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On the Solution of Spherically Symmetric Static Problem for a Fluid Sphere in General Relativity

V. V. Vasiliev[1] & L. V. Fedorov[1]

[1] Institute of Problems in Mechanics, Russian Academy of Sciences, 101 Vernadskogo, Moscow, Russia

Correspondence: V. V. Vasiliev, Institute of Problems in Mechanics, Russian Academy of Sciences, 101 Vernadskogo, Moscow 119526, Russia. Tel: 7-916-607-1281. E-mail: vvvas@dol.ru

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Abstract

The paper is devoted to the spherically symmetric static problem of General Theory of Relativity (GTR) originally solved by K. Schwarzschild in 1916 for a particular form of the line element. This classical solution specifies the metric tensor for the external and internal semi-Riemannian spaces for a perfect fluid sphere with constant density and includes the so called gravitational radius r_g which is associated with the singular behavior of the solution. The Schwarzschild solution for the external space becomes singular if the sphere radius reaches r_g which is referred to as the radius of the Black Hole event horizon. The solution for the internal space gives infinitely high fluid pressure at the center of sphere with radius equal to 9/8 r_g.

In contrast to the classical solution, the solution presented in the paper is based on the general form of line element for spherically symmetric Riemannian space in which the circumferential component of the metric tensor $\rho^2(r)$ is an arbitrary function of the radial coordinate. As shown, the solution of the static problem exists for a whole class of functions $\rho(r)$. The particular form of this function is determined in the paper under the assumption according to which the gravitation, changing the Euclidean space to the Riemannian space inside the sphere in accordance with GTR equations, does not affect the sphere mass. The solution obtained for the proposed particular form of the line element cannot be singular neither on the sphere surface nor at the sphere center. Direct comparison with the Schwarzschild solution for external and internal spaces is presented.

Keywords: general relativity, spherically symmetric problem, liquid sphere, singularity

1. Introduction

In GTR, the material properties of space are specified by the energy-momentum tensor T_i^j which must satisfy the conservation equation having the following form for a spherically symmetric static problem (Synge, 1960):

$$(T_1^1)' - \frac{2}{r}(T_2^2 - T_1^1) + \frac{h'}{h}(T_1^1 - T_4^4) = 0 \tag{1}$$

in which $(\cdot)' = d(\cdot)/dr$. Equation (1) is written for the traditional form of the line element in spherical coordinates r, θ, φ, i.e., for

$$ds^2 = g^2 dr^2 + r^2(d\theta^2 + \sin^2 \varphi d\varphi^2) - h^2 c^2 dt^2 \tag{2}$$

where $g^2(r)$ and $h^2(r)$ are the components of the metric tensor, and g is referred to as the metric coefficient. The form of the T_i^j tensor depends on the distribution of matter in space. For the sphere with radius R simulated with perfect fluid of constant density μ, the components of the energy-momentum tensor are

$$T_1^1 = T_2^2 = -p\,, \quad T_4^4 = \mu c^2 \tag{3}$$

Here, $p(r)$ is the pressure in the fluid. In the external ($r \geq R$) empty space surrounding the fluid sphere, the energy-momentum tensor vanishes, i.e.,

$$T_1^1 = T_2^2 = T_4^4 = 0 \tag{4}$$

In GTR, the tensor T_i^j is expressed in terms of the Einstein tensor E_i^j which for the spherically symmetric static problem has the following components:

$$\chi T_1^1 = E_1^1 = -\frac{1}{g^2}\left(\frac{2h'}{rh} + \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{5}$$

$$\chi T_2^2 = \chi T_3^3 = E_2^2 = -\frac{1}{g^2}\left[\frac{h''}{h} + \frac{1}{r}\left(\frac{h'}{h} - \frac{g'}{g}\right) - \frac{g'h'}{gh}\right] \tag{6}$$

$$\chi T_4^4 = E_4^4 = -\frac{1}{g^2}\left(\frac{1}{r^2} - \frac{2g'}{rg}\right) + \frac{1}{r^2} \tag{7}$$

The GTR gravitational constant

$$\chi = 8\pi G / c^4 \tag{8}$$

is linked with the classical gravitational constant G. Substitution of Equations (5)–(7) in Equation (1) satisfies this equation identically. So, only three of four Equations (1) and (5) –(7) are mutually independent. The simplest set of equations which is traditionally used for analysis includes Equations (1), (5) and (7), whereas Equation (6) is satisfied identically (Synge, 1960). Substituting Equations (3) in Equations (1), (5) and (7), we get for the internal ($0 \le r \le R$) space

$$p' + \frac{h_i'}{h_i}(p + \mu c^2) = 0 \tag{9}$$

$$\chi p = \frac{1}{rg_i^2}\left(\frac{2h_i'}{h_i} + \frac{1}{r}\right) - \frac{1}{r^2} \tag{10}$$

$$\chi \mu c^2 = \frac{1}{r^2}\frac{d}{dr}\left(r - \frac{r}{g_i^2}\right) \tag{11}$$

