Electromagnetic Pressure and the Lorentz Force

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Abstract

Two seldom used concepts, electric and magnetic pressure, have been applied to the classical problem of characterizing the force exerted on a charged particle by external electric and magnetic fields. In terms of fundamental natural laws such as the Coulomb's and magnetic ones (Lorentz), a generalization of the electrostatic and magnetostatic energy densities is obtained.

Keywords: Coulomb Law, electrostatic energy density, electric pressure

1. Introduction

The concept of electrostatic energy (Roche, 2003) is often controversial due to certain arbitrariness in the amount of charge assigned to the volume elements used in its calculation. In particular, it can be shown that the energy density (Jackson, 1975) is conceptually equivalent to the electric pressure (Boast, 1964) or, in certain cases, to the magnetic one (Boast, 1964; White, 2001) acting on charged particles moving within the respective pressure producing fields.

The term magnetic pressure is familiar to the circle of magnetic plasma confinement specialists. Plasmas (high temperature ionized gases) are the raw material of fusion nuclear energy research, whose current state of development mainly relies on tokamak reactors (Wesson, 1997). These are electromagnetic devices intended to produce, confine and preserve high temperature (~5 to 10 [KeV]) plasmas, mainly by means of magnetic fields. The equilibrium between the expansion forces of a plasma column and the magnetic forces applied to it is often expressed in terms of magnetic pressures. Yet, the term "electric pressure" (Boast, 1964) remains, at large, unknown.

Magnetic pressure is usually written as $B^2/(2\mu)$ while the electric one is $\varepsilon E^2/2$ (Boast, 1964; White, 2001). It should be emphasized that these expressions coincide with those of the respective energy densities. In other words, whenever some energy density, magnetic or electric, is present in a region then an electric or magnetic pressure is exerted on the charged particles found there.

The present work sets out to show that, in those cases of more than one degree of freedom, 3D clearly included, the factor 1/2 present both in the electric energy density and in the pressure, should be written as $d/2$ instead. Where, d indicates the degree freedom number. In (Alonso & Fin, 1967) example 19.1 points out that the energy density E_E is equivalent to the radiation pressure, there d is equal to 2 associated to a perfect absorber and equal to 4 associated to a perfect reflector. Such reconsideration, consistent with the energy equipartition theorem (Reif, 1969), is never addressed by the traditional treatment of mainstream textbooks (Jackson, 1975; Feynman, 1964; Purcell, 1965). Therefore, the fundamental Coulomb's and Lorentz magnetic part laws will be invoked in the classical case of a charged particle placed in an externally created uniform electric and magnetic fields. Then a novel treatment will follow, where the concept of electric and magnetic pressure is incorporated.

2. Charge q in an External Electric Field

The expression of electrostatic energy density most commonly found in textbooks, such as Jackson's (cf. Equation (1.55) (Jackson, 1975)) can be applied (albeit with a different coefficient, equivalent to the electric pressure (Boast, 1964)) to the expression of the force on a charge q uniformly distributed in the volume of a non conductive sphere with radius a, submerged in a uniform external electric field.

Figure 1 presents an electric charge q uniformly distributed in the volume of a non conductive sphere with radius

a, centred at the origin and submersed in a uniform external electric field $\bar{E}_0 = E_0 \hat{j}$ where $\hat{j}$ is the unit vector in the *y*-direction.

Figure 1. An electric charge *q* is immersed in a uniform external electric field $\bar{E}_0$. $\bar{E}_q = q /\left(4\pi\varepsilon a^2\right)\hat{r}$ is the radial field of the charge itself

According to Coulomb's law, the force on the charge *q* is

$$\bar{F} = q\,\bar{E}_0 \tag{1}$$

which can be calculated from the pressure exerted by the field on the sphere. The radial electric field associated to *q*, on the sphere surface and added to the external field is

$$\bar{E} = q /\left(4\pi\varepsilon a^2\right)\hat{r} + E_0\hat{j} \tag{2}$$

Here $\hat{r}$ is a radial unit vector in a spherical system of coordinates (r,θ,ϕ) concentric to the sphere, so that

$$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k} \tag{3}$$

Thus, the total electric field on the sphere surface resulting from substituting expression (3) into (2) is

$$\bar{E} = q /\left(4\pi\varepsilon a^2\right)\sin\theta\cos\varphi\,\hat{i} + \left(E_0 + q /\left(4\pi\varepsilon a^2\right)\sin\theta\sin\varphi\right)\hat{j}$$

$$+ q /\left(4\pi\varepsilon a^2\right)\cos\theta\,\hat{k} \tag{4}$$

Notice that, according to this expression, we are dealing with three degrees of freedom. Then, the total electric pressure can be written as

$$p_E = k\,E^2 \tag{5}$$

where k is a constant.

