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The Photon Energy of an Electron With Rest Mass Energy of $m_e c^2$ is $2m_e c^2$

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Abstract

Einstein's energy-momentum relationship holds for isolated systems in free space. However, this relationship cannot be applied to an electron in a hydrogen atom where potential energy exists. Thus, the author has previously derived an energy-momentum relationship applicable to an electron in a hydrogen atom using three methods. When the formula for this relationship is solved, the hydrogen atom has energy levels that can take negative values, even when described on an absolute scale. When an electron transitions to this negative energy level, it is not enough even if it releases all of the rest mass energy $m_e c^2$. Thus, the author pointed out that the electron has unknown photon energy in addition to its rest mass energy. The author predicted—but was unable to prove—the existence of this unknown energy. In this paper, the energy-momentum relationship applicable to an electron in a hydrogen atom is derived using two methods different from those used previously. The paper also elucidates the latent photon energy of the electron, and the existence of negative energy which cancels that out.

Keywords: Einstein's energy-momentum relationship, energy-momentum relationship in a hydrogen atom, ultra-low energy levels in a hydrogen atom, negative energy specific to the electron, *n*=0 energy level

1. Introduction

The following is the most famous formula discovered by Einstein, who made tremendous contributions to the development of physics in the 20th century: (Einstein, 1905a).

$$E = mc^2. \tag{1}$$

A body with mass *m* has an energy of mc^2.

According to the special theory of relativity (STR), the following relation holds between the energy and momentum of a body moving in free space (Einstein, 1961).

$$\left(mc^2\right)^2 = \left(m_0 c^2\right)^2 + c^2 p^2. \tag{2}$$

Here, $m_0 c^2$ is the rest mass energy of the body. And mc^2 is the relativistic energy.

Formula (2), which is called Einstein's energy-momentum relationship, holds for isolated systems in free space.

Also, Einstein and Sommerfeld defined the relativistic kinetic energy as follows (Sommerfeld, 1923).

$$K_{re} = mc^2 - m_0 c^2. \tag{3}$$

The "re" subscript of K_{re} stands for "relativistic."

Incidentally, Einstein's relationship (2) holds when the energy absorbed by a body is all converted to kinetic energy of that body. However, an electron in an atom acquires kinetic energy through emission of energy. Therefore, Einstein's relationship (2) cannot be applied to an electron in an atom.

Thus, the author derived the following relationship applicable to an electron in a hydrogen atom.

$$\left(m_n c^2\right)^2 + c^2 p_{\mathrm{re},n}^2 = \left(m_{\mathrm{e}} c^2\right)^2. \tag{4}$$

$m_{\mathrm{e}}c^2$ is the rest mass energy of the electron. Also, $m_n c^2$ is the relativistic energy of an electron whose principal quantum number is in the state *n*. $p_{\mathrm{re},n}$ is the momentum of an electron in that state.

The author has previously derived Formula (4) using three methods.

First, Formula (4) was derived mathematically (Suto, 2011). The third time, it was derived through consideration of an ellipse (Suto, 2020; Suto, 2024). Section 2 reviews the second method, which is comparatively easy to understand.

2. An Energy-Momentum Relationship Applicable to an Electron in a Hydrogen Atom and Its Solutions

Taking Formula (3) into account, Formula (2) can be rewritten as follows.

$$c^2 p^2 = \left(mc^2 - m_0 c^2\right)\left(mc^2 + m_0 c^2\right)$$

$$= K_{\mathrm{re}}\left(mc^2 + m_0 c^2\right). \tag{5}$$

From this, the following formula for relativistic kinetic energy can be derived.

$$K_{\mathrm{re}} = \frac{p_{\mathrm{re}}^2}{m + m_0}. \tag{6}$$

In classical quantum theory, the total mechanical energy of a hydrogen atom is defined as the sum of the kinetic energy and potential energy of the electron. That is,

$$E_n = K_n + V\left(r_n\right) = \frac{1}{2}V\left(r_n\right) = -K_n, \quad E_n < 0. \tag{7}$$

Here, *n* is the principal quantum number.

