Applied Physics Research; Vol. 7, No. 3; 2015
ISSN 1916-9639 E-ISSN 1916-9647
Published by Canadian Center of Science and Education

On the Escape Velocity in General Relativity and the Problem of Dark Matter

Valery Vasiliev[1] & Leonid Fedorov[1]

[1] Institute of Problems in Mechanics, Russian Academy of Sciences, 101 Vernadskogo, Moscow 119526, Russia

Correspondence: Valery Vasiliev, Institute of Problems in Mechanics, 101 Vernadskogo, Moscow 119526, Russia. Tel: 7-916-607-1281. E-mail: vvvas@dol.ru

Received: April 26, 2014 Accepted: May 5, 2015 Online Published: May 19, 2015

doi:10.5539/apr.v7n3p84 URL: http://dx.doi.org/10.5539/apr.v7n3p84

Abstract

The paper is concerned with analysis of the escape velocity for a spherical star within the framework of the Classical Gravitation Theory (CGT) and the General Theory of Relativity (GTR). In GTR, two possible results corresponding to the classical and the generalized Schwarzschild solutions are obtained. For the latter solution which, in contrast to the former one, is not singular, the critical radius of the star for which the particle cannot leave the surface even if the initial velocity of the particle is equal to the velocity of light is found and is associated with the radius of the Dark Star introduced in the 18-th century by J. Michel and P. Laplace. As shown, the radius of the largest visible star, i.e., the red supergiant UY Scutti, is much larger than the obtained critical radius. If the Dark Stars whose radii are smaller than the critical radius exist, they are hypothetically not visible and can concentrate the matter that cannot be observed.

Keywords: escape velocity, general relativity, dark matter

1. Introduction

To determine the escape velocity v_e corresponding to CGT, consider a solid sphere with radius R, constant density μ and mass

$$m_o = \frac{4}{3}\pi\mu R^3 \tag{1}$$

Assume that a relatively small particle with mass m_1 moves from the sphere in the radial direction r under the action of the Newton gravitation force in accordance with the following equation:

$$m_1 \frac{d^2 r}{dt^2} = -G\frac{m_e m_1}{r^2} \tag{2}$$

where $G = 6.67 \cdot 10^{-11} N \cdot m^2 / kg^2$ is the classical gravitation constant and t is time. Introducing the so-called gravitation radius

$$r_g = \frac{2Gm_o}{c^2} \tag{3}$$

in which c is the velocity of light, we can reduce Equation (2) to

$$\frac{d^2 r}{dt^2} + r_g \frac{c^2}{2r^2} = 0 \tag{4}$$

This equation is integrated under the following initial conditions:

$$r(t=0) = R\,, \quad v(t=0) = \left.\frac{dr}{dt}\right|_{t=0} = v_0 \tag{5}$$

where v_0 is the initial velocity of the particle with which it starts moving from the sphere surface $r = R$. The first integral of Equation (4)

$$v = \frac{dr}{dt} = \sqrt{C_1 + c^2 \frac{r_g}{r}} \tag{6}$$

includes the integration constant C_1 which can be found from the second condition in Equations (5), i.e.,

$$C_1 = v_0^2 - \frac{r_g c^2}{R} \tag{7}$$

Substituting C_1 from Equation (7) in Equation (6), we get

$$v = \sqrt{v_0^2 + r_g c^2 \left(\frac{1}{r} - \frac{1}{R} \right)} \tag{8}$$

Further integration of Equation (6) under the first condition in Equations (5) specifies the following dependence between the radial coordinate and time:

$$t = \frac{r_g c^2}{2C_1\sqrt{C_1}} \ln \frac{2C_1 R + r_g c^2 + 2\sqrt{C_1 R(r_g c^2 + C_1 R)}}{2C_1 r + r_g c^2 + 2\sqrt{C_1 r(r_g c^2 + C_1 r)}} - \frac{1}{C_1}\left[\sqrt{R(r_g c^2 + C_1 R)} - \sqrt{r(r_g c^2 + C_1 r)} \right] \tag{9}$$

