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On Quantum Cosmology of the R^2 Gravity Theory

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Abstract

We study the minisuperspace quantization of the theory of gravity where an R^2 term is included in the Einstein Hilbert action. Matter is described by a perfect fluid and we employ the generally relativistic hydrodynamic formulation of Schutz. We set up the Wheeler-DeWitt equation and obtain its solutions. A finite norm wave packet solution is constructed and used to calculate asymptotic values for the expectation values of the scale factor and the Ricci scalar

Keywords: higher order gravity, Wheeler-DeWitt equation, minisuperspace

1. Introduction

The difficulties encountered when attempting to construct a full quantum theory of gravity have led to the development of minisuperspace models of quantum cosmology. In this framework all but a few of the degrees of freedom which describe the gravitational field and its sources are "frozen out" (DeWitt, 1967). Numerous investigations of Friedmann-Robertson-Walker (FRW) universes were carried out within this framework (Halliwell, 1991). Of particular interest are the investigations in which matter is described as a perfect fluid and quantization is performed using both the ADM method (Arnowitt, Deser, & Misner, 1962) and the method of superspace quantization (Wheeler, 1968). Detailed analyses were undertaken by Lapchinsky and Rubakov (1977) and others (Lemos, 1996; Alvarenga & Lemos, 1998; Alvarenga, Fabris, Lemos, & Monaret, 2002) in this regard. These analyses were conducted using the canonical formulation of the generally relativistic hydrodynamics of a perfect fluid due to Schutz (1970, 1971). This formulation is characterized by ascribing dynamical degrees of freedom to the fluid. The works cited above employ the standard Einstein-Hilbert action to describe gravity.

The works cited above employ the standard Einstein-Hilbert action to describe gravity.

In the present work we use the aforementioned method to study an $f(R)$ gravity model, R being the scalar curvature. The suggestion that the gravitational lagrangian can be taken as a function of R was first made by Buchdahl (1970). Here we consider the case where the Einstein-Hilbert action is modified by the addition of an R^2 term (Starobinsky, 1980; Barth & Christensen, 1983; Hawking & Luttrell, 1984; Horowitz, 1985). Hawking and Luttrell (1984) discussed the minisuperspace quantization of this higher derivative theory while Vilenkin (1985) studied the quantization of the Starobinsky model. The author has recently studied the quantization of the R^2 theory and obtained solutions of the Wheeler-DeWitt equation in two-dimensional spacetime (Ahmed, 2012). Here we quantize the R^2 theory, set up the Wheeler-Dewitt equation and obtain its solution in four-dimensional spacetime. The material is organized as follows. In section 2 we discuss the minisuperspace canonical quantization of the model by two different methods and obtain the super-Hamiltonian in each case. In section 3 we write down the Wheeler-DeWitt equation, define the inner product and establish the boundary conditions on the wave function. Exact solutions are obtained and wave packets are constructed. Section 4 is devoted to computing some expectation values and in section 5 we present some concluding remarks.

2. Quantization of $f(R)$ Gravity

We start by writing down the FRW metric as

$$ds^2 = -N^2(t)dt^2 + h_{ij}(t)dx^i dx^j \,, \tag{1}$$

where $i,j = 1,2,3$, N is the lapse function and $h_{ij}(t)$ is given by

$$h_{ij}(t) = a^2(t)\sigma_{ij} \tag{2}$$

In Equation (2) $a(t)$ is the scale factor and σ_{ij} represent the metric for the three dimensional space of constant curvature $k = 1, 0, -1$ corresponding to spherical, flat, or hyperbolic space-like sections, respectively. The action for $f(R)$ gravity in the absence of matter reads (Weinberg, 1972)

$$I_G = -\frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,f(R)\,. \tag{3}$$