When an electron outside a hydrogen atom is taken into the atom, the electron emits photon energy and acquires kinetic energy. Formula (7) shows that these two types of energy are provided by the electron reducing potential energy.

If we let $E_{\mathrm{ph},n}$ (a different expression for $h\nu$) be the energy emitted when an electron outside an atom drops to an energy level with principal quantum number *n* in a hydrogen atom, then the relationship between $E_{\mathrm{ph},n}$ and other energy is as follows.

$$E_{\mathrm{ph},n} = h\nu = K_{\mathrm{re},n} = -E_{\mathrm{re},n}. \tag{8}$$

Here, the "ph" subscript of $E_{\mathrm{ph},n}$ stands for "photon." $E_{\mathrm{re},n}$ are the relativistic energy levels of a hydrogen atom.

$E_{\mathrm{ph},n}$ is the energy emitted when an electron with rest mass energy $m_{\mathrm{e}}c^2$ is taken into a hydrogen atom.

The relationship between the rest mass energy of the electron $m_{\mathrm{e}}c^2$ and the relativistic energy of the electron $m_n c^2$ is as follows.

$$m_n c^2 = E_{\mathrm{ab},n} = m_{\mathrm{e}} c^2 + V\left(r_n\right) + K_{\mathrm{re},n}. \tag{9}$$

$$m_n c^2 = m_e c^2 + E_{re,n} = m_e c^2 - K_{re,n}. \quad (10)$$

$$E_{re,n} = -K_{re,n} = (m_n - m_e)c^2. \quad (11)$$

$$E_{ph,n} = K_{re,n} = (m_e - m_n)c^2. \quad (12)$$

Here, $m_n c^2 (= E_{ab,n})$ is the sum of the residual part of the rest mass energy of the electron $\left(m_e c^2 + V(r_n)\right)$ and the kinetic energy $K_{re,n}$ (Suto, 2018a). The "ab" subscript of $E_{ab,n}$ stands for "absolute."

Here, the relativistic kinetic energy of an electron inside a hydrogen atom is defined as follows by referring to Formulas (3) and (6) (Suto, 2023a).

$$K_{re,n} = m_e c^2 - m_n c^2. \quad (13)$$

$$K_{re,n} = \frac{p_{re,n}^2}{m_e + m_n}. \quad (14)$$

Linking the right sides of Formulas (13) and (14) with an equals sign and rearranging, the following relationship can be derived (Suto, 2018b).

$$\left(m_n c^2\right)^2 + c^2 p_{re,n}^2 = \left(m_e c^2\right)^2. \quad (15)$$

This energy-momentum relationship is applicable to an electron inside a hydrogen atom.

Next, if Formula (15) is solved for $m_n c^2$, then the following formula can be derived.

$$m_n c^2 = \frac{m_e c^2}{\left(1 + \frac{v_n^2}{c^2}\right)^{1/2}}. \quad (16)$$

To change Formula (16) into a formula of quantum theory, the discreteness of energy must be incorporated into Formula (16).

Previously, the author has shown that the following relationship holds for an electron in a hydrogen atom (Suto, 2019; Suto2021).

$$\frac{v_n}{c} = \frac{\alpha}{n}. \quad (17)$$

Here, α is the following fine-structure constant.

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = 7.2973525693 \times 10^{-3}. \quad (18)$$

Using the relationship in Formula (17), Formula (16) can be written as follows.

$$m_n c^2 = m_e c^2 \left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2}. \quad (19)$$

Incidentally, there are positive and negative solutions to Einstein's relationship (2). In the same way, Formula (15) also has the following positive and negative solutions (Suto, 2014a; Suto2022a).

$$E_{ab,n}^{+} = m_n c^2 = m_e c^2 + E_{re,n} = m_e c^2 \left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2}, \quad n = 0, 1, 2, \cdots. \quad (20)$$

$$E_{\mathrm{ab},n}^{-} = -m_n c^2 = -m_{\mathrm{e}} c^2 \left(\frac{n^2}{n^2 + \alpha^2} \right)^{1/2}, \quad n = 0, 1, 2, \cdots. \tag{21}$$

It has already been pointed out that a state with *n*=0 exists in the energy levels of a hydrogen atom (Suto, 2014b; Suto2023b).

