Venn Diagram Approach to Heisenberg Inequalities

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Received: March 19, 2012 Accepted: April 6, 2012 Online Published: May 1, 2012

doi:10.5539/apr.v4n2p253 URL: http://dx.doi.org/10.5539/apr.v4n2p253

Abstract

We report on a proof to the Heisenberg inequalities, for both vector-like and scalar-like variables, that is based on statistical dependence of quantum events on appropriate Venn diagrams. A similar proof is provided for the "energy-velocity" uncertainty principle of Haidar (2010).

Keywords: Heisenberg uncertainty principle, statistical independence, Venn diagram

1. Introduction

Physical parameters of a particle, such as momentum **p,** position **r,** energy E and time t**,** are conceivable in quantum mechanics as values of corresponding random variables that we shall denote respectively as P, X, E and T. Their pertaining probability density functions $f_P(p)$, $f_X(r)$ **,** $f_E(E)$ and $f_T(t)$ satisfy the well known defining condition

$$\int_{-\infty}^{\infty} f_{\circ}(\gamma)\, d\gamma = 1, \tag{1}$$

$\forall \circ := P, X, E, T,$ and for all corresponding $\gamma := p, r, E, t,$ where $p = |\mathbf{p}|$ and $r = |\mathbf{r}|$.

The Heisenberg uncertainty principle (HUP), which is foundational in quantum mechanics (Fermi, 1961) is formulated for the vector-like quantities in the form

$$\triangle p \ \triangle r \ \geq \ \frac{\hbar}{2}, \tag{2}$$

and for the scalar-like quantities as

$$\triangle E \ \ \triangle t \ \geq \ \ \frac{\hbar}{2}, \tag{3}$$

where $\hbar = 1.054 \times 10^{-34}\ Js$ is the reduced Planck constant. The symbol $\triangle$ represents, according to Kennard, 1927, the standard deviation of the quantity that follows it, i.e.

$$\triangle\gamma \ = \left\{ \int_{-\infty}^{\infty} (\sigma - \gamma)^2 \ \ f_{\circ}(\sigma)\, d\sigma \right\}^{\frac{1}{2}}. \tag{4}$$

As much as in physics, the HUP turns out to hold also in mathematics (Blatter,1998) for any abstract function $\psi(x)$, that is Fourier-transformable to $\widehat{\psi}(z)$, in the form

$$\|(\sigma - x)\psi\| \ \ \left\|(\zeta - z)\widehat{\psi}\right\| \ \geq \ \frac{1}{2} \ \ \|\psi\|^2, \tag{5}$$

where $\|\psi\| = \left\{ \int_{-\infty}^{\infty} \ \psi^2(\sigma)\, d\sigma \right\}^{\frac{1}{2}}$.

Comparison of $\hbar$,in the context of (2) or (3) with $\|\psi\|^2$ in (5) indicates that the sense of these closed inequalities appears to be the uniting feature between their physical and mathematical aspects and not their right hand side limits. This fact is supported further by Bohr's reformulation (Fermi, 1961) of the HUP to the situation when $\triangle$ represents a rather unspecified generalized measure of size, and not necessarily the standard deviation as in the work of Kennard,1927. Furthermore, it has recently been demonstrated by Haidar, 2010 (see also the review of Matsuno, 2011), that the energy-time HUP collapses to an energy-velocity UP at transition between quantum mechanics and classical mechanics.