We shall specifically take $f(R)$ to be of the form (Barth & Christensen, 1983; Hawking & Luttrell, 1984; Horowitz, 1985)

$$f(R) = R + \beta R^2\,, \tag{4}$$

and describe matter by a perfect fluid with the action

$$I_m = \int d^4x\,\sqrt{-g}\;p\,, \tag{5}$$

where p is the pressure. We also employ Schutz's description of the dynamics of a relativistic fluid in interaction with the gravitational field in terms of velocity potentials (Schutz, 1970, 1971). The degrees of freedom ascribed to the fluid are five scalar potentials $\phi, \alpha, \beta, \theta, S$ in terms of which the four-velocity of the fluid is given by

$$U_\nu = \frac{1}{h}\left(\phi_{,\nu} + \alpha\beta_{,\nu} + \theta S_{,\nu}\right) \tag{6}$$

where h is the specific enthalpy and S is the specific entropy. The potentials α and β, which describe vortex motion are zero in the FRW model because of its symmetry. The potentials θ and ϕ do not have direct physical meaning. Two common choices for the lapse function N are $N = 1$ and $N = a$ (Lapchinsky & Rubakov, 1977). Choosing $N = 1$ and inserting the metric into Equations (3) and (5), the actions become

$$I_G = -\int dt\,a^3(R + \beta R^2)\,, \tag{7}$$

$$I_m = \int dt\,a^3\,p \tag{8}$$

where an overall factor of the spatial integral of $(\det\sigma)^{1/2}$ has been discarded as all functions are taken to depend on t only. We are using units such that $c = 16\pi G = 1$.

The total action is given by

$$I = I_G + I_m\,. \tag{9}$$

We first consider Equation (7) for I_G and note that the scalar curvature for the FRW metric is

$$R = -6a^{-2}(t)(a\,\ddot{a} + \dot{a}^2 + k)\,. \tag{10}$$

Now in the general case of Equation (3) and with Equation (10) for R, one cannot remove the second derivative of the scale factor a using integration by parts. In such a case, the procedure followed in canonical quantization, is to introduce beside a, a second variable which can be chosen to be the scalar curvature R (Vilenkin, 1985), and to express the action I_G in the form

$$I_G = -\int dt\,\mathcal{L}(a, \dot{a}, R, \dot{R})\,. \tag{11}$$

In our case of Equation (7) we substitute Equation (10) for the term linear in R and for one factor R in the R^2 term thus obtaining

$$I_G = 6\int dt\,a\,[\,a\,\ddot{a} + \dot{a}^2 + k + \beta R\,(a\,\ddot{a} + \dot{a}^2 + k)]. \tag{12}$$

Integrating by parts, we eliminate the $\ddot{a}$ term and obtain

$$I_G = 6\int dt\left[-(1+\beta R)\,a\dot{a}^2 - \beta\dot{R}\,\dot{a}\,a^2 + (1+\beta R)ka\right], \tag{13}$$

and the Lagrangian thus reads

$$\mathcal{L}_G = 6\left[-(1+\beta R)a\,\dot{a}^2 - \beta\dot{R}\dot{a}a^2 + (1+\beta R)ka\,\right], \tag{14}$$

which is indeed of the form indicated in Equation (11). As we shall see the quantization based on Equation (14) leads to non-trivial results for the expectation values of the Ricci scalar and the scale factor. The canonical momenta are

$$P_a = -12(1+\beta R)\dot{a}a - 6\beta a^2\dot{R},$$
$$P_R = -6\beta\dot{a}a^2\,. \tag{15}$$

Next expressing $\dot{a}$ and $\dot{R}$ in terms of P_a and P_R we obtain the following expression for the Hamiltonian

$$H_G = \frac{1}{6\beta^2a^3}\,(1+\beta R)P_R^2 - \frac{1}{6\beta a^2}P_RP_a - 6(1+\beta R)ka\,. \tag{16}$$

