubov (TDHFB) equations is a generalized coherent state (GCS). As is common to justify, by generally speaking the Hamiltonian for interacting many-body bosons should take the form:

$$\hat{H}_{=}\sum_{ij}\hat{H}_{ij}\hat{a}_i^{\dagger}\hat{a}_j + \sum_{ijkl}V_{ijkl}\hat{a}_i^{\dagger}\hat{a}_j^{\dagger}\hat{a}_k\hat{a}_l. \tag{24}$$

At zero temperature, the unnormalized generalized coherent states that belong to such extended Heisenberg-Weyl algebra for interacting bosons have the form:

$$|\psi(\tau)\rangle = \exp\left(\sum_i \alpha_i(\tau)\hat{a}_i^{\dagger} + \sum_{i,j}\beta_{ij}(\tau)\hat{a}_i^{\dagger}\hat{a}_j^{\dagger}\right)|\Omega_0\rangle. \tag{25}$$

Again we deal here with the referent vector sate $|\Omega_0\rangle$

The key ideas for using coherent states consists in the following approach. For obtaining the variational equations at zero temperature is necessary to take the Hamiltonian $\hat{H}$, and the time-dependetn wave functions $|\Omega(\tau)\rangle$, and minimize the action

$$S[\Omega(\tau)] = \int d\tau\left[i\langle\Omega(\tau)|d\Omega(\tau)/d\tau\rangle - \langle\Omega(\tau)|\hat{H}|\Omega(\tau)\rangle\right]. \tag{26}$$

By choosing a GCS form for $|\Omega(\tau)\rangle$, the resulting variational equations can be written in the Hamiltonian form for any set Ω_j of coordinates which parametrize $|\Omega\rangle$:

$$\frac{d\Omega_j}{d\tau} = \{\mathcal{H}, \Omega_j\} \tag{27}$$

where $\{\cdots\}$ denote Poisson brackets and $\mathcal{H}$ is the classical Hamiltonian defined by:

$$\mathcal{H}(\Omega) = \langle\Omega|\hat{H}|\Omega\rangle. \tag{28}$$

This scheme can be applied for instance to the well known problem variational equation of motion in the trap basis . The Hamiltonian for this case takes the form

$$H = \sum_{ij}H_{ij}\hat{a}_i^{\dagger}\hat{a}_j + \sum_{ijkl}V_{ijkl}\hat{a}_i^{\dagger}\hat{a}_j^{\dagger}\hat{a}_k\hat{a}_l + \sum_{ij}E_{ij}a_i^{\dagger}\hat{a}_j. \tag{29}$$

The matrix elements of the single particle Hamiltonian H_{ij} are given by

$$H_{ij} = \int d^3\mathbf{r}\,\phi_i^*(\mathbf{r})\left[-\frac{\hbar^2}{2m}\Delta + V_{\text{trap}}(\mathbf{r})\right]\phi_j(\mathbf{r}), \tag{30}$$

A convenient basis for trapped BEC is the eigenstates for which the H_{ij} is diagonal. The indices may also be viewed as the mode indices in a multimode quantum state. The symmetrized two particle interaction matrix elements are

$$V_{ijkl} = \frac{1}{2}\Big[\langle ij|V|kl\rangle + \langle ji|V|kl\rangle\Big], \tag{31}$$

where

$$\langle ij|V|kl\rangle = \int d^3\mathbf{r}\,d^3\mathbf{r}'\,\phi_i^*(\mathbf{r})\phi_j^*(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\phi_k(\mathbf{r}')\phi_l(\mathbf{r}), \tag{32}$$

with $V(\mathbf{r}-\mathbf{r}')$ being a arbitrary interatomic potential. We have also

$$E_{ij} \equiv \int d\mathbf{r}\phi_i^*(\mathbf{r})V_f(\mathbf{r},t)\phi_j(\mathbf{r}), \tag{33}$$

where $V_f(\mathbf{r},t)$ denotes a general time- and position-dependent external driving potential. The generalized coherent sates at zero temperature is a Gaussian state in coordinate space.