When an electron transitions to the energy level $E_{\mathrm{ab},n}^{-}$, it must emit an energy larger than the rest mass energy $m_{\mathrm{e}} c^2$.

Also, as is clear from Formula (17), only energy levels satisfying the following conditions are allowed in the hydrogen atom.

$$\frac{p_{\mathrm{re},n}}{m_n c} = \frac{\alpha}{n}. \tag{22}$$

Formula (22) gives a new quantum condition to replace Bohr's quantum condition.

Thus, the energy levels of a hydrogen atom $E_{\mathrm{re},n}$ are given by the following formula.

$$E_{\mathrm{re},n} = m_n c^2 - m_{\mathrm{e}} c^2 = m_{\mathrm{e}} c^2 \left[\left(\frac{n^2}{n^2 + \alpha^2} \right)^{1/2} - 1 \right], \quad n = 0, 1, 2, \cdots. \tag{23}$$

In addition, Butto, N. has also discussed electron spin when discussing momentum of the electron. However, electron spin is not incorporated into the formula derived in this paper.

Therefore, it may not be the final formula (Butto, 2021).

Next, when the part of Formula (23) in parentheses is expressed as a Taylor expansion,

$$E_{\mathrm{re},n} \approx m_{\mathrm{e}} c^2 \left[\left(1 - \frac{\alpha^2}{2n^2} + \frac{3\alpha^4}{8n^4} - \frac{5\alpha^6}{16n^6} \right) - 1 \right]$$

$$\approx -\frac{\alpha^2 m_{\mathrm{e}} c^2}{2n^2}. \tag{24}$$

Taking Formula (18) into account, Formula (24) can be written as follows.

$$-\frac{\alpha^2 m_{\mathrm{e}} c^2}{2n^2} = -\frac{1}{2} \left(\frac{1}{4\pi\varepsilon_0} \right)^2 \frac{m_{\mathrm{e}} e^4}{\hbar^2} \cdot \frac{1}{n^2} = E_{\mathrm{BO},n}, \quad n = 1, 2, \cdots. \tag{25}$$

Here, $E_{\mathrm{BO},n}$ are the energy levels of a hydrogen atom derived by Bohr (Bohr, 1913).

From this, it is evident that Formula (25) is an approximation of Formula (23).

Now, Formula (20) absolutely and relativistically describes the photon energy of an electron constituting a hydrogen atom. In contrast, Formula (21) indicates previously unknown energy levels.

Next, if the electron orbital radii corresponding to the energy levels in Formulas (20) and (21) are taken to be, respectively, r_n^{+} and r_n^{-} (Suto, 2017).

$$r_n^{+} = \frac{r_{\mathrm{e}}}{2} \left[1 + \frac{n}{\left(n^2 + \alpha^2 \right)^{1/2} - n} \right]. \tag{26}$$

$$r_n^- = \frac{r_e}{2}\left[1 - \frac{n}{\left(n^2+\alpha^2\right)^{1/2}+n}\right]. \tag{27}$$

Here, r_e is the classical electron radius, defined as follows.

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2}. \tag{28}$$

In Formula (27), the electron approaches toward $r_e/4$ as *n* increases.

The domain of the ordinary hydrogen atom that we all know starts from $r = r_e/2$ $\left(E_{ab}=0\right)$.

It is strange that negative energy levels exist even though energy is described with an absolute scale. To resolve this contradiction, the author has previously predicted the existence of photons with negative energy (Suto, 2020). (Figure 1)