Two ways can be used to determine the escape velocity. First, as follows from Equation (9), the motion is real if $C_1 \geq 0$. Then, Equation (7) yields

$$v_0 \geq v_e = c\sqrt{\frac{r_g}{R}} \tag{10}$$

To apply the second approach, put $r \to \infty$ in Equation (8) to get

$$v_\infty = \sqrt{v_0^2 - c^2 \frac{r_g}{R}} \tag{11}$$

and specify v_e as the initial velocity v_0 for which $v_\infty = 0$. Then, Equation (11) gives the same expression for v_e that follows from Equation (10).

Taking $v_0 = v_e$ (or $C_1 = 0$) and integrating Equation (6), we arrive at the following dependence of the radial coordinate on time:

$$r = R\left(1 + \frac{3v_e}{2R} t \right)^{\frac{2}{3}}$$

As follows from Equation (10), for the sphere with radius $R = r_g$ the escape velocity becomes equal to the velocity of light. According to the interpretation of this result proposed by J. Michel in 1783 and P. Laplace in 1796 (Thorne, 1994) the star with radius r_g can be referred to as the Dark Star which is not visible. Later, the conception of Dark Stars based on the Classical Gravitation Theory and the corpuscular model of light has been abandoned by Laplace and the subsequent authors (Thorne, 1994). However, the idea of Dark Stars becomes attractive in the light of the modern problem of Dark Matter. The dramatic discrepancy between the mass of the observed space objects and the mass following from the gravitation effects corresponding to these objects allows us to suppose the existence of hidden mass or Dark Matter which is estimated to constitute about 85% of the total matter in space. The present paper contains an attempt to revive the idea of Dark Stars within the framework of the General Theory of Relativity.

2. Equation of the Radial Motion in GTR

To determine the escape velocity in GTR, introduce the spherical coordinate frame $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, $x^4 = ct$ and the corresponding spherically symmetric form of the line element

$$ds^2 = g_{11}dr^2 + g_{22}(d\theta^2 + \sin^2\theta d\phi^2) - g_{44}c^2dt^2 \tag{12}$$

Here, the components of the metric tensor g_{ij} depend on the radial coordinate r only. Consider the equation for geodesic lines in the Riemannian space (Weinberg, 1972)

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0 \tag{13}$$

and perform some transformations. First, write Equation (13) for the radial coordinate $x^1 = r$. Second, using the procedure described elsewhere (Weinberg, 1972), change the variable s to the time coordinate t. The resulting equation is

$$\frac{d^2 r}{dt^2} = (2\Gamma_{14}^4 - \Gamma_{11}^1)\left(\frac{dr}{dt}\right)^2 - c^2 \Gamma_{44}^1 \tag{14}$$

where

$$\Gamma_{11}^1 = \frac{g_{11}'}{2g_{11}}, \quad \Gamma_{44}^1 = \frac{g_{44}'}{2g_{11}}, \quad \Gamma_{14}^4 = \frac{g_{44}'}{2g_{44}} \tag{15}$$

and $(...)' = d(...)/dr$. The coefficients of the metric tensor which enter Equations (15) can be found from the basic GTR equations for the components of the Einstein tensor E_i^j which for the problem under study can be presented in the following form:

$$E_1^1 = \frac{1}{g_{22}} - \frac{g_{22}'}{2g_{11}g_{22}}\left(\frac{g_{22}'}{2g_{22}} + \frac{g_{44}'}{g_{44}}\right) \tag{16}$$

$$E_4^4 = \frac{1}{g_{22}} - \frac{1}{g_{11}}\left[\frac{g_{22}''}{g_{22}} - \frac{1}{4}\left(\frac{g_{22}'}{g_{22}}\right)^2 - \frac{g_{11}'g_{22}'}{2g_{11}g_{22}}\right] \tag{17}$$

