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About Convection in the Core and Mantle of the Earth

Boris P. Kondratyev[1]

[1] Sternberg Astronomical Institute, Lomonosov Moscow State University, Central Astronomical Observatory of Russian Academy of Sciences at Pulkovo, St. Petersburg, Russia

Correspondance: Boris P. Kondratyev, Sternberg Astronomical Institute, Lomonosov Moscow State University, Central Astronomical Observatory of Russian Academy of Sciences at Pulkovo, St. Petersburg, Russia. E-mail: work@boris-kondratyev.ru

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Abstract

Problem- The mystery of origin Earth's magnetic field is connected with the solution of the problem of convection in the mantle and in the outer liquid core. However, the mathematical analysis of the stability conditions of convection in the core and the mantle of the Earth still insufficiently developed. Even in the classical formulation of the problem the main roots of the dispersion equation has not been studied. **Purpose-** It should be more fully and rigorously consider the problem of convection in rotational thermodynamically inhomogeneous viscous liquid. In this task, you must obtain a general dispersion equation and perform a mathematical analysis of the roots of this equation. You also need find the shape of convective cells in the bowels of the Earth. **Approach-** We start from the Navier-Stokes equations for thermodynamically inhomogeneous viscous fluid. In the Boussinesq approximation, neglecting the square of the velocity of fluid flow was obtained and carefully studied the cubic dispersion equation. The roots of the dispersion equation are investigated in order to detect the areas of stability and instability of convective currents. **Findings-** In the complex domain is made analysis of roots of the cubic dispersion equation. It is shown that on the chart "angular velocity – temperature gradient" there are areas of stable and unstable fluid motion. The Earth's daily rotation does not affect the convection in the mantle, where convective cells are almost cubic with a characteristic size $\approx 150\,km$. But the convection in the Earth's outer liquid core is divided into a plurality of thin "pipes." These tubes are parallel to the Earth's axis, and currents up and down them alternate. The role of the daily rotation of the Earth in the creation of such mechanism of convection is very important. **Implications-** The method is applied to the study of convection in the mantle and liquid core of the Earth. We find some important properties of the convective cells in the mantle and the liquid core. The results are in satisfactory agreement with geological data. We draw attention to the fact that the structure of "tubes" in the liquid core is such that it stands out particularly equatorial toroidal ring. In this ring the convection is absent and, most likely, the liquid of the core there is in a highly turbulent state. We assume that this is the zone of turbulence ring may be responsible for the emergence of Earth's magnetic field. **Originality-** These results are new in mathematical physics and dynamics of the Earth's shells.

Keywords: Navier-Stokes equations, thermodynamically inhomogeneous viscous liquid, Boussinesq approximation, dispersion equation, stable (unstable) motion, convection, magnetic field

1. Introduction

As you know, in the Earth's depths can really exists the thermal convection of the mantle material. Laboratory simulations of the convection was carried out by many researchers, see, eg, Jacoby (1976). Now there are also many works on numerical modeling of convection (Aurnou et al., 2003; Brown & Ahlers, 2006; Trubitsyn, 2009; Benerji Babu et al., 2011; Carcia et al., 2014).

However it is unreasonable to build too detailed models of convection as we don't know full information about the Earth's interior. It is more important to carry out calculations that reflect orders of magnitude, associated with the convection. That is why it is important to once again draw attention to the fundamental work Chandrasekhar (1961), where Chandrasekhar was studied the stability of convection in a rotating layer thermodynamically inhomogeneous fluid. But Chandrasekhar in his book has not studied the main roots of the dispersion equation.

The purpose of this article is as follows. We start from the Navier-Stokes equations for thermodynamically inhomogeneous viscous fluid. In the Boussinesq approximation, neglecting the square of the velocity of fluid flow and omitting the inertia terms, was obtained the cubic dispersion equation. The roots of the dispersion equation are investigated in order to detect the stability and instability zones of convective currents. Based on a thorough analysis of the roots, we find the geometric shape of convective cells in the mantle and outer liquid core of the Earth.

Now let us make a few general remarks. We take into account compressibility of fluid, like Chandrasekhar (1961), in the linear Boussinesq's approximation, when are considered only small changes of density, which are caused by temperature variation. Some comments see also in the review Stacey (2010). Here we use the following notation:

$\Omega\left[s^{-1}\right]$ – the angular velocity of rotation of the system around the z axis,

$\nu\left[\dfrac{m^2}{s}\right]$ – the coefficient of kinematic viscosity,

$g\left[\dfrac{m}{s^2}\right]$ – the acceleration of gravity,

$\chi\left[\dfrac{m^2}{s}\right]$ – the heat diffusivity coefficient,

$\alpha\left[\textit{град}^{-1}\right]$ – the linear coefficient of thermal expansion,

$h\left[\dfrac{\textit{град}}{m}\right]$ – the initial gradient of temperature.