The TDHFB equations in real space may be transformed to the corresponding trap basis by using the following relations between the real space basis and the trap basis variables:

$$z(\mathbf{r}) = \sum_i z_i\phi_i(\mathbf{r}), \quad \rho(\mathbf{r},\mathbf{r}') = \sum_{ij}\rho_{ij}\phi_i^*(\mathbf{r})\phi_j(\mathbf{r}'), \quad \kappa(\mathbf{r},\mathbf{r}') = \sum_{ij}\kappa_{ij}\phi_i(\mathbf{r})\phi_j(\mathbf{r}'), \tag{34}$$

along with the definition of the tetradic matrix V_{ijkl}

$$V_{ijkl} = \frac{1}{2}\Big[\langle ij|V|kl\rangle + \langle ji|V|kl\rangle\Big], \tag{35}$$

with

$$\langle ij|V|kl\rangle = \int d^3\mathbf{r}\,d^3\mathbf{r}'\,\phi_i^*(\mathbf{r})\phi_j^*(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\phi_k(\mathbf{r}')\phi_l(\mathbf{r}), \tag{36}$$

where $V(\mathbf{r}-\mathbf{r}')$ being a general interatomic potential.The TDHFB equations take the form Chernyak et al.

$$\begin{aligned}
i\hbar\frac{dz}{dt} &= \mathcal{H}_z z + \mathcal{H}_{z*}z^* + Ez && (37)\\
i\hbar\frac{d\kappa}{dt} &= (h\kappa + \kappa h^*) + (\left[\sqrt{\frac{1}{4}I + \kappa^\dagger\kappa} - \frac{1}{2}I + \delta\rho\right]\Delta + \Delta\left[\sqrt{\frac{1}{4}I + \kappa^\dagger\kappa} - \frac{1}{2}I + \delta\rho\right] + \Delta, && (38)\\
i\hbar\frac{d\delta\rho}{dt} &= [h, \sqrt{\frac{1}{4}I + \kappa^\dagger\kappa} - \frac{1}{2}I + \delta\rho] - (\kappa\Delta^* - \Delta\kappa^*) - i\hbar\frac{1}{\sqrt{I + 4\kappa^\dagger\kappa}}\left(\frac{d\kappa^\dagger}{dt}\kappa + \kappa^\dagger\frac{d\kappa}{dt}\right) && (39)
\end{aligned}$$

where

$$\begin{aligned}
[\mathcal{H}_z]_{ij} &= H_{ij} + \sum_{kl}V_{iklj}\left[z_k^*z_l + 2\sqrt{\frac{1}{4}\delta_{lk} + \sum_m \kappa_{lm}^*\kappa_{mk}} - \delta_{lk} + 2\delta\rho_{lk}\right] && (40)\\
[\mathcal{H}_{z*}]_{ij} &= \sum_{kl}V_{iklj}\kappa_{kl} && (41)\\
h_{ij} &= H_{ij} + 2\sum_{kl}V_{iklj}\left[z_k^*z_l + \sqrt{\frac{1}{4}\delta_{lk} + \sum_m \kappa_{lm}^*\kappa_{mk}} - \frac{1}{2}\delta_{lk} + \delta\rho_{lk}\right] + E_{ij} && (42)\\
\Delta_{ij} &= \sum_{kl}V_{ijkl}\left[z_kz_l + \kappa_{kl}\right]. && (43)\\
& && (44)
\end{aligned}$$

At $T = 0$, $\delta\rho = 0$ and $\rho = \sqrt{\frac{1}{4}I + \kappa^\dagger\kappa} - \frac{1}{2}I$. By direct differentiation of ρ

$$i\hbar\frac{d\rho}{dt} = i\hbar\frac{1}{\sqrt{I + 4\kappa^\dagger\kappa}}\left(\frac{d\kappa^\dagger}{dt}\kappa + \kappa^\dagger\frac{d\kappa}{dt}\right), \tag{45}$$

Finally the famous Gross - Pitaevsky Equation (GPE), which is a zero temperature theory for a coherent state ansatz, could be simply obtained from Eq. (37) by setting $\rho_{ij} = \kappa_{ij} = 0$:

$$i\hbar\frac{dz_i}{dt} = \sum_j\left[H_{ij} + \sum_{kl}V_{iklj}z_k^*z_l + E_{ij}\right]z_j. \tag{46}$$

4.4 Generalized Coherent States for the DNA-Protein Interaction Dynamics

As the last example we expose the approach on how the generalized coherent states could help us to understand the internal dynamics of DNA interacting with a protein. In this case, the base pairs of DNA are represented as classical spins in a coupled system of spins or spin ladder, and the protein amino acids as point masses on a linear chain.

First we briefly summarize the model of Takeno-Homma (Takeno & Homma, 1983, 1984). Considering the DNA model of Watson and Crick with the helical axis taken on the z-direction, Takeno and Homma considered the B-form of DNA.

Three fundamental considerations were employed for studying the structure and dynamics of DNA:

1. The essential points can be obtained solely paying attention to the bases of the double strands..
2. The fluctuations of the bases positions are given through their rotational motion around the points where they're attached to the strands.
3. The fluctuations give rise, under certain circumstances, to breaking of hydrogen bonding of the bases, inducing the unwinding of the double strands to which they're attached.