For the empty and infinite external space surrounding the sphere with radius R (i.e., for $r \geq R$), $E_1^1 = E_4^4 = 0$. Introducing a new function $\rho(r)$ as $g_{22} = \rho^2(r)$ and putting $\rho(r \to \infty) = r$, we can integrate Equations (16), (17) and express the metric coefficients g_{11} and g_{44} in terms of $\rho(r)$ as (Vasiliev & Fedorov, 2014)

$$g_{11} = \frac{(\rho')^2}{1 - r_g/\rho}, \quad g_{44} = 1 - \frac{r_g}{\rho} \tag{18}$$

For $\rho(r \to \infty) = r$, which is the case for the solutions discussed further, and $r \to \infty$, Equations (18) reduce to the corresponding solutions of CGT. Note that the function $\rho(r)$ in Equations (18) can be appropriately chosen as discussed by Vasiliev and Fedorov (2015).

3. Schwarzschild Solution

The classical Schwarzschild solution follows from Equations (18) if we take $\rho = r$ and has the following form (Synge, 1960):

$$g_{11} = \frac{1}{1 - r_g/r}, \quad g_{44} = 1 - \frac{r_g}{r} \tag{19}$$

in which r_g is specified by Equation (3). For the metric coefficients in Equations (19), the motion Equation (14) reduces to

$$\frac{d^2 r}{dt^2} - \frac{3r_g}{2r^2(1 - r_g/r)}\left(\frac{dr}{dt}\right)^2 + \frac{c^2 r_g}{2r^2}\left(1 - \frac{r_g}{r}\right) = 0 \tag{20}$$

For the gravitation field with relatively low intensity, we can neglect the ratio r_g/r in comparison with unity and reduce Equation (20) to Equation (4) corresponding to CGT.

To solve Equation (20) which includes the unknown function $r(t)$, interchange the variables and transform it to the following equation for the function $t(r)$:

$$\frac{d^2 t}{dr^2} - \frac{r_g c^2}{2r^3}(r - r_g)\left(\frac{dt}{dr}\right)^2 + \frac{3r_g}{2r(r - r_g)}\frac{dt}{dr} = 0$$

Introducing a new variable $u(r)$ as (Kamke, 1959)

$$\frac{dt}{dr} = u(r)\exp\left[-3r_g \int \frac{dr}{r(r-r_g)}\right]$$

we can reduce this equation to the first order differential equation

$$\frac{1}{u^3}\frac{du}{dr} = \frac{r_g c^2}{2r^2}(r-r_g)\exp\left[-3r_g \int \frac{dr}{r(r-r_g)}\right]$$

apply separation of variables and arrive after rather cumbersome transformations at the following first integral:

$$\frac{dr}{dt} = c\left(1-\frac{r_g}{r}\right)f(r) \tag{21}$$

in which

$$f(r) = \sqrt{\frac{r_g}{r} + C_2\left(1-\frac{r_g}{r}\right)}$$

and C_2 is the integration constant. Equation (21) specifies the so-called coordinate velocity. The actual velocity can be found with aid of Equations (19) and (21) as (Landau & Lifshitz, 1962)

$$v = \sqrt{\frac{g_{11}}{g_{44}}}\frac{dr}{dt} = cf(r) \tag{22}$$

Taking in accordance with initial conditions $v(r=R) = v_0$, we get

$$C_2 = \frac{1}{1-r_g/R}\left(\frac{v_0^2}{c^2} - \frac{r_g}{R}\right) \tag{23}$$

and reduce Equation (22) to the following final form:

$$v = c\sqrt{\frac{1}{1-r_g/R}\left[\frac{v_0^2}{c^2}\left(1-\frac{r_g}{r}\right) + r_g\left(\frac{1}{r}-\frac{1}{R}\right)\right]} \tag{24}$$