Subscript "i" specifies the internal space. The solution of Equation (9) must satisfy the boundary condition on the sphere surface

$$p(r = R) = 0 \tag{12}$$

For the external ($r \ge R$) space, Equations (4) and (5), (7) yield

$$\frac{1}{g_e^2}\left(\frac{2h_e'}{h_e} + \frac{1}{r}\right) - \frac{1}{r} = 0\,, \quad \frac{d}{dr}\left(r - \frac{1}{g_e^2}\right) = 0 \tag{13}$$

Subscript "e" specifies the external space. The solution of Equations (10) and (11) must satisfy the regularity condition at the sphere center $r = 0$, whereas the solution of Equations (13) must reduce to the solution corresponding to the Newton gravitation theory for $r \to \infty$. Moreover, the metric coefficient g for the internal and external spaces must be continuous on the sphere surface, i.e.,

$$g_i(r = R) = g_e(r = R) \tag{14}$$

The solution of Equations (9)–(11) and (13) which was obtained by K. Schwarzschild in 1916 is well known and is described, e.g., by Synge (1960). To demonstrate the properties of this solution calling for the necessity to generalize it, brief derivation of this solution is presented below.

2. Analysis of the Schwarzschild Solution

Consider first the external space. The general solution of Equations (13) is

$$g_e^2 = \frac{1}{1 + C_1 / r}\,, \quad h_e^2 = C_2\left(1 + \frac{C_1}{r}\right) \tag{15}$$

Determine the integration constants C_1 and C_2. For $r \to \infty$, Equations (15) must reduce to the solution of the Newton gravitation theory (Landau & Lifshitz, 1988), which depends on the sphere mass m through the gravitational potential φ_e, i.e.,

$$g_e^2 = 1 - \frac{2\varphi_e}{c^2}\,, \quad h_e^2 = 1 + \frac{2\varphi_e}{c^2}\,, \quad \varphi_e = -\frac{Gm}{r} \tag{16}$$

Then, Equations (15) become

$$g_e^2 = \frac{1}{1 - r_g / r}, \qquad h_e^2 = 1 - \frac{r_g}{r} \tag{17}$$

where

$$r_g = \frac{2mG}{c^2} \tag{18}$$

is the gravitational radius. As known (Thorne, 1994), the expression for r_g in Equation (18) was originally derived by J. Michel in 1783 and P. Laplace in 1796 from the classical expression for the escape velocity for the sphere with mass m and radius R, i.e.,

$$v = \sqrt{2mG / R}$$

Taking $v = c$, we get $R = r_g$ and can conclude that r_g is the radius of the sphere for which the escape velocity is equal to the velocity of light. Thus, one of the fundamental GTR parameters, i.e. the gravitational radius, actually follows from the Newton gravitation theory which does not look natural.

As follows from Equations (17), g_e becomes singular and h_e is zero if r reaches the gravitational radius r_g which is referred to as the radius of the Black Hole event horizon (Frolov & Zelnikov, 2011). The history of this effect is discussed elsewhere (Thorne, 1994). Particularly, Einstein did not consider the singularity as a physical phenomenon and restricted GTR equations to the spaces for which $r \geq 1.5\, r_g$. As has been further proved (Singe, 1960; Vasiliev, & Fedorov, 2012), the surface with radius r_g is located inside the sphere. Hence, the solution in Equations (17) which are valid for the external space only is not singular. Moreover, the singularity in the space component of the metric tensor in Equations (17) can be eliminated by the proper selection of the coordinate frame (Feinman, Morinigo & Wagner, 1995). Nonsingular solutions of the problem under study (e.g., Logunov, 2006; Hynecek, 2012) have been obtained within the framework of the gravitational theories which are different from GTR.