Our aim in this communication is to provide a novel proof to the HUP, for both the vector-like and scalar like variables, that can be based on quite elementary statistical arguments. We start by considering any of the previously mentioned four γ variables and assume for it an increase in its uncertainty $\triangle\gamma$, as given by (4), subject to satisfaction of (1). Such an increase is widely known (see, e.g. Blatter, 1998) to be associated with a flattening reduction in the pertaining $f_\circ(\gamma)$. In this situation, the probability of observing the particle momentum, say, within a sphere A in $\mathbb{R}^3$, of radius G around $\mathbf{p}$, which is

$$P(A) = \int_A f_P(p)\, dp = \int_0^G f_P(|\sigma - \mathbf{p}|)\, d(|\sigma - \mathbf{p}|), \tag{6}$$

will obviously decrease. Subsequently $\triangle p \backsim \left[\frac{1}{P(A)} - 1\right]$ and this can always be approximated for small enough G by

$$\triangle p = \frac{K_p}{P(A)}\left[1 - P(A)\right], \tag{7}$$

where $K_p = \frac{P(A)}{[1-P(A)]}\triangle p \neq 0$ constant that has the same units as p.

In a similar fashion, the probability of observing the particle position within a sphere B in $\mathbb{R}^3$, of radius R around $\mathbf{r}$, can be conceived as

$$P(B) = \int_B f_X(r)\, dr = \int_0^R f_X(|\sigma - \mathbf{r}|)\, d(|\sigma - \mathbf{r}|). \tag{8}$$

Here also, an increase in $\triangle r$, subject to satisfaction of (1) will lead to a reduction in $P(B)$ expressible by $\triangle r \backsim \left[\frac{1}{P(B)} - 1\right]$ and

$$\triangle r = \frac{K_r}{P(B)}\left[1 - P(B)\right], \tag{9}$$

where $K_r = \frac{P(B)}{[1-P(B)]}\triangle r \neq 0$ has the same units as r. It should be noted here that G and R can have the same geometrical units in an abstract Venn diagram space (of arbitrary size) despite the fact that they represent different variables in the real physical space.

2. Theorem 1

Statistical dependence of the events of observing a quantum particle with a momentum $\boldsymbol{p}$ *and position* $\boldsymbol{r}$ *requires that*

$$\triangle p \ \triangle r \ \geq \ 4\, K_p K_r\,, \tag{10}$$

where $K_p K_r$ *is a nonzero constant in* Js *units.*

Proof. In classical mechanics, when $\hbar \to 0$ can formally be assumed in (2), the commutativity relations (Weaver, 2001) between $\mathbf{p}$ and $\mathbf{r}$ vanish due to an established perfect correlation between these variables for the same particle. Consequently, A and B are statistically independent events, i.e.

$$P(A)P(B) = P(A \cap B). \tag{11}$$

In contrast, the existing commutativity relations between the $\mathbf{p}$ and $\mathbf{r}$ variables, as a result of their lack of correlation in quantum mechanics, makes A and B dependent statistical events, for which

$$P(A)P(B) \neq P(A \cap B), \tag{12}$$

and if we substitute (7) and (9) in (12) we obtain $\triangle p \, \triangle r = \frac{K_p K_r}{P(A)P(B)} \, [1 - P(A)] \, [1 - P(B)]$.

For quantum particles however $P(A)$and $P(B) << 1$, and

$$\triangle p \, \triangle r = \frac{K_p K_r}{P(A)P(B)}. \tag{13}$$

Imagine further that A and B as two spherically shaped abstract events of a sample space S, of an arbitrary volume within a three-dimensional Venn diagram, i.e. $A, B \subset S \subset \mathbb{R}^3$. Correspondingly

$$P(S) = 1, P(A) = \frac{4}{3}\frac{\pi G^3}{V}, and \; P(B) = \frac{4}{3}\frac{\pi R^3}{V}. \tag{14}$$

Making use of (14) in the substitution of (12) in (13) leads to

$$\triangle p \, \triangle r = K_p K_r \, \frac{9V^2}{16 \, \pi^2 G^3 R^3}. \tag{15}$$

The minimal value physically and mathematically conceivable for V corresponds to a situation when $V = \frac{4}{3}\pi(G^3 + R^3)$.Substitution of this in (15) transforms it to

$$\triangle p \, \triangle r = K_p K_r \left[2 + \frac{G^3}{R^3} + \frac{R^3}{G^3}\right]. \tag{16}$$