One can alternatively approach canonical quantization using a method due to Vilenkin (1985) where one regards Equation (10) as a constraint and write the gravitational action as

$$I'_G = -\int dt\,\{(R+\beta R^2)a^3 - \lambda[R + 6a^{-2}(a\ddot{a} + \dot{a}^2 + k)]\}\,, \tag{17}$$

where λ is a Lagrange multiplier. By varying I'_G with respect to R we determine λ

$$\lambda = a^3(1+2\beta R)\ . \tag{18}$$

Substituting Equation (18) back into Equation (17) and integrating by parts to eliminate $\ddot{a}$, we obtain

$$I'_G = \int dt\big[\beta R^2 a^3 - 6(1+2\beta R)\dot{a}^2 a - 12\beta\dot{a}\dot{R}a^2 + 6(1+2\beta R)ka\big]\,, \tag{19}$$

and hence the Lagrangian reads

$$\mathcal{L}'_G = \beta R^2 a^3 - 6(1+2\beta R)\dot{a}a - 12\beta\dot{a}\dot{R}a^2 + 6(1+2\beta R)ka\,. \tag{20}$$

The canonical momenta are given by

$$P'_a = -12(1+2\beta R)\dot{a}a - 12\beta\dot{R}a^2\,, \tag{21}$$

$$P'_R = -12\beta\dot{a}a^2 \tag{22}$$

and the corresponding Hamiltonian reads

$$H'_G = \tfrac{1}{24}(1+2\beta R)\beta^{-2}a^{-3}{P'_R}^2 - \tfrac{1}{12}\beta^{-1}a^{-2}P'_R P'_a - \beta R^2 a^3 - 6(1+2\beta R)ka\,. \tag{23}$$

We first continue with the quantization of the theory based on the gravitational Hamiltonian given by Equation (16). Using the equation of state of a perfect fluid $p = \omega\rho$, with ρ being the energy density, and following the procedure described by Lapchinsky and Rubakov (1977), we obtain the following expression for the action of the combined system of gravitational field plus matter

$$I = \int dt\,\left(\dot{a}P_a + \dot{R}P_R + \dot{\phi}P_\phi + \dot{S}\,P_s - \mathcal{H}\right), \tag{24}$$

where the super Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H} = \mathcal{H}_G + P_\phi^{\omega+1}a^{-3\omega}e^S, \tag{25}$$

and

$$P_\phi = a^3(1+\omega)^{-1/\omega}(\dot{\phi} + \theta\dot{S})^{1/\omega}e^{-S/\omega}\,. \tag{26}$$

Specializing now to the case of radiation filled universe with $=\frac{1}{3}$, we make following canonical transformation to new variables (Lapchinsky & Rubakov, 1977)

$$\begin{aligned} T &= -e^{-S}P_S P_\phi^{-\frac{4}{3}}, \qquad P_T = P_\phi^{\frac{4}{3}}e^S\,, \\ \Phi &= \phi + \tfrac{4}{3}P_S P_\phi^{-1}, \qquad P_\Phi = P_\phi\,. \end{aligned} \tag{27}$$

One then easily shows that

$$P_\phi\dot{\phi} + P_S\dot{S} = \dot{T}P_T - \tfrac{1}{3}\dot{P}_s + P_\Phi\dot{\Phi}\,, \tag{28}$$

and the action becomes

$$I = \int dt\,\left(\dot{a}P_a + \dot{R}P_R + \dot{\Phi}\,P_\Phi + \dot{T}P_T - \mathcal{H}\right) \tag{29}$$

where we have dropped the $\dot{P_S}$ term since it a total derivative. The super-Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H} = \frac{1}{6\beta^2a^3}(1+\beta R)P_R^2 - \frac{1}{6\beta\,a^2}P_R P_a - 6(1+\beta R)ka + \frac{P_T}{a}. \tag{30}$$

The variable T is a global time since it does not involve the canonical momenta (Alvarenga & Lemos, 1998; Hajicek, 1986; Beluardi & Ferraro, 1995). A similar analysis can be carried out for the model based on H'_G leading to the following super-Hamiltonian

$$\mathcal{H}' = \frac{1}{24\beta^2a^3}(1+2\beta R){P'}_R^2 - \frac{1}{12\beta a^2}{P'}_R^2\,P'_a - \beta R^2a^3 - 6(1+2\beta R)ka + \frac{P_T}{a}\ . \tag{31}$$