Consider the internal space for which $\mu = const$, and the solution of Equation (11) is

$$g_i^2 = \frac{1}{1 - (\chi \mu c^2 / 3) r^2 + C_3 / r} \tag{19}$$

From the regularity condition at $r = 0$ we get $C_3 = 0$ (Misner, Thorne, & Wheeler, 1973). The metric coefficient g must be continuous on the sphere surface $r = R$, i.e., it must satisfy Equation (14). Substituting g from Equations (17) and (19) in Equation (14), we get the following relationship:

$$\frac{\chi}{3} \mu c^2 R^3 = r_g \tag{20}$$

Then, Equation (19) can be reduced to

$$g_i^2 = \frac{1}{1 - r_g r^2 / R^3} \tag{21}$$

For $r_g = 0$, Equations (17) and (21) specify the metric coefficients of the Euclidean space. If we substitute r_g from Equation (18) in Equation (20) and take into account Equation (8) for χ, we arrive at the following expression for sphere mass:

$$m = (4/3)\pi \mu R^3 \tag{22}$$

This expression corresponds to the Euclidean space. However, the space inside the sphere is not Euclidean in GTR. For the metric coefficient in Equation (21), we get the following expression for the sphere mass:

$$m = 2\mu \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\varphi d\varphi \int_0^R g_i r^2 dr = \frac{2\pi}{\bar{r}_g} \mu R^3 \left(\frac{1}{\sqrt{\bar{r}_g}} \sin^{-1} \sqrt{\bar{r}_g} - \sqrt{1 - \bar{r}_g} \right) \tag{23}$$

in which $\bar{r}_g = r / R$. If $\bar{r}_g$ is much less than unity, we have approximately

$$m = \frac{4}{3} \pi \mu R^3 F(\bar{r}_g) \tag{24}$$

where

$$F(\bar{r}_g) = 1 + \frac{3}{10}\bar{r}_g + \frac{9}{56}\bar{r}_g^2 + \cdots$$

As follows from Equation (24), the sphere mass can be specified by Equation (22) only if $\bar{r}_g = 0$ which corresponds to the Euclidean space. But the sphere mass in GTR equations must correspond to the Riemannian space and be specified by Equation (23) rather than by Equation (22). However, if we apply Equation (23) to calculate the mass, the compatibility condition at the sphere surface in Equation (14) cannot be satisfied by the solutions in Equations (17) and (19). Note that these conditions can be formally satisfied for the sphere with the mass in Equation (22) if the expressions for the gravitational constant and radius in Equations (8) and (18) are generalized as (Vasiliev & Fedorov, 2013)

$$\chi = \frac{8\pi G}{c^4}F(\bar{r}_g), \quad \bar{r}_g = \frac{2mG}{Rc^2}F(\bar{r}_g)$$

where m is given by Equation (22).

Thus, if Equation (8) for the gravitational constant and Equations (18) and (22) following from the Schwarzschild solution are valid, either the metric coefficients of the external and internal spaces are compatible, but the sphere mass corresponds to the Euclidean space, or the sphere mass corresponds to the Riemannian space, but the metric coefficient g is not continuous on the sphere surface.

Finally, determine the pressure acting inside the fluid sphere. Using Equations (10) to express h'/h and Equation (21) for g_i, we can transform Equation (9) to the following form:

$$p' + \frac{\mu c^2 r_g r}{2(R^3 - r_g r^2)}\left(1 + \frac{3p}{\mu c^2}\right)\left(1 + \frac{p}{\mu c^2}\right) = 0 \tag{25}$$

Assume that $p/\mu c^2$ is much less than unity and that r_g is much less than R. Neglecting small terms, we can simplify Equation (25) as

$$p' + \frac{\mu c^2 r_g r}{2R^3} = 0 \tag{26}$$

This equation corresponds to the Newton gravitation theory (Love, 1927). The solution of Equation (26) which satisfies the boundary condition in Equation (12) is

$$p = \frac{\mu c^2 r_g}{4R}\left(1 - \frac{r^2}{R^2}\right) \tag{27}$$

Return to the general case. The general solution of Equation (25) can be written as

$$p(r) = -\mu c^2 \frac{\sqrt{1 - r_g r^2 / R^3} - C_4}{\sqrt{1 - r_g r^3 / R^3} - 3C_4} \tag{28}$$

Satisfying the boundary condition in Equation (12), we arrive at the following final expression for the pressure (Synge, 1960):

$$p(r) = -\mu c^2 \frac{\sqrt{1 - r_g r^2 / R^3} - \sqrt{1 - r_g / R}}{\sqrt{1 - r_g r^2 / R^3} - 3\sqrt{1 - r_g / R}} \tag{29}$$

Consider the pressure at the sphere center. Taking $\bar{r} = 0$ in Equation (29), we get

$$p(0) = -\mu c^2 \frac{1 - \sqrt{1 - r_g / R}}{1 - 3\sqrt{1 - r_g / R}} \tag{30}$$