A differential of force on the surface is then

$$d\bar{F} = p_E d\bar{s} \tag{6}$$

This differential of area being, in spherical coordinates,

$$d\overline{s} = a^2 \sin\theta \, d\theta \, d\phi \, \hat{r} \tag{7}$$

then the square electric field from Equation (4) becomes

$$E^2 = \left[q / \left(4\pi\varepsilon a^2 \right) \right]^2 + E_0^2 + 2\left[q / \left(4\pi\varepsilon a^2 \right) \right] E_0 \sin\theta \sin\phi \tag{8}$$

the electric pressure can be expressed by

$$p_E = k\left\{ \left[q / \left(4\pi\varepsilon a^2 \right) \right]^2 + E_0^2 + 2\left[q / \left(4\pi\varepsilon a^2 \right) \right] E_0 \sin\theta \sin\phi \right\} \tag{9}$$

Meanwhile, by substituting Equation (3) into (7) and then into (6), an explicit differential of force is obtained as

$$dF = ka^2 \{ \, [\left(q / 4\pi\varepsilon a^2 \right)^2 \sin^2\theta \cos\varphi + E_0^2 \sin^2\theta \cos\varphi$$

$$+ 2E_0 \left(q / 4\pi\varepsilon a^2 \right) \sin^3\theta \sin\varphi \cos\varphi \,] d\theta \, d\varphi \, \hat{i}$$

$$+ [\left(q / 4\pi\varepsilon a^2 \right)^2 \sin^2\theta \sin\phi + E_0^2 \sin^2\theta \sin\phi$$

$$+ 2E_0 \left(q / 4\pi\varepsilon a^2 \right) \sin^3\theta \sin^2\varphi \,] d\theta \, d\varphi \, \hat{j}$$

$$+ [\left(q / 4\pi\varepsilon a^2 \right)^2 \cos\theta \sin\theta + E_0^2 \cos\theta \sin\theta$$

$$+ 2E_0 \left(q / 4\pi\varepsilon a^2 \right) \sin^2\theta \cos\theta \sin\phi] \, d\theta \, d\phi \, \hat{k} \} \tag{10}$$

The integral of most of the terms in all three directions $\hat{i}$, $\hat{j}$ and k disappear, only one term in the direction $\hat{j}$ survives. The integral of the $\hat{i}$ component turns out to be

$$ka^2 \{ \left[\left(q / 4\pi\varepsilon a^2 \right)^2 + E_0^2 \right] \int_0^\pi \sin^2\theta \, d\theta \int_0^{2\pi} \cos\varphi \, d\varphi +$$

$$2E_0 \left(q / 4\pi\varepsilon a^2 \right) \int_0^\pi \sin^3\theta \, d\theta \int_0^{2\pi} \sin\phi \cos\phi \, d\phi \} \tag{11}$$

Given that $\int_0^{2\pi} \cos\phi \, d\phi = 0$ and $\int_0^{2\pi} \sin\phi \cos\phi \, d\phi = 0$, the integral of the $\hat{i}$ component is zero. The integral of the $\hat{k}$ component is

$$ka^2 \{ \left[\left(q / 4\pi\varepsilon a^2 \right)^2 + E_0^2 \right] \int_0^\pi \cos\theta \sin\theta \, d\theta \int_0^{2\pi} d\varphi +$$

$$2E_0 \left(q / 4\pi\varepsilon a^2 \right) \int_0^\pi \sin^2\theta \cos\theta \, d\theta \int_0^{2\pi} \sin\phi \, d\phi \} \tag{12}$$

Now, again $\int_0^\pi \cos\theta \sin\theta \, d\theta = 0$ and $\int_0^{2\pi} \sin\phi \, d\phi = 0$, the integral of the $\hat{k}$ component is zero.