3. Quantization and the Wheeler-DeWitt Equation

We first deal with the model based on the super-Hamiltonian given in Equation (30). The Wheeler-DeWitt quantization scheme requires the standard representation of canonical momenta into operators

$$P_a \to -i\frac{\partial}{\partial a}\,, \qquad P_R \to -i\frac{\partial}{\partial R}\,, \qquad P_T \to -i\frac{\partial}{\partial T}\,, \tag{32}$$

in order to form the operator $\hat{H}$. The Wheeler-Dewitt equation reads

$$\hat{H}\Psi = 0, \tag{33}$$

where Ψ is the wave function of the Universe. In our case this equation reads

$$i\frac{\partial\Psi}{\partial T} = -\frac{(1+\beta R)}{6\beta^2 a^2}\frac{\partial^2\Psi}{\partial R^2} + \frac{1}{6\beta a}\frac{\partial^2\Psi}{\partial R\partial a} - 6(1+\beta R)ka^2\Psi \tag{34}$$

Equation (34) has the form of a Schroedinger equation with T playing the role of the time. In writing Equation (34) we have ordered the factors so that the variables a and R appear on the left of the derivative operators. We shall seek solutions of Equation (34) of the form

$$\Psi(a,R,T) = e^{-iET}\psi(a,R)\,, \tag{35}$$

where E is a constant. The wave function ψ satisfies the equation

$$\frac{(1+\beta R)}{6\beta^2 a^2}\frac{\partial^2\psi}{\partial R^2} - \frac{1}{6\beta a}\frac{\partial^2\psi}{\partial R\partial a} + [6(1+\beta R)ka^2 + \mathrm{E}]\psi = 0\,. \tag{36}$$

Equation (36) is a second order hyperbolic linear partial differential equation. Now in order to interpret T as a true time and Equation (34) as a genuine Schroedinger equation, the operator

$$\hat{\mathrm{H}} = -\frac{(1+\beta R)}{6\beta^2 a^2}\frac{\partial^2}{\partial R^2} + \frac{1}{6\beta a}\frac{\partial^2}{\partial R\partial a} - 6(1+\beta R)ka^2, \tag{37}$$

must be self-adjoint. The scale factor a is restricted to the domain $a > 0$, so that the minisuperspace quantization deals with the wave functions defined for a restricted to the half-line $(0,\infty)$. As is well known certain conditions have to be imposed on the wave functions in order to ensure the self-adjointness property. To do this we first define the inner product by

$$(\psi,\phi) = \int_{-\infty}^{\infty} dR \int_0^{\infty} da\, a\, \psi^*(a,R)\phi(a,R)\,. \tag{38}$$

The factor a in Equation (38) will remove the a^{-1} factor in front of the differential operator $\partial^2/\partial R\partial a$ thereby rendering establishing its self-adjointness dependent only on the boundary conditions on the wave function. It is the simplest way to achieve self-adjointness for the second term on the right hand side of Equation (37) and also obviates the need for further factor ordering beyond that indicated in Equation (34) for that term. Thus the conditions of square integrability, which are necessary to ensure the existence of the inner product, as well as either of the two conditions

$$\psi(0,R) = 0\,, \tag{39}$$

or

$$\frac{\partial\psi}{\partial R}(0,R)0 \tag{40}$$

are required.