As known (Weinberg, 1972), the denominator of this expression becomes zero at $R = R_s = 9/8 r_g = 1.125 r_g$, and the pressure at the sphere center is infinitely high if the sphere radius becomes equal to R_s. It could be natural to expect that the singularity occurs if the sphere radius is equal to the gravitational radius. But $R_s > r_g$, and no explanation of this result can be found in the existing literature. Moreover, the pressure specified by Equation (30) formally exists if the sphere radius is less than R_s. For example for $R = 1.1\ r_g$, Equation (30) yields $p(0) = -8.36\ \mu c^2$. This result has no physical meaning, because both the pressure and the density must be positive. And finally, consider the case $R = r_g$. Taking $r = R$ in Equation (28), we find $p = -\mu c^2/3$. Thus, the pressure cannot be zero on the sphere surface as is required by the boundary condition in Equation (12). To explain this result, consider Equation (26) which is the equilibrium equation corresponding to the Newton gravitation theory. The second term in this equation is the body force induced by the gravitation field. Comparing Equations (26) and (25), we can conclude that the second term of Equation (25) is analogous to the body force in Equation (26), and this term becomes infinitely high on the sphere surface if the radius of this surface becomes equal to r_g. However, according to the boundary condition in Equation (12), the pressure must be zero on the sphere surface. Thus, the body force on the surface is infinitely high, whereas the pressure induced by this force must be zero. Naturally, the problem in which the structure of the governing equation is in direct contradiction with the boundary condition has no solution. Usually, such situation takes place if the mathematical model of the phenomenon under study does not correspond to its physical nature. The mathematical model of the Schwarzschild solution is determined by Equation (2) specifying the geometry of the semi-Riemannian space. The space is not quite Riemannian because the circumferential metric coefficient is equal to r^2 which corresponds to the Euclidean space. So, it looks natural to apply a more general form of the line element to study the problem.

3. General Solution of a Spherically Symmetric Static Problem

Consider the following form of the line element generalizing Equation (1) as

$$ds^2 = g^2dr^2 + \rho^2(d\theta^2 + \sin^2\varphi d\varphi^2) - h^2c^2dt^2 \tag{31}$$

Here, $\rho(r)$ is some unknown function which, in general, is not equal to r. For the metric form in Equation (31), the governing equations for the internal space ($0 \le r \le R$), Equations (9)–(11), become (Singe, 1960)

$$p' + \frac{h_i'}{h_i}(p + \mu c^2) = 0 \tag{32}$$

$$\chi p = \frac{1}{g_i^2}\frac{\rho_i'}{\rho_i}\left(\frac{\rho_i'}{\rho_i} + \frac{2h_i'}{h_i}\right) - \frac{1}{\rho_i^2} \tag{33}$$

$$\chi\mu c^2 = -\frac{1}{g_i^2}\left[\left(\frac{\rho_i'}{\rho_i}\right)^2 + \frac{2\rho_i''}{\rho_i} - \frac{2\rho_i' g_i'}{\rho_i g_i}\right] + \frac{1}{\rho_i^2} \tag{34}$$

Consider Equation (34) which can be reduced to

$$\chi\mu c^2\rho_i^2 = 1 - \frac{1}{\rho_i'}\frac{d}{dr}\left[\frac{(\rho_i')^2\rho_i}{g_i^2}\right] \tag{35}$$

Integration yields the following general solution:

$$g_i^2 = \frac{\rho_i(\rho_i')^2}{\rho_i - (\chi\mu c^2/3)\rho_i^3 + C_5} \tag{36}$$

Here, C_5 is the integration constant. Assume that $\rho_i(r = 0) = 0$. Then, using, as earlier, the regularity condition at the sphere center, we get $C_5 = 0$, and Equation (36) reduces to

$$g_i^2 = \frac{(\rho_i')^2}{1 - (\chi\mu c^2/3)\rho_i^2} \tag{37}$$

Using Equation (33), we get the following equation for the coefficient h_i:

$$\frac{h_i'}{h_i} = \frac{\rho_i}{2\rho_i'}\left[g_i^2\left(\frac{1}{\rho_i^2} + \chi p\right) - \left(\frac{\rho_i'}{\rho_i}\right)^2\right] \tag{38}$$

Consider the external space ($r \ge R$) for which $\mu = 0$ and Equations (13) are generalized as (Singe, 1960)

$$\frac{\rho_e'}{g_e^2}\left(\frac{2h_e'}{h_e} + \frac{\rho_e'}{\rho_e}\right) - \frac{1}{\rho_e} = 0, \quad \frac{d}{dr}\left[\rho_e'\left(\rho_e - \frac{\rho_e'}{g_e^2}\right)\right] = 0$$