Integration of Equation (21) under the initial condition $r(t=0) = R$ yields

$$\begin{aligned} t = \frac{1}{c}\Bigg\{ & C_2[rf(r) - Rf(R)] - r_g \ln\frac{(R-r_g)[r+r_g+C_2(r-r_g)+2rf(r)]}{(r-r_g)[R+r_g+C_2(R-r_g)+2Rf(R)]} + \\ & + \frac{r_g}{2\sqrt{C_2}}(3C_2-1)\ln\frac{r_g+C_2(2r-r_g)+2rf(r)\sqrt{C_2}}{r_g+C_2(2R-r_g)+2Rf(R)\sqrt{C_2}}\Bigg\} \end{aligned} \tag{25}$$

As follows from this solution, the motion is real if $C_2 \geq 0$. Then, Equation (23) gives

$$v_0 \geq v_e = c\sqrt{\frac{r_g}{R}} \tag{26}$$

Formally, this result coincides with Equation (10) for CGT. However, in contract to Equation (10) which is valid for $R = r_g$, Equation (26) does not follow in this case from Equation (23), because the denominator of this equation becomes zero for $R = r_g$. This means that the escape velocity cannot be found for the Shwarzschild solution if the sphere radius is equal to the gravitation radius.

To proceed, assume that the initial velocity has the maximum possible value, i.e., that $v_0 = c$. Then, Equation (23) yields $C_2 = 1$. Taking $C_2 = 1$ in Equation (25), we arrive at

$$t = \frac{1}{c}\left(r - R + r_g \ln \frac{r - r_g}{R - r_g} \right)$$

Solving this equation for r, we get

$$(r - r_g)e^{\frac{r}{r_g}} = (R - r_g)e^{\frac{ct+R}{r_g}} \tag{27}$$

Putting here $R = r_g$, we find that $r = r_g = R$. Thus, the particle cannot leave the surface of the sphere with radius $R = r_g$ even if its initial velocity is equal to the velocity of light.

The sphere with radius $R = r_g$ is referred to as the Black Hole. To evaluate the mass of the sphere, consider the internal space ($0 \le r \le R$). For a solid sphere with constant density μ, the component of the Einstein tensor in Equation (17) is

$$E_4^4 = \chi \mu c^2 \quad , \qquad \chi = \frac{8\pi G}{c^4} \tag{28}$$

Taking $g_{22} = r^2$, we arrive at the following equation

$$1 - \frac{1}{g_{11}}\left(1 - \frac{g_{11}^{/}}{g_{11}}\right) = \chi \mu c^2 r^2$$

The solution of this equation which satisfies the regularity condition at the sphere center $r = 0$ is (Synge, 1960)

$$g_{11}^i = \frac{1}{1 - \chi \mu c^2 r^2 / 3} \tag{29}$$

Index " i " corresponds to the internal space. The metric coefficient g_{11} must be continuous on the sphere surface, i.e., $g_{11}^i(R) = g_{11}(R)$ in which g_{11} is specified by the first Equation (19). As a result, we have

$$\frac{1}{3}\chi \mu c^2 R^3 = r_g \tag{30}$$

and Equation (29) takes the following final form:

$$g_{11}^i = \frac{1}{1 - r_g r^2 / R^3} \tag{31}$$

As follows from Equations (19) and (31), the obtained solution is singular - g_{11} and g_{11}^i become infinitely high on the sphere surface $r = R$ if $R = r_g$.