2. Derivation of the Dispersion Equation

The Navier-Stokes equations in view the temperature gradient have the form

$$\begin{aligned}
&\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x_1}+u_2\frac{\partial u_1}{\partial x_2}+u_3\frac{\partial u_1}{\partial x_3}=-\frac{1}{\rho}\frac{\partial p'}{\partial x_1}+\nu\left(\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_2^2}+\frac{\partial^2 u_1}{\partial x_3^2}\right)+2\Omega u_2,\\
&\frac{\partial u_2}{\partial t}+u_1\frac{\partial u_2}{\partial x_1}+u_2\frac{\partial u_2}{\partial x_2}+u_3\frac{\partial u_2}{\partial x_3}=-\frac{1}{\rho}\frac{\partial p'}{\partial x_2}+\nu\left(\frac{\partial^2 u_2}{\partial x_1^2}+\frac{\partial^2 u_2}{\partial x_2^2}+\frac{\partial^2 u_2}{\partial x_3^2}\right)-2\Omega u_1,\\
&\frac{\partial u_3}{\partial t}+u_1\frac{\partial u_3}{\partial x_1}+u_2\frac{\partial u_3}{\partial x_2}+u_3\frac{\partial u_3}{\partial x_3}=-\frac{1}{\rho}\frac{\partial p'}{\partial x_3}+\nu\left(\frac{\partial^2 u_3}{\partial x_1^2}+\frac{\partial^2 u_3}{\partial x_2^2}+\frac{\partial^2 u_3}{\partial x_3^2}\right)+\alpha gT'.
\end{aligned}\tag{1}$$

Heat transfer is described by the equation

$$\frac{\partial T'}{\partial t}+u_1\frac{\partial T'}{\partial x_1}+u_2\frac{\partial T'}{\partial x_2}+u_3\frac{\partial T'}{\partial x_3}=\chi\left(\frac{\partial^2 T'}{\partial x_1^2}+\frac{\partial^2 T'}{\partial x_2^2}+\frac{\partial^2 T'}{\partial x_3^2}\right).\tag{2}$$

We are working, in addition to the assumption of smallness of the temperature gradient, in the linear approximation with respect to the velocities, so that the inertial terms further are rejected, and instead of (1) we have

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= -\frac{1}{\rho}\frac{\partial p'}{\partial x_1} + \nu\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right) + 2\Omega u_2, \\
\frac{\partial u_2}{\partial t} &= -\frac{1}{\rho}\frac{\partial p'}{\partial x_2} + \nu\left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2}\right) - 2\Omega u_1, \\
\frac{\partial u_3}{\partial t} &= -\frac{1}{\rho}\frac{\partial p'}{\partial x_3} + \nu\left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2}\right) + \alpha g T'.
\end{aligned} \tag{3}$$

The temperature of initial state will take in the form

$$T' = T_0 - h\,x_3. \tag{4}$$

Linearization of the equation of thermal conductivity gives

$$\frac{\partial T'}{\partial t} - h\,x_3 = \chi\left(\frac{\partial^2 T'}{\partial x_1^2} + \frac{\partial^2 T'}{\partial x_2^2} + \frac{\partial^2 T'}{\partial x_3^2}\right). \tag{5}$$

Further, in our analysis we assume that all quantities are proportional to exponent

$$e^{ikx+i\mu z+\lambda t} \tag{6}$$

Here k and μ are the wave numbers in directions x and z respectively. Then, the linearized equation (3) will take algebraic form

$$\begin{aligned}
\lambda u_1 &= -\frac{ik}{\rho}p' - \nu\left(k^2+\mu^2\right)u_1 + 2\Omega u_2, \\
\lambda u_2 &= -\nu\left(k^2+\mu^2\right)u_2 - 2\Omega u_1, \\
\lambda u_3 &= -\frac{i\mu}{\rho}p' - \nu\left(k^2+\mu^2\right)u_3 + \alpha T', \\
\lambda T' - h\,x_3 &= -\chi\left(k^2+\mu^2\right)T'.
\end{aligned} \tag{7}$$