The general solutions of these equations are

$$g_e^2 = \frac{\rho_e(\rho_e')^2}{\rho_e + C_6}, \qquad h_e^2 = C_7\frac{\rho_e + C_6}{\rho_e} \tag{39}$$

Assume that $\rho_e(r) \to r$ for $r \to \infty$. Then, Equations (39) must reduce to Equations (16) if $r \to \infty$. Determining the integration constants C_6 and C_7, we get the following final expressions for the metric coefficients of the external space:

$$g_e^2 = \frac{\rho_e(\rho_e')^2}{\rho_e - r_g}, \qquad h_e^2 = 1 - \frac{r_g}{\rho_e} \tag{40}$$

in which r_g is specified by Equation (18).

As can be seen, the components of the metric tensor metric coefficients g^2 and h^2 entering Equation (31) are expressed in terms of function $\rho(r)$. It is important that GTR does not provide the equation for this function. The fact that GTR equations do not allow us to obtain the unique solution for the metric tensor is known. Moreover, some additional coordinate conditions (e.g., de Donder-Fock conditions) are used to supplement the GTR equations for the external space (Belinfanter, 1955; Fock, 1959).

Thus, we need to determine the continuous function $\rho(r)$ which allows us to satisfy the boundary conditions similar to Equations (14), i.e.,

$$\rho_i(r=R) = \rho_e(r=R), \quad g_i(r=R) = g_e(r=R) \tag{41}$$

The particular solution which satisfies these conditions is presented below.

4. Particular Solution

Recall that the conditions in Equations (41) or (14) are met by the traditional solutions in Equations (17) and (21) if χ and r_g are specified by Equations (8) and (18) and satisfy Equation (20) which, in turn, results in Equation (22) for the sphere mass corresponding to the Euclidean space. In connection with this, introduce the following basic assumption: gravitation changes the space geometry inside the solid in accordance with GTR equations, but does not affect the solid mass. This assumption can be supported by the following reasoning. Consider a solid body whose internal space is Riemannian because of gravitation and assume that the second solid body appears in the vicinity of the first one. Then, the gravitation induced by the second body changes the geometry of space inside the first body, but it does not change the body mass. For the metric form in Equation (31), the sphere mass is

$$m = 2\mu \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\varphi d\varphi \int_0^R g_i \rho_i^2 dr$$

If we take here

$$g_i = \frac{r^2}{\rho_i^2}, \tag{42}$$

the sphere mass becomes equal to the mass specified by Equation (22) corresponding to the Euclidean space. Then, Equation (20) is valid, and Equation (37) takes the form

$$g_i^2 = \frac{(\rho_i')^2}{1 - r_g\rho_i^2 / R^3} \tag{43}$$

Now, we can use Equations (42) and (43) to determine the function $\rho_i(r)$ from the following equation:

$$\frac{\rho_i'\rho_i^2}{\sqrt{1 - r_g\rho_i^2 / R^3}} = r^2 \tag{44}$$

The solution of this equation is

$$\frac{1}{\sqrt{\bar{r}_g}} \sin^{-1}(\bar{\rho}_i\sqrt{\bar{r}_g}) - \bar{\rho}_i\sqrt{1 - \bar{\rho}_i^2\bar{r}_g} = \frac{2}{3}\bar{r}_g\bar{r}^3 + f_i(\bar{r}_g) \tag{45}$$

where $f_i(\bar{r}_g)$ is the integration function and

$$\bar{r}_g = \frac{r_g}{R}, \quad \bar{\rho} = \frac{\rho}{R}, \quad \bar{r} = \frac{r}{R} \tag{46}$$

Presenting the left-hand side of Equation (45) with the power series, we get

$$\bar{r}_g\left(\frac{2}{3}\bar{\rho}_i^3 + \frac{1}{5}\bar{r}_g\bar{\rho}_i^5 + \ldots\right) = \frac{2}{3}\bar{r}_g\bar{r}^3 + f_i(\bar{r}_g) \tag{47}$$

Using the condition at the sphere center, i.e. $\bar{\rho}_i(\bar{r}=0)=0$, we can conclude that $f_i(\bar{r}_g)=0$. Then, Equation

(47) yields $\bar{\rho}_i = \bar{r}$ for $\bar{r}_g = 0$ which corresponds to the Euclidean space. Introducing a new variable

$$u = \sin^{-1}(\bar{\rho}_i \sqrt{\bar{r}_g}), \tag{48}$$

we can transform Equation (45) to the following final form:

$$u - \frac{1}{2}\sin 2u = \frac{2}{3}\bar{r}_g \bar{r}^3 \sqrt{\bar{r}_g} \tag{49}$$