Next we make the following replacement in the first term of Equation (37) for $\hat{\mathrm{H}}$

$$R\frac{\partial^2}{\partial R^2} \to \hat{O} = \frac{1}{2}\left(R\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial R^2}R\right). \tag{41}$$

This replacement is dictated by the need to make $\hat{\mathrm{H}}$ self-adjoint. It is of course not the most general choice but we choose it for its simplicity. We readily conclude that the requirement of square integrability for the wave function and its first and second derivatives are enough to ensure self-adjointness for the operator $\hat{O}$. Needless to say the factor a introduced in the definition of the inner product above does not affect the operation in Equation (41) which is needed to ensure self-adjointness properties of the first term on the right hand side of Equation (37). Making the replacement of Equation (41) in $\hat{\mathrm{H}}$, Equation (36) gets replaced by

$$\frac{(1+\beta R)}{6\beta^2}\frac{\partial^2\Psi}{\partial R^2} - \frac{a}{6\beta}\frac{\partial^2\Psi}{\partial R\partial a} + \frac{1}{6\beta}\frac{\partial\psi}{\partial R} + a^2[6(1+\beta R)ka^2 + \mathrm{E}]\Psi = 0\,. \tag{42}$$

Equation (42) is again hyperbolic and we proceed to reduce it to a canonical form. We introduce new coordinates

$$\xi = \xi(R,a)\,, \qquad \eta = \eta(R,a)\,, \tag{43}$$

and readily show that the canonical form is

$$\frac{1}{6}\frac{\partial^2\psi}{\partial\xi\partial\eta} - 6k\xi e^{\eta}\psi = E\psi\,, \tag{44}$$

with

$$\xi(R,a) = a, \quad \eta(R,a) = \ln(a|1+\beta R|)\,. \tag{45}$$

We look for a factorizable solution of Equation (44)

$$\psi(\xi,\eta) = X(\xi)Y(\eta)\,, \tag{46}$$

and obtain

$$\frac{1}{6}e^{-\eta}\frac{X'(\xi)Y'(\eta)}{X(\xi)Y(\eta)} - 6k\xi e^{\eta} = E\,, \tag{47}$$

where the prime indicates differentiation with respect to the argument. Clearly we can separate the ξ and η dependencies only for $k = 0$. For $k = \pm 1$ we cannot disentangle the ξ and η dependencies and the method of obtaining factorizable solutions in the manner of Equation (46) does not work. For the case $k = 0$, we obtain the equation

$$\frac{1}{6}e^{-\eta}\frac{Y'(\eta)}{Y(\eta)} = \frac{E}{X'(\xi)/X(\xi)} = c\,, \tag{48}$$

where c is a constant. This leads to the solutions

$$X(\xi) = X_1 \exp\left(\frac{E}{c}\,\xi\right), \tag{49}$$

$$Y(\eta) = Y_1 \exp(6ce^{\eta})\,. \tag{50}$$

where X_1, Y_1 are constants. In terms of the original variables a, R the solution reads

$$\psi(a,R) = Z\exp\left[\frac{E}{c}a + 6ca(1+\beta R)\right], \tag{51}$$

where Z is a constant. If $E > 0$ we must have $c < 0$ in order for the wave function to vanish as $a \to \infty$. Taking $\beta > 0$, good asymptotic behaviour also obtains for $R \to \infty$. However the wave function blows up for $R \to -\infty$. On the other hand if $E < 0$, then good asymptotic behaviour in a requires $c > 0$. In this case the wave function vanishes for $R \to -\infty$ but blows up for $R \to \infty$. We note that the wave function satisfies the boundary condition of Equation (40) dictated by the requirement of self-adjointness.

The stationary solutions of Equation (51) thus have infinite norm and to obtain finite norm solutions we have to construct wave packets by superposing these stationary solutions.