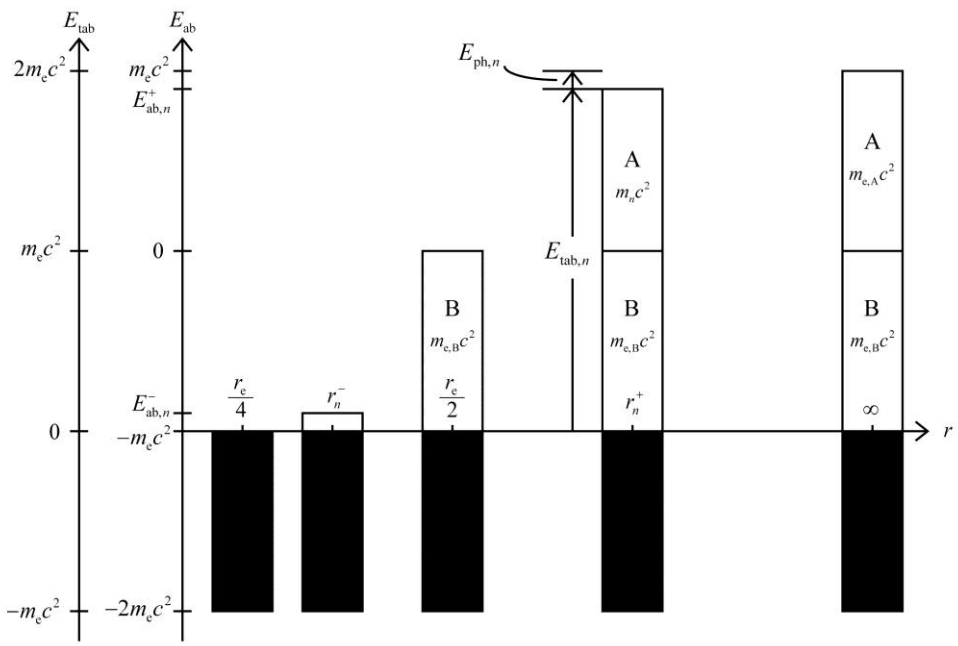

Figure 1. Photon energies of electrons in different states, and negative energy

The photon energy of an electron corresponds to the white rectangle in the diagram (A+B). Energy A is an energy we understand well. This paper asserts the existence of the B part. Also, the negative energy specific to the electron $-m_ec^2$ corresponds to the black rectangle. This figure shows that the original photon energy of an electron with rest mass energy m_ec^2 is $2m_ec^2$. (However, this figure is just a conceptual illustration. The *r* coordinate on the *x*-axis is not accurate).

Incidentally, Daviau, C. has already discussed the cloud of photons of an electron. For details, please see that paper (Daviau, 2024).

In the state $E_{ab}=0$, the photon energy $m_{e,B}c^2$ and negative energy $-m_ec^2$ cancel each other out, resulting in a state where energy is zero. An electron in the state where $E_{\mathrm{ab}} = 0$ still has photon energy, so it can emit another photon and drop to a negative energy level.

The author has previously defined the residual energy $E_{\mathrm{tab},n}$ of an electron that has emitted the photon energy $E_{\mathrm{ph},n}$ as follows (Suto, 2020).

$$E_{\mathrm{tab},n} = \left(m_{\mathrm{e,A}} + m_{\mathrm{e,B}}\right)c^2 - E_{\mathrm{ph},n} = E_{\mathrm{ab},n} + m_{\mathrm{e,B}}c^2 = \left(m_n + m_{\mathrm{e,B}}\right)c^2. \tag{29}$$

The “tab” subscript of this energy indicates the true, absolute photon energy. The descriptor “tab” is applied because absolute energy $E_{\mathrm{ab},n}$ has already been defined.

In section 2, existence of the energy $m_{\mathrm{e,B}}c^2$ was predicted, but it was not possible to prove that that energy exists. In sections 3 and 4, Formula (15) will be derived through two methods different from the previous methods. Then the existence of the photon energy $m_{\mathrm{e,B}}c^2$ of the electron will be elucidated.

3. Formula (15) Derived With the Fourth Method

The author previously derived Formula (15) by placing line segments of an ellipse into correspondence with physical quantities (Suto, 2020; Suto, 2024).

This paper examines this ellipse again, derives Formula (15) using a method different from that used before, and shows the existence of $m_{\mathrm{e,B}}c^2$. (Figure 2)

First, consider the Cartesian coordinate system $\mathrm{O}-xy$. Letting F and F' be the points $x = \pm f$, an ellipse is drawn taking those 2 points as foci. Let A and A' be the points where the ellipse intersects the *x*-axis, and let B and B' be the points where the ellipse intersects the *y*-axis. Also, let 2*a* be the length of the line segment $\overline{\mathrm{AA'}}$, 2*b* be the length of the line segment $\overline{\mathrm{BB'}}$, and 2*f* be the length of the line segment $\overline{\mathrm{FF'}}$.