Consider the external space for which we need to determine the function $\rho_e(r)$ and to satisfy the compatibility conditions in Equations (41). To meet these conditions, assume that Equation (42) is valid not only for the internal space, but for the external space as well, i.e.,

$$g_e = \frac{r^2}{\rho_e^2} \tag{50}$$

This result, in conjunction with Equation (42), shows that if the first condition in Equations (41) is satisfied, the second condition is satisfied automatically. Thus, we need to meet only one compatibility condition on the sphere surface, i.e., $\rho_i = \rho_e$. Substituting the first equation of Equations (40) in Equation (52), we arrive at the following equation for ρ_e:

$$\frac{\rho_e^/ \rho_e^2}{\sqrt{1 - r_g / \rho_e}} = r^2 \tag{51}$$

In the notations specified by Equations (46), the solution of this equation is

$$\left(\frac{1}{3}\bar{\rho}_e^2 + \frac{5}{12}\bar{r}_g \bar{\rho}_e + \frac{5}{8}\bar{r}_g^2\right)\sqrt{\bar{\rho}_e(\bar{\rho}_e - \bar{r}_g)} + \frac{5}{8}\bar{r}_g^3 \ln\left(\sqrt{\bar{\rho}_e} + \sqrt{\bar{\rho}_e - \bar{r}_g}\right) = \frac{1}{3}\bar{r}^3 + f_e(\bar{r}_g) \tag{52}$$

To determine the integration function $f_e(\bar{r}_g)$, apply the first compatibility condition in Equations (41). Taking $\bar{r} = 1$ and $\bar{\rho}_i(\bar{r} = 1) = \bar{\rho}_e(\bar{r} = 1) = \bar{\rho}_1$, we get the following expression:

$$f_e(\bar{r}_g) = \left(\frac{1}{3}\bar{\rho}_1^2 + \frac{5}{12}\bar{r}_g \bar{\rho}_1 + \frac{5}{8}\bar{r}_g^2\right)\sqrt{\bar{\rho}_1(\bar{\rho}_1 - \bar{r}_g)} + \frac{5}{8}\bar{r}_g^3 \ln\left(\sqrt{\bar{\rho}_1} + \sqrt{\bar{\rho}_1 - \bar{r}_g}\right) - \frac{1}{3} \tag{53}$$

Dividing Equation (52) by $\bar{\rho}_e^3$, we can prove that $\bar{\rho}_e \to \bar{r}$ for $\bar{\rho}_e \to \infty$ which, in conjunction with Equation (50) providing $g_e(\rho_e \to r) = 1$, means that the external space reduces to the Euclidean space with the infinite increase of the radial coordinate.

Thus, Equations (42), (48), (49) and (50), (52), (53) satisfy the GTR equations, as well as required asymptotic and boundary conditions. Perform the analysis of the obtained solution. Consider first Equation (49). Taking $\bar{r} = 1$ and putting $\bar{\rho}_i(\bar{r} = 1) = \bar{\rho}_1$, we can plot the dependence of the metric coefficient $\bar{\rho}_1$ on the normalized gravitational radius $\bar{r}_g$ which is shown in Figure 1. Second, consider Equation (53). As follows from this equation, the function $f_e(\bar{r}_g)$ is real if $\bar{\rho}_1 \geq \bar{r}_g$. The line corresponding to $\bar{r}_g$ is also plotted in Figure 1. The coordinates of the intersection point $\bar{\rho}_1^u = \bar{r}_g^u = 0.896789$ specify the ultimate values of $\bar{\rho}_1$ and $\bar{r}_g$. These coordinates correspond, in accordance with Equations (46), to the sphere radius $R_g = 1.115\ r_g$. For $R < R_g$, the obtained solution becomes imaginary.

So, we have found some critical radius of the sphere $R = R_g$ which is similar to the gravitational radius r_g. However, first, R_g does not coincide with r_g and, second, if the sphere radius R becomes equal or less than R_g, the solution does not demonstrate singular behavior – the real solution does not exist. Dependences $\bar{\rho}(\bar{r})$ are shown in Figure 2(a) for $\bar{r}_g = 0.5$ and $\bar{r}_g = 0.88$ which is close to $\bar{r}_g^u$. The straight line $\bar{r}$ corresponds to the Schwarzschild solution. Dependences $g(\bar{r})$ are presented in Figure 2(b) for $\bar{r}_g = 0$ (Euclidean space), $\bar{r}_g = 0.5$ and $\bar{r}_g = 0.88$. Solid lines correspond to the obtained solution, whereas dashed lines – to the Schwarzschild solution. As can be seen, the obtained solution does not demonstrate the tendency to singular behavior on the sphere surface.