The incompressibility condition

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 \tag{8}$$

gives the ratio

$$ku_1 + \mu u_3 = 0. \tag{9}$$

So, the components u_2 and u_3 are expressed through u_1

$$u_2 = -\frac{2\Omega\,u_1}{\lambda + \nu\left(k^2+\mu^2\right)}; \quad u_3 = \frac{k}{\mu}u_1. \tag{10}$$

Eliminating the pressure from the first and third equations in (7), we find

$$T' = \frac{2\Omega\mu u_2 - \left[\lambda + \nu\left(k^2+\mu^2\right)\right]\left(\mu u_1 - k u_3\right)}{k\alpha}. \tag{11}$$

Here

$$\mu u_2 = -\frac{2\mu\Omega u_1}{\lambda+\nu\left(k^2+\mu^2\right)}; \quad \mu u_1 - k u_3 = \frac{k^2+\mu^2}{\mu}u_1. \tag{12}$$

Therefore, the equation (11) takes the form

$$k\alpha T' = \left\{-\frac{4\mu\Omega^2}{\lambda+\nu\left(k^2+\mu^2\right)} - \frac{\left(\mu^2+k^2\right)\left[\lambda+\nu\left(\mu^2+k^2\right)\right]}{\mu}\right\}u_1. \tag{13}$$

Substituting into the left side (13) the value T' from (11), after reducing u_1 and some transformations, we obtain

$$\left[\lambda+\chi\left(k^2+\mu^2\right)\right]\left\{-\frac{4\mu\Omega^2}{\lambda+\nu\left(k^2+\mu^2\right)} - \frac{\left(\mu^2+k^2\right)\left[\lambda+\nu\left(\mu^2+k^2\right)\right]}{\mu}\right\} + \frac{hk^2\alpha g}{\mu} = 0, \tag{14}$$

or, after the multiplication on μ and division on $\lambda+\chi\left(k^2+\mu^2\right)$:

$$\frac{4\Omega^2\mu^2}{\lambda+\nu\left(k^2+\mu^2\right)} - \frac{h\alpha gk^2}{\lambda+\chi\left(k^2+\mu^2\right)} + \left(k^2+\mu^2\right)\left[\lambda+\nu\left(k^2+\mu^2\right)\right] = 0. \tag{15}$$

This is the general dispersion equation (cubic relatively increment λ) in this problem, obtained previously by Chandrasekhar (1961). Next, we will deal with mathematical analysis of the dispersion equation (15).

3. Analysis of the Roots of the DISPERSION Equation

The study of the equation (15) is convenient to start with separation of the roots in complex plane λ, or more precisely, their placement in the zones separated by vertical lines

$$\operatorname{Re}\lambda = -\nu\left(k^2+\mu^2\right); \quad \operatorname{Re}\lambda = -\chi\left(k^2+\mu^2\right). \tag{16}$$

Specifically, this is achieved by relatively simple substitution into the left side (15) the values $\lambda = -\nu\left(k^2+\mu^2\right)+i\lambda'$, or $\lambda = -\chi\left(k^2+\mu^2\right)+i\lambda'$ with real values λ', and it is proved that (1) never satisfied. If we are also using the continuity condition, then it turns out the following accommodations of the roots:

a). $\nu > \chi$. Two roots at $\operatorname{Re}\lambda < -\nu\left(k^2+\mu^2\right)$ and one root at $\operatorname{Re}\lambda > -\chi\left(k^2+\mu^2\right)$; (17)

b). $\nu < \chi$. One root at $\operatorname{Re}\lambda < -\chi\left(k^2+\mu^2\right)$ and two roots at $\operatorname{Re}\lambda > -\nu\left(k^2+\mu^2\right)$. (18)

In the middle zone no roots, and the roots in the left lane is not important for our purposes because the corresponding damped processes, and the ones are insignificant in the analysis of stability of the system.

It remains to verify the possibility of moving the roots through the imaginary axis λ. Results are as follows. In the case a) the root responsible for instability can only be real. The instability starts from the value $\lambda = 0$ that is equivalent to the condition

$$H\left(\Omega,h\right) \equiv -\frac{4\Omega^2\mu^2}{\nu\left(k^2+\mu^2\right)} + \frac{h\alpha gk^2}{\chi\left(k^2+\mu^2\right)} - \nu\left(k^2+\mu^2\right)^2 > 0. \tag{19}$$

In the case b) situation is more complicated. The roots lie on the imaginary axis at

$$\frac{h\alpha gk^2\nu}{\chi} - \nu\left(k^2+\mu^2\right)^3 > 4\Omega^2\mu^2. \tag{20}$$