Further we will follow the main results obtained in the paper (Agüero, Belyaeva, & Serkin, 2011). For constructing the Hamiltonian it is used the analogy with the spin model Hamiltonian discussed before for ferromagnetism. Thus, the whole Hamiltonian will be constructed taking into consideration the following parts . The first one is the pure quasi-spin contribution H_1

$$H_1 = \sum_n[J(\mathbf{S}_n\cdot\mathbf{S}_{n+1} + \mathbf{S}'_n\cdot\mathbf{S}'_{n+1}) + \mu(\mathbf{S}_n\cdot\mathbf{S}'_n)]$$

that are wrote in terms of the quasi-spin operator $S_n = (S_n^x, S_n^x, S_n^x)$ with

$$S_n^x = sin\theta_n cos\varphi_n, \quad S_n^y = sin\theta_n sin\varphi_n, \quad S_n^z = cos\theta_n$$

The double helical chain of DNA is represented in a model of two coupled quasi spin chains. The proportional terms to J correspond to the stacking interaction between the nth base and their nearest neighboring interaction of two strands, and the last term corresponds to inter strand interaction or the hydrogen bonding energy between the complementary strands. On equilibrium, the parameter μ is expected to be less than zero.

Also, as the functions of DNA change under biological temperatures, it's necessary to include in the Hamiltonian the phonons contribution and the coupling between the oscillation of the hydrogen atom due to the thermal fluctuation and the rotation of bases.

$$H_2 = \left[\frac{p_n^2}{2m_1} + k_1(X_n - X_{n+1})^2 + \alpha_1(X_{n+1} - X_{n-1})(\mathbf{S}_n\cdot\mathbf{S}'_n)\right]$$

Here X_n denotes the displacement of the bases along the hydrogen bonding in the n site and p_n represents the momentum of displacement.

Likewise when the DNA is interacting with other molecule as a protein, it could be added a new contribution to the Hamiltonian

$$H_3 = \left[\frac{q_n^2}{2m_2} + k_2(Y_n - Y_{n+1})^2 + \alpha_2(Y_{n+1} - Y_{n-1})(S^z_n S^z_n)\right]$$

and y_n denotes the displacement of the nth peptide group in this protein molecule. The total Hamiltonian is given by the sum of all the contributions

$$H = H_1 + H_2 + H_3 \tag{47}$$

Making an analogous procedure as before we can use the coherent state of the $SU(2)/U(1)$ group to pass from the pseudo-quantum case to the classical case. An observation, for many too obvious but very remarkable to study the behavior of DNA under average physiological temperatures for living beings (300 K, or a little more for the humans) most of the degrees of freedom are as though they were frizzed due to the considerations of the quantum mechanics (the excitation energy will be higher), and ,therefore, a lot of bonds between the atoms can be considered as rigid; on the other hand, there are degrees of freedom for which the excitation energy is much lower that in the thermal scale, and they can be considered with a classical behavior.

Moving forth, we recall that the coherent states in the $SU(2)$ group have the form

$$|\psi_j >= (1 + |\psi_j|^2)^{-S} e^{\psi_j \hat{S}^\dagger_j} |S, -S >_j$$

And considering the fact that the spin operators $\hat{S}^\dagger_j$ commute on neighboring sites of the DNA strands, the coherent state for the entire lattice is the direct product of the generalized coherent states taken in different sites

$$|\psi >= \prod_j |\psi >_j; \quad j = 1, 2, 3, ...N$$

we have for the averaged values of spin

$$< \psi|\hat{S}^\dagger_j \hat{S}^\dagger_{j+1}|\psi >=< \psi|\hat{S}^\dagger_j|\psi >< \psi|\hat{S}^\dagger_{j+1}|\psi >$$

The averaged values of the quasi-spin operators $\mathbf{S} = (S^x, S^y, S^z)$ employing the coherent states of the $SU(2)$ group can be written in the following ways of their stereographic projections, those will be used to average the lattice of Hamiltonian. Because of the length of excitations in DNA is much greater than the internal distance a of the sites between neighboring nucleotides, we can make an approximation to the continuous limit. Introducing the fields $X_n \rightarrow X(z, t)$, $Y_n \rightarrow y(z, t)$ with $z = na$ and making standard approximations and applying the forms of the stereographic projections one can obtain a new classical Hamiltonian

$$\mathbf{H} = \int \Big\{ \frac{aJ}{2}\Big(\frac{|\psi_z|^2}{1 + |\psi|^2} + \frac{|\xi_z|^2}{1 + |\xi|^2}\Big) - \frac{1}{4a}(\mu - 2\alpha_1 \alpha X_z)\Big(\frac{2(\psi\bar{\xi} + \bar{\psi}\xi) + (1 - |\xi|^2)(1 - |\psi|^2)}{(1 + |\xi|^2)(1 + |\psi|^2)}\Big)$$

$$+ \frac{\alpha_2}{2}\frac{(1 - |\xi|^2)(1 - |\psi|^2)}{(1 + |\xi|^2)(1 + |\psi|^2)} y_z + \frac{p^2}{2am_1} + \frac{q^2}{2am_2} + k_1 a(X_z)^2 + k_2 a(y_z)^2 \Big\} dz + const$$