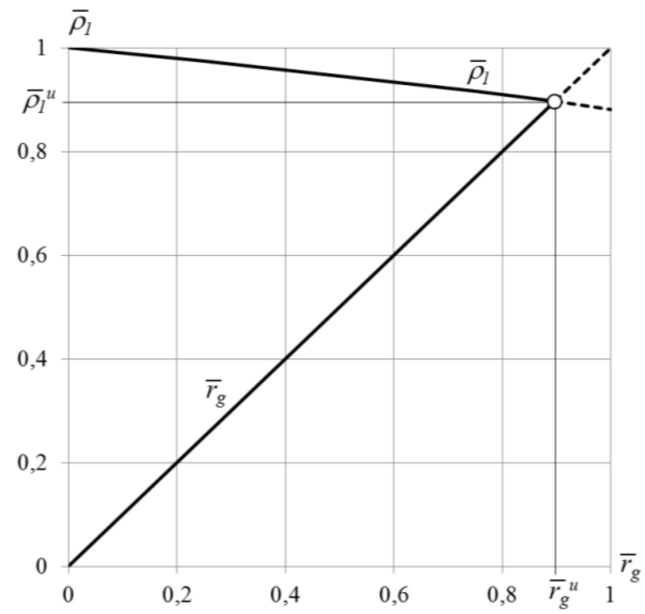
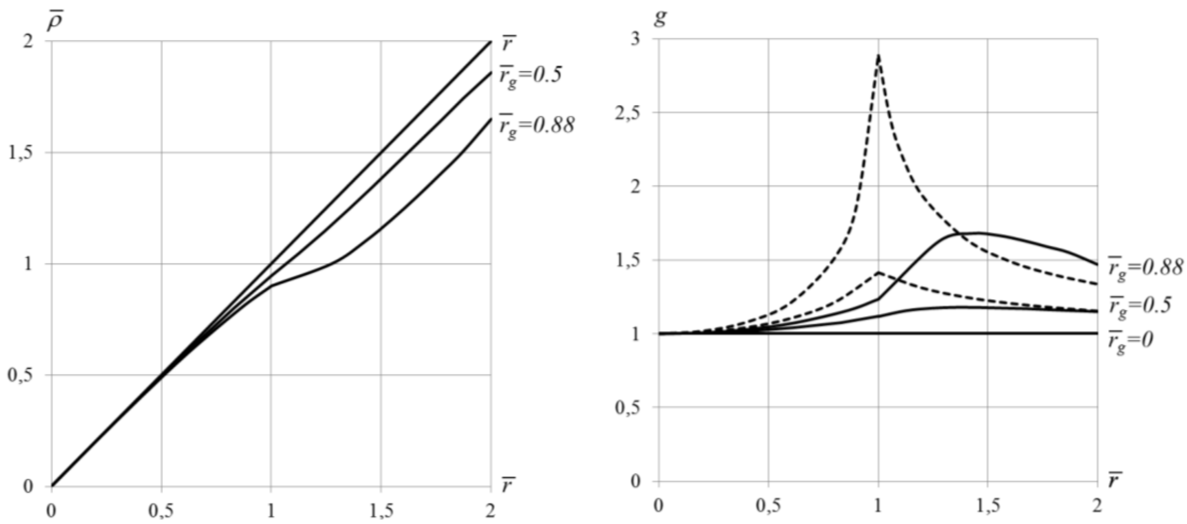


Figure 1. Dependence of $\overline{\rho}_1$ on $\overline{r}_g$



(a) (b)

Figure 2. Dependences of the metric coefficients (a) $\overline{\rho}$ and (b) g on $\overline{r}$

Finally, determine the pressure acting in the fluid sphere. The conservation equation which is analogous to Equation (25) and follows from Equations (32), (38) and (43) with allowance for Equations (8) and (20) can be presented as

$$\frac{dp}{d\rho_i}+\frac{\mu c^2 r_g \rho_i}{2(R^3-r_g\rho_i^2)}\left(1+\frac{3p}{\mu c^2}\right)\left(1+\frac{p}{\mu c^2}\right)=0 \tag{54}$$

The solution of Equation (54) which has the same form that the traditional equation, Equation (25), but includes ρ_i instead of r as the independent variable is similar to Equation (28), i.e.,

$$p(\rho_i)=-\mu c^2\,\frac{\sqrt{1-r_g\rho_i^2/R^3}-C_8}{\sqrt{1-r_g\rho_i^2/R^3}-3C_8}$$