To do this we first note from Equation (48) that we can write

$$\frac{X'(\xi)}{X(\xi)} = -\mu\,, \tag{52}$$

where μ is a constant and we have $\frac{E}{c} = -\mu$. Taking $E > 0$ for definiteness, it follows that $\mu > 0$ since $c < 0$. Upon including the T −dependence we can write our solution in terms of E and μ as

$$\Psi(\mathrm{a},\mathrm{R},\mathrm{T}) = \mathrm{Z}\exp\left[-\mu\mathrm{a} - \frac{6\mathrm{E}}{\mu}\mathrm{a}(1+\beta\mathrm{R}) - \mathrm{iET}\right], \tag{53}$$

and construct the wave packet as,

$$\Psi_w = Z\int_0^{\infty} dE\, g(E)\exp\left[-\mu a - \frac{6E}{\mu}a(1+\beta R) - iET\right], \tag{54}$$

where $g(E)$ is suitably chosen. As an illustration we choose

$$g(E) = \sqrt{E}\,e^{-pE^2}\,; \qquad Re\,p > 0\,. \tag{55}$$

Equation (55) for $g(E)$ represents a Gaussian function modulated by the factor $\sqrt{E}$. This choice is found to give rise to a wave function that has the requisite asymptotic behaviour which would ensure the finiteness of the inner product and hence normalizability.

Defining

$$q = \frac{6a}{\mu}(1+\beta R) + i\,T\,, \tag{56}$$

we express Ψ_w as

$$\Psi_w = Z\int_0^{\infty} dE\,\sqrt{E}\,\exp(-\mu a - pE^2 - qE)\,. \tag{57}$$

The integration is carried out (Prudinov, Brychkov, & Marichev, 1986) yielding

$$\Psi_w = \frac{Z}{8}e^{-\mu a}\left(\frac{q}{p}\right)^{3/2}\exp\left(\frac{q^2}{8p}\right)\left[K_{3/4}\left(\frac{q^2}{8p}\right) - K_{1/4}\left(\frac{q^2}{8p}\right)\right], \tag{58}$$

where $K_\nu(z)$ is the modified Bessel function. Using the asymptotic expansion of $K_\nu(z)$ (Bender & Orszag, 1999)

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\left(1 + \frac{4\nu^2-1}{8} + \ldots\ldots\right), \; z \to \infty; \qquad |arg z| < \frac{3}{2}\pi\,, \tag{59}$$

we obtain the following asymptotic expression for Ψ_w

$$\Psi_w \sim \frac{1}{2}\sqrt{\pi}\, Z\, e^{-\mu a}\,[36\,\mu^{-2}a^2(1+\beta R)^2 - T^2 + 12\, i\, \mu^{-1} a\,(1+\beta R) T]^{-3/4}\,, \tag{60}$$

as $q^2 \to \infty$ with

$$|\tan^{-1}\left(\frac{12\mu a(1+\beta R)T}{36a^2(1+\beta R)^2-\mu^2T^2}\right)| < \frac{3}{2}\pi\,, \tag{61}$$

and we have taken p to be real for simplicity. The asymptotic behaviour of Ψ_w expressed by Equation (60) ensures the finiteness of the inner product and hence the normalizability of the wave function given in Equation (58). In fact since the limit $q^2 \to \infty$ can be achieved by any of the variables going to infinity we have

$$\Psi_w \sim a^{-3/2}\, e^{-\mu a}\,, \qquad a \to \infty\,, \qquad R, T \;\; fixed\,, \tag{62}$$

$$\Psi_w \sim R^{-3/2}\, e^{-\mu a}\,, \qquad |R| \to \infty\,, \qquad a, T \;\; fixed\,. \tag{63}$$

Next we turn to the Hamiltonian $\mathcal{H}'$, given by Equation (31). Following a similar procedure and writing the wave function Φ in the form of Equation (35) as $\Phi = \exp(-iEt)\phi$ leads to

$$\frac{(1+2\beta R)}{24\beta^2}\frac{\partial^2\phi}{\partial R^2} - \frac{a}{12\beta}\frac{\partial^2\phi}{\partial R\partial a} + \frac{1}{12\beta}\frac{\partial\phi}{\partial R} + a^2[\beta R^2 a^4 + 6(1+2\beta R)ka^2 + E]\phi = 0\,. \tag{64}$$