Using the Hamilton equation of motion $\dot{X} = \frac{\partial H}{\partial p}$ and $\dot{y} = \frac{\partial H}{\partial q}$ and their canonical conjugate counterparts we can find the equations of motion for $X_{X(z,t)}$ and $y_{X(z,t)}$. The equation of motion for $\psi_{z,t}$ and $\xi_{x,t}$ will have a common classical form

$$i\xi_t = -(1 + |\xi|^2)^2 \frac{\delta \mathbf{H}}{\delta \bar{\xi}}$$

The same sort of equations are built for the second field variable ψ. By solving the system of equations finally it is obtained the Generalized Nonlinear Schrodinger Equation of saturable type

$$i\psi_t = -\frac{Ja}{2}\psi_{zz} + aJ\frac{\psi_z^2\bar{\psi}}{1+|\psi|^2} + \tag{48}$$

$$+\left(\frac{2\mu}{a} - \beta\frac{1+|\xi|^4-6|\psi|^2}{(1+|\psi|^2)^2} - \gamma\left(\frac{1-|\psi|^2}{1+|\psi|^2}\right)^2\right)\frac{1-|\psi|^2}{1+|\psi|^2}\psi \tag{49}$$

with the following parameter values $\beta = \frac{2\alpha_1^2}{m_1 v_1^2 - 2ak_1}$, $\gamma = \frac{\alpha_2^2}{2(m_2 v_2^2 - 2ak_2)}$

It's useful to emphasize that in the study of the behavior of a lineal molecule (in this case the DNA) is convenient introduce the enzymatic effect. Due to this cause topological or geometric consequences exist, as long as the molecule tips aren't free. This approach of studying how the enzymes act on the DNA where the topology and the geometry make a contribution is known in literature as *topological approach to enzymology*. Because of the non-existence of an direct observational method for studying the local action of an enzyme -neither the local DNA configuration nor its secundary structure are observable- we have to look for some indirect method. The global topology and geometry of the molecule make a contribution, detecting the change these provoke. These changes are appreciated -by means of an electronic microscope- at upper level to the secundary structure and lead to *winding* and *knotting* on the molecule central axis.

5. Conclusion

We devoted our attention to coherent states associated with groups, specially we concentrate our review to two types of coherent states the first one are based on the Heisenberg Weyl group and the second on SU(2) lie group. The examples depicted are meant to draw throughout this work one of the major features of these quantum states: that is, we're able to go from a quantum system to a classical one and vice-versa. Going from the description of Kennard squeezed states to the Hubbard model, from the cubic-quintic non-linear Schrödinger equation to DNA-protein interaction dynamics, we only caught a glimpse of the paramount importance of these states in modern physics. Using the GCS ansatz, it has been derived variationally the TDHFB equations equations of motion for BEC, which are known to be valid in the collisionless regime. This implies that the GCS ansatz should be applicable in the lower temperature, collisionless regime. In various of the system studied we put the emphasis on the final result that was the obtaining of some generalization of the classical Nonlinear Schrodinger equation that describes the collective excitations in each of them.

By starting with a similar treatment employed by Glauber it is possible to obtain the coherent states, also known as standard coherent states where we see clearer the meaning of coherence when we refer to these quantum states. In a very natural way we show the generalization for these method to obtain the so called *generalized coherent states*. The basis of this development for Lie groups was to connect closely the coherent states with the group dynamics of each physical problem. Next we made a review of the generalized coherent states based on the most popular Lie groups. By means of seeing clearly how powerful this states are we proceed to summarize examples represent in a very suitable and comprehensive way the major features of these states.
Hopefully, this description will stimulate the curiosity of the marvelous and indispensable topic coherent states represent. Moreover, in words of Gazeau, these states are the language of quantum mechanics. We cannot underestimate the main role group theory play in the development, and that's also a increasing area of study in mathematical physics. The study of deformed algebras and the connection of coherent states with polynomials is worrisome but just outlining these examples is worth of another review.

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