The boundary condition on the sphere surface is $p(\rho_1) = 0$ in which, as earlier, $\rho_1 = \rho(r = R)$. Determining the constant C_8, we finally get

$$p(\rho_i) = -\mu c^2 \frac{\sqrt{1 - r_g \rho_i^2 / R^3} - \sqrt{1 - r_g \rho_1^2 / R^3}}{\sqrt{1 - r_g \rho_i^2 / R^3} - 3\sqrt{1 - r_g \rho_1^2 / R^3}} \tag{55}$$

The pressure at the sphere center $(\rho_i = 0)$ is

$$p(0) = -\mu c^2 \frac{1 - \sqrt{1 - r_g \rho_1^2 (r_g) / R^3}}{1 - 3\sqrt{1 - r_g \rho_1^2 (r_g) / R^3}} \tag{56}$$

The dependence $\rho_1(r_g)$ entering this equation is presented in Figure 1. Normalized pressure at the sphere center $\overline{p} = p / \mu c^2$ is shown in Figure 3 as a function of the dimensionless gravitational radius. Dashed curve corresponds to the Schwarzschild solution in Equation (30) which demonstrates singular behavior for $\overline{r}_g = 8/9$. Solid line corresponds to the solution in Equation (56). The maximum pressure takes place for $\overline{r}_g = \overline{r}_g^u$ and is finite ($\overline{p} = 0.725 \mu c^2$). Distribution of the normalized pressure on the radial coordinate for $\overline{r}_g = 0.5$ and $\overline{r}_g = 8/9$ is presented in Figure 4. Dotted line corresponds to Equation (27), i.e., to the Newton theory, dashed lines correspond to the Schwarzschild solution in Equation (29), and solid lines – to Equation (55). As can be seen, the obtained solution does not demonstrate singular behavior.

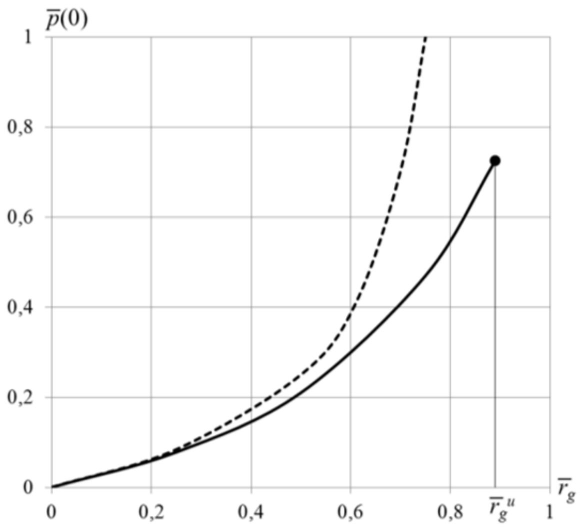


Figure 3. Dependences of pressure at the sphere center on $\overline{r}_g$

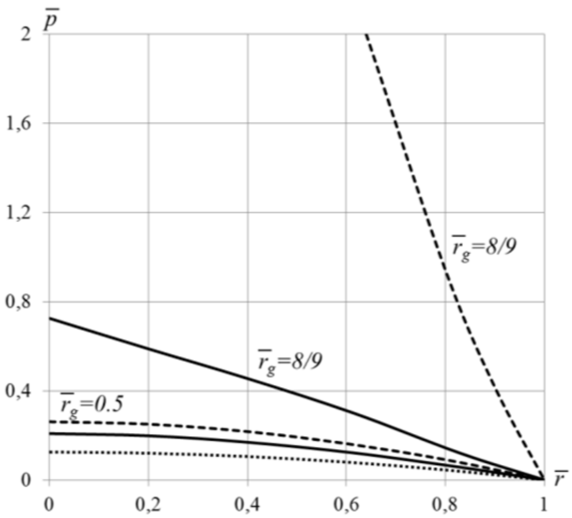


Figure 4. Distribution of pressure over the sphere radius

5. Conclusion

Thus, the general line element form of the spherically symmetric Riemannian space in Equation (31), in contrast to the traditional form in Equation (2), allows us to construct the space whose metric coefficients and pressure in the fluid sphere do not demonstrate singular behavior. The solution specifes the minimum possible radius R_g of the fluid sphere which does not coincide with the classical gravitational radius r_g. For the sphere with radius $R < R_g$, the solution of GTR equations becomes imaginary.

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