Moreover, since G and R are very small numbers which are of the same order, if not equal, then it is physically and mathematically quite reasonable to assume in (16) that $\frac{G^3}{R^3} + \frac{R^3}{G^3} = 2$, and this yields the lowest limit

$$\triangle p \, \triangle r = 4 \, K_p K_r \, , \tag{17}$$

for (15). Clearly then the arbitrariness of V for any fixed G and/or R means that $\triangle p \, \triangle r$ is, as a must, in excess of the limit in (17), and here the proof of the sense of the inequality (10) completes.

3. Remark 1

If $K_p K_r = \frac{1}{8} \, \hbar = 0.13175 \times 10^{-34} Js$, then the result (10) of theorem1 becomes identical to the HUP (2).

The energy-time HUP can be analyzed in a similar but distinct fashion. Here the probability of observing the particle energy within an interval I in $\mathbb{R}$, of width 2a around E, is

$$P(I) = \int_I f_E(E) \, dE = \int_{-a}^{a} f_E(\nu - E) \, d\nu, \tag{18}$$

and the fact that $\triangle E \backsim \left[\frac{1}{P(I)} - 1\right]$ allows, for small enough a, to write

$$\triangle E = \frac{K_E}{P(I)} \, [1 - P(I)] \, , \tag{19}$$

where $K_E = \frac{P(I)}{[1-P(I)]} \, \triangle E \neq 0$ constant that has the same units as E.

As for the time variable which is not an observable quantity in physics, the probability of existence of the particle at a time within an interval J in $\mathbb{R}$, of width $2b$ around t, is

$$P(J) = \int_J f_T(t) \, dt = \int_{-b}^{b} f_T(\nu - t) \, d\nu, \tag{20}$$

and satisfies

$$\triangle t = \frac{K_t}{P(J)}\left[1 - P(J)\right], \tag{21}$$

with $K_t = \frac{P(J)}{[1-P(J)]}\triangle t \neq 0$ having the same units as t. Here also a and b can have the same geometrical units in an abstract Venn diagram space (of arbitrary size).

4. Theorem 2

Statistical dependence of the events of existence of a quantum particle with an energy E and time t requires that

$$\triangle E \ \triangle t \ \geq \ 4\, K_E K_t\,, \tag{22}$$

where $K_E K_t$ is a nonzero constant in Js units.

Proof. The same ingredients of the proof of theorem1 are applicable in this proof as well. The lack of correlation between the scalar-like variables in quantum mechanics implies that I and J are dependent statistical events with

$$P(I)P(J) \neq P(I \cap J). \tag{23}$$

Substitution of (19) and (21) in (23) leads to $\triangle E\ \triangle t = \frac{K_E K_t}{P(I)P(J)}\left[1 - P(I)\right]\left[1 - P(J)\right]$.

Here also $P(I)$, $P(J) << 1$, and

$$\triangle E\ \triangle t = \ \frac{K_E K_t}{P(I)P(J)}. \tag{24}$$

In relation (24) I and J are two segmental events of a sample space S, of an arbitrary length $L = l(S)$, within a one-dimensional Venn diagram, i.e.

$I, J \subset S \subset \mathbb{R}$. Correspondingly

$$P(S) = 1, P(I) = \frac{2a}{L}, and\ \ P(J) = \frac{2b}{L}. \tag{25}$$

We then make use of (25) in the substitution of (23) in (24) to obtain

$$\triangle E\ \triangle t = \ K_E K_t\ \frac{L^2}{4\ ab}. \tag{26}$$

The minimal value conceivable for L corresponds to a situation when $L = 2(a + b)$,which upon substitution in (26) transforms it to

$$\triangle E\ \triangle t = \ K_E K_t\ \left[2 + \frac{a}{b} + \frac{b}{a}\right]. \tag{27}$$

Since a and b are very small numbers which are of the same order, if not equal, then it is physically and mathematically quite reasonable to assume in (27) that $\frac{a}{b} + \frac{b}{a} = 2$, and consequently

$$\triangle E\ \triangle t = 4\ \ K_E K_t\,, \tag{28}$$

is lowest limit for (26).