Equation (30) allows us to find the sphere mass. Indeed, substituting r_g from Equation (3) and χ from the second Equation (28), we arrive at Equation (1) corresponding to the Euclidean space. However, this result does not correspond to the basic GTR concept according to which the space inside the sphere is Riemannian. Particularly, for the metric coefficient in Equation (31), we have

$$m = 4\pi\mu \int_0^R \sqrt{g_{11}}\, r^2 dr = \frac{2\pi}{r_g}\mu R^4 \left(\sqrt{\frac{R}{r_g}} \sin^{-1}\sqrt{\frac{r_g}{R}} - \sqrt{1 - \frac{r_g}{R}} \right) = m_0 \left(1 + \frac{3r_g}{10R} + \frac{9r_g^2}{56R^2} + \ldots \right) \tag{32}$$

The last part of this equation is the decomposition of the second part into the power series with respect to r_g / R. As follows from Equation (32), $m = m_0$ only if $r_g = 0$. Thus, for the mass of the sphere with radius $R = r_g$, we have two expressions following from Equations (1) and (32), i.e.,

$$m_0 = \frac{4}{3}\pi \mu r_g^3 , \quad m = 2\pi \mu r_g^3 \tag{33}$$

which do not coincide. It should be noted that both results are of relatively low value, because the Black Hole with singular geometry can hardly be simulated with a sphere of constant density.

4. Generalized Solution

To derive the generalized solution, assume that the metric coefficients of the external space are specified by Equations (18) in which ρ is an unknown function of r. Then, the motion Equation (14) takes the form

$$\frac{d^2r}{dt^2}-\left[\frac{3r_g\rho^{/}}{2\rho^2(1-r_g/\rho)}-\frac{\rho^{//}}{\rho^{/}}\right]\left(\frac{dr}{dt}\right)^2+\frac{c^2r_g}{2\rho^2\rho^{/}}\left(1-\frac{r_g}{\rho}\right)=0$$

Applying the transformation used to solve Equation (20), we finally arrive at the following solution which is valid for an arbitrary function $\rho(r)$:

$$\frac{d\rho}{dt}=c\left(1-\frac{r_g}{\rho}\right)\sqrt{\frac{r_g}{\rho}+C_3\left(1-\frac{r_g}{\rho}\right)}$$

Note, that this solution coincides with Equation (21) if we change ρ to r. The actual velocity of the particle similar to Equation (22) is

$$v=\sqrt{\frac{g_{11}}{g_{44}}}\frac{dr}{dt}=\frac{1}{\rho^{/}}\sqrt{\frac{g_{11}}{g_{44}}}\frac{d\rho}{dt}=c\sqrt{\frac{r_g}{\rho}+C_3\left(1-\frac{r_g}{\rho}\right)} \tag{34}$$

Assume that the sphere surface $r=R$ corresponds to $\rho=\mathrm{P}$, i.e., that $\rho(R)=\mathrm{P}$. Then, the integration constant C_3 can be found from the initial condition $v(\mathrm{P})=v_0$, i.e.,

$$C_3=\frac{1}{1-r_g/\mathrm{P}}\left(\frac{v_0^2}{c^2}-\frac{r_g}{\mathrm{P}}\right) \tag{35}$$

Equation (25) for $t(r)$ is also valid if we change r, R and C_2 to ρ, P and C_3. As earlier, the motion is real if $C_3\geq 0$ and the escape velocity follows from the following expression:

$$v_0\geq v_e=c\sqrt{\frac{r_g}{\mathrm{P}}}$$

As earlier, this result is valid if $\mathrm{P}>r_g$. To study the case $\mathrm{P}=r_g$, apply the approach described in Section 4, i.e., take $v_0=c$. Then, Equation (35) yields $C_3=1$ and the following motion equation similar to Equation (27) specifies the dependence between ρ and t:

$$(\rho-r_g)e^{\frac{\rho}{r_g}}=(\mathrm{P}-r_g)e^{\frac{ct+\mathrm{P}}{r_g}} \tag{36}$$

As can be seen, the situation is the same that for the Black Hole discussed in Section 3. For $\mathrm{P}=r_g$, we have $\rho=r_g=\mathrm{P}$ and the particle cannot leave the sphere surface $\rho=\mathrm{P}$ even if its velocity is equal to the velocity of light.