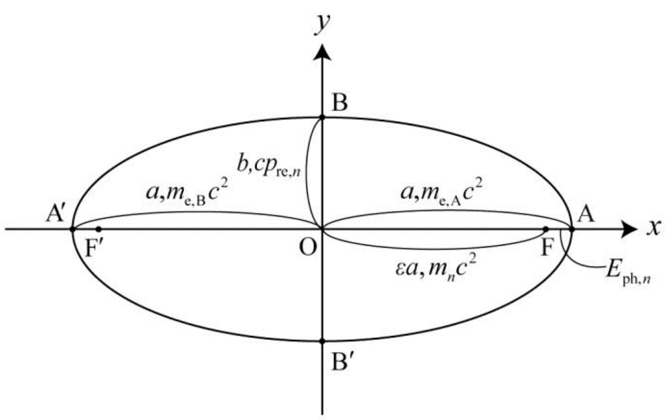

Figure 2

First, $m_{\mathrm{e,A}}c^2$ is placed into correspondence with $\overline{\mathrm{OA}}$. Then m_nc^2 corresponds to $\overline{\mathrm{OF}}$, and $m_{\mathrm{e,B}}c^2$

corresponds to $\overline{\mathrm{OA'}}$. Also, $E_{\mathrm{ph},n}$ corresponds to $\overline{\mathrm{FA}}$, and $cp_{\mathrm{re},n}$ corresponds to $\overline{\mathrm{OB}}$.

The eccentricity of the ellipse in this case is defined as follows.

$$\varepsilon = \frac{f}{a}. \tag{30}$$

The eccentricity of the ellipse can also be expressed using the following formula.

$$\varepsilon = \left(1 - \frac{b^2}{a^2}\right)^{1/2}. \tag{31}$$

The following equation can be derived from Formula (31).

$$\left(a + \varepsilon a\right)\left(a - \varepsilon a\right) = b^2. \tag{32}$$

The following relationship is known to hold in an ellipse.

$$\overline{\mathrm{A'F}} \cdot \overline{\mathrm{FA}} = \overline{\mathrm{OB}}^2. \tag{33}$$

Here, the line segments $\overline{\mathrm{OA}}$ and $\overline{\mathrm{OA'}}$ are taken to correspond to the energy $m_{\mathrm{e}}c^2$. Let us express these as follows.

$$\overline{\mathrm{OA}} = m_{\mathrm{e,A}}c^2. \tag{34}$$

$$\overline{\mathrm{OA'}} = m_{\mathrm{e,B}}c^2. \tag{35}$$

Also, assume the following relation.

$$\varepsilon = \left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2}. \tag{36}$$

If Formula (20) is taken into account, $m_n c^2$ corresponds to εa. That is,

$$\overline{\mathrm{OF}} = \varepsilon a = m_n c^2. \tag{37}$$

Also, the following energy corresponds to $\overline{\mathrm{FA}}$.

$$\overline{\mathrm{FA}} = a - \varepsilon a = E_{\mathrm{ph},n} = m_{\mathrm{e,A}}c^2 - m_n c^2. \tag{38}$$

Taking these points into account, Formula (33) can be written as follows.

$$\left(m_n c^2 + m_{\mathrm{e,B}}c^2\right)\left(m_{\mathrm{e,A}}c^2 - m_n c^2\right) = c^2 p_{\mathrm{re},n}^2. \tag{39}$$

Rearranging Formula (39), the following relationship can be derived.

$$m_{\mathrm{e,A}}c^2 \cdot m_{\mathrm{e,B}}c^2 = \left(m_n c^2\right)^2 + c^2 p_{\mathrm{re},n}^2. \tag{40}$$

Also, Formula (40) can be written as follows because $m_{\mathrm{e,A}}c^2 = m_{\mathrm{e,B}}c^2$.

$$\left(m_{\mathrm{e}}c^2\right)^2 = \left(m_n c^2\right)^2 + c^2 p_{\mathrm{re},n}^2. \tag{41}$$

In section 3, an energy-momentum relationship applicable to an electron in a hydrogen atom was derived with a fourth method. However, section 3 did not simply derive Formula (41). The newly derived Formula (40) showed that the electron has a latent photon energy of $m_{\mathrm{e,B}}c^2$.