The canonical form for $k = 0$ reads

$$\frac{1}{6}e^{-\eta'}\frac{\partial^2\phi}{\partial\xi\partial\eta'} - \left[\frac{\xi^2}{4\beta}\left(e^{\eta'} - \xi\right)^2 + E\right]\phi = 0 \tag{65}$$

where $\xi = a$ as before and η' is given by

$$\eta' = \ln(a|1+2\beta R|)\,. \tag{66}$$

Equation (65) does not possess factorizable solutions. We seek solutions for large ξ at fixed η'. Approximating the equation to read

$$\frac{1}{6}e^{-\eta'}\frac{\partial^2\phi}{\partial\xi\partial\eta'} - \left(\frac{\xi^4}{4\beta} + E\right)\phi \approx 0 \tag{67}$$

we obtain the solution

$$\phi \approx F\exp\left(6\,D\,e^{\eta'} + \frac{\xi^5}{20\beta D} + \frac{E\xi}{D}\right), \tag{68}$$

where D and F are constants. In terms of the variables a and R we have

$$\phi \approx G\exp\left[6Da\,(1+2\beta R) + \frac{a^5}{20\beta D} + \frac{Ea}{D}\right]. \tag{69}$$

We can also obtain approximate solutions of Equation (65) for $a \to 0$. We get

$$\phi \approx G\exp\left[\frac{Ea}{M} + \;6\,Ma\,(1+2\beta R)\right], \tag{70}$$

where G and M are constants. This solution differs from that of Equation (51) in that the coefficient of R is doubled and in the fact that it holds for $a \to 0$ only in the model at hand.

4. Computing Expectation Values

The expectation value of an operator O in the state Ψ_w is given by

$$< O > = \frac{(\Psi_w, O\,\Psi_w)}{(\Psi_w, \Psi_w)} \tag{71}$$

where the inner product is defined by Equation (38). We are particularly interested in computing the expectation values for the scale factor and the Ricci scalar. However it is difficult to do this in general for Ψ_w given in Equation (58). Instead we shall consider two limiting values of T, namely $T = 0$ and $T \to \infty$ where computations become manageable. We denote by $q(0)$ the value of the variable q given by Equation (56) when $T = 0$ and perform a change of variables from (a, R) to (u, v) given by

$$u = a, \qquad \mathrm{v} = q^2(0) = \frac{36a^2}{\mu^2}(1+\beta R)^2. \tag{72}$$

The Jacobian of the transformation is easily calculated to be

$$\frac{\partial(a,R)}{\partial(u,v)} = \frac{\mu \mathrm{v}^{-1/2}}{12\beta u} \tag{73}$$

and the inner product then reads

$$(\Psi_w, \Psi_w) = \frac{\mu}{12\beta}\int_0^\infty d\mathrm{v}\,\int_0^\infty du\,\mathrm{v}^{-\frac{1}{2}} f(u, \mathrm{v})\,, \tag{74}$$

where

$$f(u,\mathrm{v}) = \frac{|Z|^2}{64}\left(\frac{\mathrm{v}}{p^2}\right)^{3/2} \exp\left(-2\mu u + \frac{\mathrm{v}}{4p}\right)\left[K_{3/4}\left(\frac{\mathrm{v}}{8p}\right) - K_{1/4}\left(\frac{\mathrm{v}}{8p}\right)\right]^2 \ . \tag{75}$$

The factorization of the integrand in Equation (74) into a function of u multiplied by a function v makes the calculations of $< a >$ very simple as it obviates the need of computing the integration over the variable v. The integral over v simply cancels between the numerator and denominator in Equation (71) and we find for $< a >_0$, the expectation value at $T = 0$, the result

$$< a >_0 = \frac{1}{2\mu}\,. \tag{76}$$

We remark that this result is exact and we can interpret this equation to mean that the constant μ is determined by the expectation value of the scale factor at $T = 0$. Calculating $< R >_0$ in this fashion however is not feasible as we do not have the factorization property for the integrand alluded to above.