Thus, the problem is to determine the function $\rho(r)$ and to find $\mathrm{P}=\rho(R)$. To solve this problem and to avoid the nonunique value of mass in Equations (33) found for the Schwarzschild solution, assume that gravitation, changing Euclidean geometry inside the interactive solids to Riemannian geometry in accordance with the basic GTR concept, does not affect the solids mass (Vasiliev & Fedorov 2014, 2015).

For the sphere with constant density μ, the internal geometry is specified by Equation (17) in which E_4^4 is given by Equation (28). Taking $g_{22}=\rho^2$, we arrive at the following solution (Vasiliev & Fedorov, 2014):

$$g_{11}^i=\frac{(\rho_i^{/})^2}{1-\chi\mu c^2\rho_i^2/3} \tag{37}$$

The sphere mass can be expressed as

$$m = 4\pi\mu \int_0^R \sqrt{g_{11}^i}\,\rho_i^2\, dr \tag{38}$$

Because it is assumed that gravitation does not affect the sphere mass, whereas in the absence of gravitation the mass is specified by Equation (1), suppose that for the internal space

$$\sqrt{g_{11}^i}\,\rho_i^2 = r^2 \tag{39}$$

Thus, Equation (38) reduces to Equation (1) and $m = m_0$. Then, Equation (30) is valid and we can transform Equation (37) to the final form

$$g_{11}^i = \frac{(\rho_i^{/})^2}{1 - r_g \rho_i^2 / R^3} \tag{40}$$

Using Equations (39) and (40), we arrive at the following equation for $\rho_i(r)$:

$$\rho_i^{/} = \frac{r^2}{\rho_i^2}\sqrt{1 - \frac{r_g}{R^3}\rho_i^2} \tag{41}$$

The solution of this equation which satisfies the regularity condition at the sphere center is

$$F(\rho_i) = \frac{2 r_g r^3}{3R^4} \tag{42}$$

in which

$$F(\rho_i) = \sqrt{\frac{R}{r_g}} \sin^{-1}\left(\frac{\rho_i}{R}\sqrt{\frac{r_g}{R}}\right) - \frac{\rho_i}{R}\sqrt{1 - \frac{r_g}{R^3}\rho_i^2}$$

Taking $r = R$ and putting $\rho_i = \mathrm{P}$ in Equation (41), we obtain the following equation for P:

$$F_i(P) = \frac{2r_g}{3R} \tag{43}$$

This equation allows us to find P if the ratio r_g / R is known. However, this is not the case yet. As follows from the analysis of the Schwarzschild solution in Equations (19), R cannot be lower than r_g. To establish the analogous constraint for the solution under study, consider the external space for which the metric coefficients are specified by Equations (18). The main problem is to provide the boundary conditions on the sphere surface according to which

$$\rho(r = R) = \rho_i(r = R) = \mathrm{P}, \quad g_{11}(\mathrm{P}) = g_{11}^i(\mathrm{P}) \tag{44}$$

Assume that for the external space the following equation which is similar to Equation (39) is valid

$$\sqrt{g_{11}}\,\rho^2 = r^2 \tag{45}$$

Matching Equations (39) and (45), we can conclude that if the first of the conditions in Equations (44) is satisfied, then the second condition is satisfied automatically. Thus, we need to satisfy only the first boundary condition in Equations (44). Substituting g_{11} from the first Equation (18) in Equation (45), we arrive at the following equation for $\rho(r)$ corresponding to the external space:

$$\rho^{/} = \frac{r^2}{\rho^2}\sqrt{1 - \frac{r_g}{\rho}} \tag{46}$$

The solution of this equation which satisfies the first boundary condition in Equations (44) is

$$\left(\frac{\rho^2}{3}+\frac{5r_g\rho}{12}+\frac{5}{8}r_g^2\right)\sqrt{\rho(\rho-r_g)}+\frac{5}{8}r_g^3\ln\left(\sqrt{\frac{\rho}{R}}+\sqrt{\frac{\rho-r_g}{R}}\right)-$$
$$-\left(\frac{\mathrm{P}^2}{3}+\frac{5r_g\mathrm{P}}{12}+\frac{5}{8}r_g^2\right)\sqrt{\mathrm{P}(\mathrm{P}-r_g)}-\frac{5}{8}r_g^3\ln\left(\sqrt{\frac{\mathrm{P}}{R}}+\sqrt{\frac{\mathrm{P}-r_g}{R}}\right)=\frac{1}{3}(r^3-R^3) \tag{47}$$