The border is

$$K\left(\Omega,h\right) \equiv h\alpha gk^2\left(\nu+\chi\right) - 8\Omega^2\mu^2\nu - 2\nu\left(\nu+\chi\right)^2\left(k^2+\mu^2\right)^3 = 0. \tag{21}$$

On the diagram $\left(\Omega^2, h\right)$ (Figure 1) the both lines $H\left(\Omega^2, h\right) = 0$ and $K\left(\Omega, h\right) = 0$ are straight. For reasons of continuity we obtain the following location of the specific areas

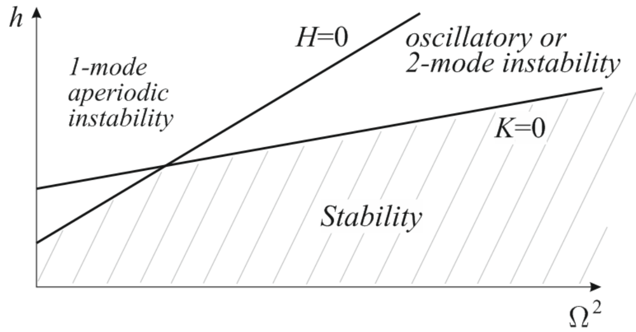

Figure 1. Location of zones of stability (shaded) and instability for a rotating thermodynamically inhomogeneous liquid mass. The lines $H\left(\Omega, h\right) = 0$ and $K\left(\Omega, h\right) = 0$ are found from the equations (19) and (21)

4. The Case without Rotation

Let us first consider the simpler case of a non-rotating system $\left(\Omega = 0\right)$. Thus there can be only 1-modal aperiodic instability. Criterion of its occurrence is $H > 0$, or

$$h\alpha g k^2 > \nu\chi\left(k^2 + \mu^2\right)^3. \tag{22}$$

Further assume isotropic wave propagation inside the cells, i.e., $\mu \approx k$. Then we will receive critical value of wave number

$$k^* = \left(\frac{h\alpha g}{8\nu\chi}\right)^{\frac{1}{4}}. \tag{23}$$

We see the following. If the convective cells were roughly cubic $\mu \sim k$, the role of the rotation for the Earth's shells would always be negligible, because even for the approximately identical ν and χ we have following estimate

$$h\mu g >> 4\Omega^2. \tag{24}$$

For the evaluation we take numerical values of parameters for the Earth's mantle by Turcotte & Schubert (1982)

$$h = 20\frac{grad}{km}, \quad \alpha = 2\cdot 10^{-6} grad^{-1}, \quad \chi = 5\cdot 10^{-6}\frac{m^2}{s}, \quad g = 10\frac{m}{s^2}, \quad \nu = 0,25\cdot 10^{17}\frac{m^2}{s}. \tag{25}$$

Then, the critical wave number from (23) is equal to

$$k^* \approx 4\cdot 10^{-5} m^{-1}, \tag{26}$$

and the corresponding wavelength is

$$l^* = \frac{2\pi}{k^*} \approx 1,5\cdot 10^5 m = 150\, km, \tag{27}$$

what is in satisfactory agreement with modern geological data.

For the liquid core of the Earth is radically changing only the viscosity, and other parameters remain the same. Then, according to (26), you can just assume that the critical wavelength varies as $\nu^{\frac{1}{4}}$. If we take for the core $\nu_1 = 1.2\cdot 10^{-5}\frac{м^2}{c}$, as given by Jacobs (1987), then the ratio

$$\left(\frac{\nu_1}{\nu}\right)^{\frac{1}{4}} = \left(\frac{0,2\cdot 10^{-5}}{0,25\cdot 10^{17}}\right)^{\frac{1}{4}} \approx \left(0,8\cdot 10^{-22}\right)^{\frac{1}{4}} = 3\cdot 10^{-6}, \tag{28}$$

and the new critical length

$$l^* = 3\cdot 10^{-6}\cdot 150km = 0,45m \quad \text{при} \quad k^* \approx 13m^{-1} \tag{29}$$

mean that the Earth's liquid core can be formed from relatively narrow columns. But we emphasize that adoption of the critical values is not required in practice: the differential equation admits any larger waves.

5. The Case of Rotation

In the presence of system rotation, most likely, only possible aperiodic instability (when the numerical parameters are taken the excess ν over χ not so great). The instability criterion under (15) now is:

$$\frac{h\alpha g k^2 \nu}{\chi} - \nu^2\left(k^2+\mu^2\right)^3 > 4\Omega^2\mu^2. \tag{30}$$

Assuming again cubic cells $\mu \approx k$, then the role of rotation is negligible, because even for approximately coincident ν and χ цу have the estimate

$$h\alpha g >> 4\Omega^2. \tag{31}$$

Thus, for the Earth’s liquid core the role of rotation in this case is too small, and for the mantle, where $\nu >> \chi$, the inequality (31) is further enhanced.