Next we turn to the case of $T \to \infty$. For this purpose we use the asymptotic form of Ψ_w given by Equation (60). In the expression for the inner product we encounter the following integral over R

$$I = \int_{-\infty}^{\infty} dR\, X(a,R,T) \ , \tag{77}$$

where

$$X(a,R,T) = \{[Da^2(1+\beta R)^2 - T^2]^2 + G^2T^2a^2(1+\beta R)^2\}^{-3/4} \tag{78}$$

and

$$D = \frac{36}{\mu^2} \ , \qquad G = \frac{12}{\mu} \ . \tag{79}$$

We change variable to $R' = 1 + \beta R$ and furthermore let

$$y = \frac{6a}{\mu T} R' \ . \tag{80}$$

The integral I then reduces to the following

$$I = \frac{\mu}{6aT^2}\int_0^{\infty} dy \frac{1}{(1+y^2)^{3/2}}\,, \tag{81}$$

which is readily evaluated to give

$$I = \frac{\mu}{6aT^2} \ . \tag{82}$$

Proceeding in this manner we finally arrive at the following results for the expectation values in the asymptotic limit $T \to \infty$

$$< a > \sim \frac{1}{2\mu} \tag{83}$$

and

$$< R > \sim -\frac{1}{\beta} \ . \tag{84}$$

We remark that $< a >$ in general has non-trivial T dependence as the expression for it in terms of the modified Bessel functions shows. It is however interesting to find that asymptotically, as $T \to \infty$, $< a >$ assumes the same value it did at $T = 0$. It is also interesting that the quantity β which appears in Equation (4) as an arbitrary parameter, acquires a physical meaning in terms of the inverse of the expectation value of the Ricci scalar in the limit $T \to \infty$.

5. Conclusions

In this work we have studied quantization of the R^2 gravity model within the minisuperspace framework with matter being described by a perfect fluid. The velocity potential formalism of Schutz (1970, 1971) was used to describe the dynamics of the fluid. When performing the canonical quantization we employed the scale factor a and the curvature scalar R as our variables and eliminated the second derivative of the scale factor $\ddot{a}$ through integration by parts. This method of quantization gave rise to a Wheeler-DeWitt equation that we were able to solve exactly in the case of flat space-like sections. A different method of quantization due to Vilenkin (1985) was also employed. In this method the equation expressing R in terms of a, Equation (10), is regarded as a constraint and is introduced into the expression for the action using a Lagrange multiplier. The Hamiltonian obtained in this way is different from that arrived at by the first method and we are led to conclude that the two methods of quantization are not equivalent. Moreover the resulting Wheeler-DeWitt equation in the second method does not possess factorizable solutions even for the case of flat space-like sections. We managed to obtain solutions only for asymptotic values of the variables a and R.

The exact wave functions obtained by first method, however, did not possess the requisite behavior as $R \to -\infty$ and hence it was necessary to superpose them in order to obtain wave functions that describe physical states. We managed to construct a wave function that has finite norm and used it to calculate the expectation value of the scale factor $< a >$ at $T = 0$. It proved however to be difficult to compute expectation values for finite non-zero T but we were able to calculate $< a >$ and $< R >$ for large T. It is interesting to note that for $T \to \infty, < a >$ assumes same value it did for $T = 0$. This seems to suggest a contraction scenario in which the universe returns to its initial size when $T \to \infty$.

It is important to clarify the issue of presence or absence of singularities. Since we could not calculate $< a >$ for arbitrary values of $T,$ we cannot deduce that it never vanishes. However the existence or non-existence of singularities may be addressed from another view point (Christodoulakis & Papadopoulos, 1988). Recalling our definition of the inner product given by Equation (38) we can obtain the probability density

$$P = a^2 |\Psi|^2 \tag{85}$$

for the stationary solutions given by Equation(51). It is clear that $P \to 0$ as $a \to 0$ and thus according to this criterion the singularity is avoided for the model at hand.

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