Thus, the generalized solution is specified by Equations (42) and (47) which allow us to find the functions $\rho_i(r)$ and $\rho(r)$ for the internal and external spaces. Asymptotic and numerical analysis of this solution (Vasiliev & Fedorov, 2015) shows that it reduces to the Schwarzschild solution for $r=0$ and $r\to\infty$.

As follows from Equation (47), the obtained solution is real if the minimum value of the function ρ which is $\rho(R)=\mathrm{P}$ is not less than r_g, i.e., if $\mathrm{P}\ge r_g$. In the limiting case, substituting $r_g=\mathrm{P}$ in Equation (43), we arrive at the equation for minimum possible value of the sphere radius which yields $R=R_g=1.115r_g$. Applying Equation (3), we can conclude that the solution exists if the sphere radius R satisfies the following condition:

$$R\ge R_g=2.23\frac{mG}{c^2} \tag{48}$$

For $R<R_g$, the solution becomes imaginary.

5. The Problem of Dark Matter

Determining the nature of Dark Matter discussed in the Introduction is one of the most important problems in modern cosmology. Up to now, the structure of Dark Matter remains speculative. According to the existing hypotheses, Dark Matter can be concentrated in heavy normal objects such as Black Holes, neutron stars and white dwarfs, or can be composed of a not yet identified type of particles which interact with ordinary matter via gravity only.

The simplest possible interpretation of this phenomenon was proposed in the 18-th century by J. Michel and P. Laplace long before the problem of Dark Matter appeared. Reproducing the respective reasoning (Laplace, 1799) consider Equation (10) for the escape velocity and transform it with the aid of Equation (3) for the gravitation radius to

$$v_e=R\sqrt{\frac{8}{3}\pi\mu G} \tag{49}$$

For Earth with average density $\mu_E=5520kg/m^3$ and radius $R_E=6371032m$, Equation (49) yields $v_e=11000m/s$ which is 27270 times less than the velocity of light. Because the escape velocity in Equation (49) is proportional to the radius, increasing R_E up to $1.62\cdot10^{11}m$ (which is 249.6 times larger than the radius of Sun, $R_S=6.96\cdot10^8m$), we arrive at $v_e=c$. This calculation allowed Laplace to conclude that the star with the density of Earth and radius which is about 250 times larger than the radius of Sun is not visible and can be referred to as the Dark Star. Based on the Newton gravitation theory, the idea of Dark Stars has been later abandoned.

However, Equation (48) follows from GTR. For the star with the density μ_E, we get $R_g=233R_S$ which is close to the result obtained by P. Laplace. As follows from the analysis associated with Equation (36), a particle cannot leave the surface of the sphere with radius $R=R_g$ even its initial velocity is equal to the velocity of light. As shown in Section 3, the Black Hole possesses the same property and, as known, is not visible (Thorne, 1994). So, we can suppose that the star with radius $R<R_g$ is also not visible and following J. Michel and P. Laplace denote it as the Dark Star. In contrast to the Black Hole, the geometry of the Dark Star is not singular. Applying Equations (18), (41) and (46), we have

$$g_{11}=\frac{r^4}{\rho^4}\quad,\qquad g_{11}^i=\frac{r^4}{\rho_i^4} \tag{50}$$

in which $\rho_i(r)$ and $\rho(r)$ are specified by Equations (42) and (47). Equations (50) are valid if $R\ge R_g$ and no singularity is observed if $R=R_g$ (Vasiliev & Fedorov, 2015). For the sphere with radius $R<R_g$, the solution becomes imaginary which means that this sphere cannot be considered within the framework of GTR. In contrast to Black Holes, no gravitational collapse can be associated with Dark Stars.