Only one term survives in the integral of the $\hat{j}$ component that turns out to be

$$ka^2 \left\{ \left[\left(q/4\pi\varepsilon a^2 \right)^2 + E_0^2 \right] \int_0^\pi \sin^2\theta \, d\theta \int_0^{2\pi} \sin\varphi \, d\varphi + \right.$$

$$\left. 2E_0 \left(q/4\pi\varepsilon a^2 \right) \int_0^\pi \sin^3\theta \, d\theta \int_0^{2\pi} \sin^2\phi \, d\phi \right\} \tag{13}$$

Given that $\int_0^{2\pi} \sin\phi \, d\phi = 0$, the first two terms are zero. In the last term, evaluating the two integrals as $\int_0^\pi \sin^3\theta \, d\theta = 4/3$ and $\int_0^{2\pi} \sin^2\phi \, d\phi = \pi$ whereby

$$F = k\, 2E_0 q / \left(3\varepsilon \right) \tag{14}$$

Back to Coulomb's law Equation (1), one can compare it to Equation (14) so to conclude that

$$k = 3\varepsilon / 2 \tag{15}$$

Thus, the electric pressure (and, by an entirely equivalent procedure, the magnetic one) can be written in three degrees of freedom as

$$p_E = 3\varepsilon E^2 / 2 \tag{16}$$

which, in CGS units, evolves into

$$p_E = 3\left(E^2 / \left(4\pi \right) \right) / 2 \tag{17}$$

This expression should correspond to Jackson's (1.55) (Jackson, 1975).

3. Charge *q* in an External Magnetic Field

The complementary magnetic pressure compelling the charge q to move with constant velocity $|\overline{v}|$ in a uniform external magnetic field will be proven to be

$$p_B = 3B^2 / \left(2\mu \right) \tag{18}$$

given the magnetic contribution from the Lorentz force

$$\overline{F} = q\overline{v} \times \overline{B}_0 \tag{19}$$

playing the role of Equation (1) in the electric pressure case. It is immediate that the analogous to Equation (5) would now be

$$p_B = \mathrm{K}B^2 \tag{20}$$

We can proof that the magnetic pressure, identical to the magnetic energy density, must contain a factor 3 in a three degree of freedom problem, as follows:

Assume a charged sphere shown in Figure 1, which moves with a non relativistic velocity $\overline{v} = v\hat{j}$ within a

uniform external magnetic field $\bar{B}_0 = B_0\,\hat{k}$. The charge in motion produces on the sphere a magnetic field (Feynman, 1964; Alonso, 1967)

$$\bar{B}_q = Q\cos\theta\,\hat{i} - Q\sin\theta\cos\phi\,\hat{k} \tag{21}$$

whose field lines form circumferences concentric to the vector $\bar{v}$, so that $Q = \mu q v / \left(4\pi a^2\right)$, as seen below in Figure 2.

Figure 2. Electric charge moving with non relativistic velocity $\bar{v}$ in a uniform magnetic field $\bar{B}_0$

The magnetic field on the sphere that results from adding the field produced by its own charge to the uniform field is

$$\bar{B} = Q\cos\theta\,\hat{i} + \left(B_0 - Q\sin\theta\cos\phi\right)\hat{k} \tag{22}$$

whereby

$$B^2 = B_0^{\ 2} + Q^2\cos^2\theta - 2B_0 Q\sin\theta\cos\phi + Q^2\sin^2\theta\cos^2\phi \tag{23}$$

In the electric case of Figure 1 the resulting field on the right hand side is greater than that on the opposite side. The charge accelerates to the right, towards the high field region. That is why the electric force is a suction force, whose differential form on the surface is Equation (6). Notice that the force follows the same sense as $\overline{ds}$ given that it is a suction force.