As in the proof of theorem 1, the arbitrariness of L for any fixed a and/or b means that $\triangle E\ \triangle t$ is, as a must, in excess of the limit in (28), where the proof of the sense of the inequality (22) ends.

5. Remark 2

If $K_E K_t = \frac{1}{8}\hbar$, then the result (22) of theorem 2 becomes identical to the HUP (3). We are able then to refer to any of the previously mentioned events with the symbol $A_\circ$, i.e. $A_\circ : A, B, I, J$, and to state for them the final result that follows.

Claim 1.

$$K_\gamma = P(A_\circ)\triangle\gamma \backsim \sqrt{\frac{1}{8}}\hbar, \forall\gamma. \tag{29}$$

Proof. By virtue of relations (7), (9), (19) and (21), and as a consequence of the validity of remarks 1 and 2.

At transition between quantum and classical mechanics it is possible to assume that $\mathbf{p} = m\ \mathbf{v}$, where m is a nonrelativistic mass, in order to prove the "energy-velocity" uncertainty principle (between scalar and vector quantities) which was recently advanced in (Haidar, 2010). For the energy variable, as in (18)-(19), we have $\triangle E = \frac{K_E}{P(I)}\left[1 - P(I)\right]$.

Moreover, the probability of observing the particle velocity $\mathbf{v}$, within a sphere C in $\mathbb{R}^3$, of radius Q around $\mathbf{v}$,which is $P(C) = \int_C f_V(v)\, dv = \int_0^Q f_P(|\sigma - \mathbf{v}|)\, d(|\sigma - \mathbf{v}|)$, is accompanied by $\triangle v \backsim \left[\frac{1}{P(C)} - 1\right]$; and this can always be approximated for small enough Q by $\triangle v = \frac{K_v}{P(C)}\left[1 - P(C)\right]$, where $K_v = \frac{P(C)}{[1-P(C)]}\triangle v \neq 0$ constant that has the same units as v.

6. Theorem 3

Statistical independence of the events of observing a particle with an energy E and velocity $\mathbf{v}$ *requires that*

$$\triangle E \quad \triangle v \quad = 0\,. \tag{30}$$

Proof. Here I and C are statistically independent events, i.e.

$$P(I)P(C) = P(I \cap C). \tag{31}$$

Substitution of (31) in $\triangle E\ \triangle v = \frac{K_E K_v}{P(I)P(C)}\left[1 - P(I)\right]\left[1 - P(C)\right] = \frac{K_E K_v}{P(I)P(C)}\left[1 - P(I) - P(C) + P(I)P(C)\right]$, leads to $\triangle E\ \triangle v = \frac{K_E K_v}{P(I\cap C)}\left[1 - P(I) - P(C) + P(I \cap C)\right]$, which, by the inclusion-exclusion principle, becomes

$$\triangle E\ \triangle v = \frac{K_E K_v}{P(I \cap C)}\left[1 - P(I \cup C)\right]. \tag{32}$$

Imagine further I and C as two abstract events of a sample space S, of

an arbitrary volume $V = l(S)$, with

$$P(S) = 1 = P(I \cup C). \tag{33}$$

Since I and C are not mutually-exclusive, then the required result holds regardless of the actual values of K_E and K_v. Here the proof ends.

In conclusion, the theoretical result (29) could be quite valuable in experimental particle physics for estimating any one of its $P(A_\circ)$ or $\triangle\gamma$ factors when the other one is known. Moreover, the simplicity of the reported proofs for the HUP and for theorem 3 should have a methodological interest by themselves.

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