Two hypothetical scenarios can be considered. Assume that the star with the radius R which satisfies the inequality in Equation (48) exists and is visible. Possessing high gravitation, the star attracts the matter from space which results in the growth of the star mass and in respective increase of R_g in Equation (48). First, assume that the additional mass increases the density of the star, whereas its radius does not change and becomes closer to the increasing critical radius R_g. When R_g reaches R, the star becomes invisible. Second, the buildup of the star mass can be accompanied by the growth of its radius R, whereas the star density μ does not change. As follows from Equations (1) and (48), R is proportional to $m^{1/3}$, whereas R_g is proportional to m. So, with the growth of the star mass, R_g increases faster than R and the condition $R = R_g$ can be satisfied if the star mass becomes

$$m = 0.26c^3\sqrt{\frac{\pi\mu}{G^3}}$$

If Dark Stars exist, the radii of visible stars must be larger than R_g in accordance with Equation (48). The largest of the observed by now visible stars – the red supergiant UI Scuti is characterized with radius $R = 11.9 \cdot 10^{11} m$ and mass $m = 64 \cdot 10^{30} kg$ (Arroyo-Torres et al., 2013). Using Equation (48), we can calculate the critical radius $R_g = 52900m$ which is much lower than R.

6. Conclusion

The analysis of the escape velocity undertaken within the framework of the Classical Gravitation Theory and General Theory of Relativity has shown that the solutions corresponding to both theories demonstrate the hypothetical existence of Dark Stars which concentrate the matter that cannot be directly observed. In contrast to the Black Holes which follow from the classical Schwarzschild singular solution, are hypothesized as a result of gravitational collapse and whose internal structure cannot be identified within the framework of GTR, the Dark Stars are characterized with regular geometry which follows from GTR. The critical radius of the Dark Star determines the limit under which the GTR solution becomes imaginary and the theory can be treated as not valid. The existence of Dark Stars, in case it takes place in reality, allows us to propose a reasonable explanation of the Dark Matter phenomenon.

References

Thorn, K. S. (1994). *Black Holes and Time Warps – Einstein's Outrages Legacy.* New York, London, W.W. Norton and Company.

Weinberg, S. (1972). *Gravitation and Cosmology.* New York, Willey.

Vasiliev, V. V., & Fedorov, L. V. (2014). On the solution of spherically symmetric static problem for a fluid sphere in general relativity. *Applied Physics Research,* 6(3), 40-49. http://dx.doi.org/10.5539/apr.v6n3p40

Vasiliev V. V., & Fedorov, L. V. (2015). Possible forms of the solution for spherically symmetric static problem in general relativity. *Applied Physics Research, 7*(3), 10-17. http://dx.doi.org/10.5539/apr.v7n3p10

Synge, J. L. (1960). *Relativity: the General Theory.* Amsterdam, North Holland.

Landau, L. D., & Lifshitz, E. M. (1962). *Field Theory.* Moscow, Nauka (in Russian).

Kamke, E. (1959). *Differentialgleichungen.* Leipzig.

Arroyo-Torres, D, Wittkovski, M., Marcaide, J. M., & Hauschildt, P. H. (2013). The atmospheric structure and fundamental parameters of the red supergiants AH Scorpii, UY Scuti, and KW Sagittarii. *Astronomy and Astrophysics, 554*(A76), 1-10.

Laplace, P. S. (1799). Proof of the theorem that the attractive force of a heavenly body could be so large that the light could not flow out of it. *Allgemeine Geographischer Ephemeriden,* Weimer, IV, Bd. 1 St.