In the magnetic case (Figure 2) the particle accelerates towards the lower field region. Thus, the magnetic force can be deemed a compression one with its differential on the sphere given by the form

$$d\bar{F} = -P_B d\overline{s} \tag{24}$$

The force compresses the sphere instead of sucking at it. Then, by substituting Equation (20) into (24),

$$d\bar{F} = -KB^2 d\overline{s} \tag{25}$$

Replacing $\overline{ds}$ in Equation (24) with Equations (7) and (3), one obtains

$$\bar{F} = -Ka^2\{[B_0^{\ 2}\int_0^\pi \sin^2\theta\,d\theta\int_0^{2\pi}\cos\phi\,d\phi + Q^2\int_0^\pi \sin^2\theta\cos^2\theta\,d\theta\int_0^{2\pi}\cos\phi\,d\phi$$

$$-2B_0Q\int\limits_0^\pi \sin^3\theta\,d\theta \int\limits_0^{2\pi}\cos^2\phi\,d\phi + Q^2\int\limits_0^\pi \sin^4\theta\,d\theta \int\limits_0^{2\pi}\cos^3\phi\,d\phi]\hat{i}$$

$$+[B_0^{\,2}\int\limits_0^\pi \sin^2\theta\,d\theta \int\limits_0^{2\pi}\sin\phi\,d\phi + Q^2\int\limits_0^\pi \sin^2\theta\cos^2\theta\,d\theta \int\limits_0^{2\pi}\sin\phi\,d\phi$$

$$-2B_0Q\int\limits_0^\pi \sin^3\theta\,d\theta \int\limits_0^{2\pi}\sin\phi\cos\phi\,d\phi + Q^2\int\limits_0^\pi \sin^4\theta\,d\theta \int\limits_0^{2\pi}\sin\phi\cos^2\phi\,d\phi]\hat{j}$$

$$+[B_0^{\,2}\int\limits_0^\pi \sin\theta\cos\theta\,d\theta \int\limits_0^{2\pi}d\phi + Q^2\int\limits_0^\pi \sin\theta\cos^3\theta\,d\theta \int\limits_0^{2\pi}d\phi$$

$$-2B_0Q\int\limits_0^\pi \sin^2\theta\cos\theta\,d\theta \int\limits_0^{2\pi}\cos\phi\,d\phi + Q^2\int\limits_0^\pi \sin^3\theta\cos\theta\,d\theta \int\limits_0^{2\pi}\cos^2\phi\,d\phi]\hat{k}\}$$

It is not difficult to show that the $\hat{j}$ and $\hat{k}$ components of this expression vanish while only the third term of the $\hat{i}$ component prevails. Hence,

$$\bar{F} = -Ka^2[-2B_0Q\int\limits_0^\pi \sin^3\theta\,d\theta \int\limits_0^{2\pi}\cos^2\phi\,d\phi]\hat{i} \qquad (26)$$

now, provided that

$$\int\limits_0^\pi \sin^3\theta\,d\theta = 4/3$$

and

$$\int\limits_0^{2\pi}\cos^2\phi\,d\phi = \pi$$

then

$$\bar{F} = -Ka^2[-2B_0Q(4/3)(\pi)]\hat{i} = Ka^2[2B_0(\mu qv/4\pi a^2)(4/3)(\pi)]\hat{i} \qquad (27)$$

which, after grouping and substituting $\hat{i} = \hat{j}\times\hat{k}$, becomes

$$\bar{F} = (K2\mu/3)q\,\bar{v}\times B_0\bar{k} \qquad (28)$$

This expression retrieves the magnetic part of the Lorentz force only if

$$K = 3/(2\mu)$$

Consequentially, Equation (20) can be finally put in the form

$$P_B = 3\,B^2/(2\mu)$$

This shows that both the expression of the magnetic energy density and that of the electric one must include factor 3 in a three degree of freedom problem.

5. Some Final Remarks

By way of an unconventional procedure relying on the concept of electric and magnetic pressure combined with the fundamental Coulomb's and Lorentz magnetic contribution laws, the concept of energy density can be generalized.

It should be emphasized that the above development is missing from the best known textbooks on the subject.

One additional fact may turn out to be more relevant yet: the electric field becomes greater on the right hand side hemisphere of the charge distribution, shown in Figure 1, than on the left one. In other words, the electric (magnetic) pressure appears to be greater (lower) in the direction that a positive electric (magnetic) charge moves and accelerates. It is as if two opposing imbalanced 'negative' ('positive') electric (magnetic) pressure 'sucked' ('compressed') the charge**.** In this manner**,** the magnetic field represents a compressive force.

As a final comment, we do not pretend to establish these results as a universal truth. We are but putting them on a discussion table as the product of a clear and formal although unconventional mathematical analysis.

Acknowledgements

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