So, for the Earth's mantle daily rotation is absolutely insignificant. For the core situation is more complex, as the geometry of the cubic cells inherent in the mantle, in the core is not obligatory. Consider this question.

6. Convection in the Earth's Liquid Core

From (30) it follows that if the instability condition is satisfied for large values, then, by continuity, it will work on small values of the wave number. As stated above, for the core is allowed a small horizontal extent of the convective cell.

Therefore, convection in the core must occur in cells that are strongly stretched along the Earth’s rotation axis (Figure 2). Moreover, convection is known to dramatically accelerates the transfer of heat in comparison to a simple thermal conductivity, therefore, the temperature gradient should sharply drop almost to the level of beginning of the instability. (More precisely, it is about exceeding the magnitude of the gradient over adiabatic.) Thus, we need to introduce an amendment to the previously received value h and reduce it to that at which (30) becomes an equality at the lowest possible μ (when the cell is extended to approximately the inner solid core), and the most unfavorable for the stability k (and for all other k and μ we have in (30) the opposite sign <).

The usual finding the maximum of the left part (30) for k leads to

$$\frac{h\alpha g \nu}{\chi} = 3\nu^2\left(k^2+\mu^2\right)^2 \tag{32}$$

and

$$2\nu^2\left(k^2+\mu^2\right)^2 = 4\Omega^2\mu^2. \tag{33}$$

In the second case we can neglect μ^2 in comparison with k^2, and at $\mu \approx 1,5\cdot 10^{-6} м^{-1}$ with the above $\nu = 0,2\cdot 10^{-5}\,\frac{м^2}{c}$ we get now

$$k \approx 3,3\cdot 10^{-3} м^{-1}, \quad \frac{2\pi}{k} \approx 2км, \tag{34}$$

that quality is not contrary to the original hypothesis that the convection cells in the outer core are elongated along the rotation axis of the Earth.

On the influence of the tidal influence of the Moon on the Earth's inner core see Kondratyev (1989).

7. Conclusions

The riddle of origin of Earth's magnetic field is connected with solution of the problem of convection in the mantle and in the outer liquid core**.** We start from the Navier-Stokes equations for thermodynamically inhomogeneous viscous fluid. In the Boussinesq's approximation, neglecting the square of the velocity of fluid flow and omitting the inertia terms, was obtained the general dispersion equation. The roots of the cubic dispersion equation are carefully investigated and the zones of stability and (aperiodic and oscillatory) instability of convective currents are detected. Based on a rigorous analysis of the roots, we then make a real assessment of the type of convective cells in the mantle and the outer liqued core of the Earth.

We come to three important conclusions.

1) Found the stability zones of the convection currents in the Earth's mantle and liquid core.
2) Convection in the Earth's liquid core is divided into a plurality of thin "pipes." These tubes are parallel to the Earth's axis, and flow up and down them alternate. The role of the Earth's daily rotation in creation of such convection mechanism is *very important*.

3) Conversely, the daily rotation of the Earth does not affect the convection in the mantle, where the convective cells are almost cubic with a characteristic size $\approx 150\,km.$ Here, the centrifugal force is simply added to the pressure effect.

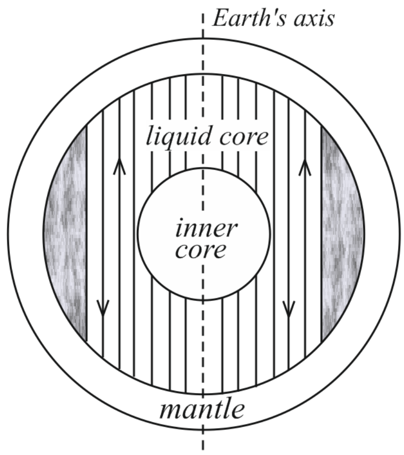

Figure 2. The scheme of convective cells in the liquid core of the Earth. Alternating current in the cells indicated by arrows. Emphasis toroidal ring with turbulent fluid motion

8. Final Comments

Besides, we make some additional comments. The structure of the "tubes" in the liquid core of the Earth is that it stands out the particularly equatorial toroidal ring with strongly turbulent fluid flow. Thus, in the core there is a special ring zone, wherein said the convection is absent. There is a fundamental question: is the annular zone with strongly turbulent fluid flow is responsible for the creation of Earth's magnetic field?

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