4. Formula (40) Derived With the Second Method

In section 4, Formula (40) is derived with a method different from section 3.

According to Maxwell's electromagnetism, the following relationship holds between the momentum *p* and energy *E* of light.

$$E = cp. \tag{42}$$

Also, Einstein asserted, based on consideration of the photoelectric effect, that light has a particle nature, although it had previously been regarded as a wave.

If a photon as a single particle is assumed to have a frequency *ν*, Einstein concluded it has the following energy (Einstein, 1905b).

$$E = h\nu\ . \tag{43}$$

Here, *h* is the Planck constant. Also Formula (43) can be written as follows using the angular frequency ω.

$$E = \hbar\omega, \quad \hbar = \frac{h}{2\pi}. \tag{44}$$

ω is defined as follows.

$$\omega = 2\pi\nu\ . \tag{45}$$

The following formula can be derived from Formulas (42) and (43).

$$\lambda = \frac{c}{\nu} = \frac{h}{p}. \tag{46}$$

Thus, de Broglie thought that, if light —previously thought to be a wave —has a particle nature, then perhaps the electron—thought to be a particle—has a wave nature. Thus, he applied Formula (46) to matter.

Incidentally, the electron's phase velocity $v_{p,n}$ is given by the following formula.

$$v_{p,n} = \lambda_n \nu_n. \tag{47}$$

Here, $v_{p,n}$ is the phase velocity of the electron wave when the principal quantum number is in the *n* state. Also,

λ_n and ν_n are the wavelength and frequency of the electron wave.

Formula (47) can be written as follows using the relationship of Formulas (46) and (43).

$$v_{p,n} = \lambda_n \nu_n = \frac{h}{p_{\mathrm{re},n}} \frac{K_{\mathrm{re},n}}{h} = \frac{K_{\mathrm{re},n}}{p_{\mathrm{re},n}}. \tag{48}$$

Incidentally, Formula (39) can be written as follows because $E_{\mathrm{ph},n} = K_{\mathrm{re},n}$.

$$\left(m_n + m_{\mathrm{e,B}}\right)c^2 \cdot K_{\mathrm{re},n} = c^2 p_{\mathrm{re},n}^2. \tag{49}$$

Based on this, the following formula can be derived.

$$K_{\mathrm{re},n} = \frac{p_{\mathrm{re},n}^2}{m_n + m_{\mathrm{e,B}}}. \tag{50}$$

It was thus found that the true nature of m_{e} in Formula (14) is $m_{\mathrm{e,B}}$.

Next, the following relationship can be derived from Formula (48) by using Formula (50).

$$v_{p,n} = \frac{K_{\mathrm{re},n}}{p_{\mathrm{re},n}} = \frac{p_{\mathrm{re},n}^2}{m_n + m_{\mathrm{e,B}}} \frac{1}{p_{\mathrm{re},n}} = \frac{p_{\mathrm{re},n}}{m_n + m_{\mathrm{e,B}}} = \frac{m_n}{m_n + m_{\mathrm{e,B}}} v_{g,n}. \tag{51}$$

Here, the following five formulas will be checked as part of deriving Formula (40) again.

First, the photon energy $E_{\mathrm{ph},n}$ emitted when an electron with mass m_{e} in free space transitions to an energy level with principal quantum number *n* in a hydrogen atom was defined as follows.

$$E_{\mathrm{ph},n} = K_{\mathrm{re},n} = \left(m_{\mathrm{e,A}} - m_n\right)c^2. \tag{52}$$

Also, when Formula (51) is considered, the momentum and kinetic energy of an electron in a state with principal quantum number *n* is given by the following formula.

$$p_{\mathrm{re},n} = \left(m_n + m_{\mathrm{e,B}}\right)v_{p,n}. \tag{53}$$

$$K_{\mathrm{re},n} = \left(m_n + m_{\mathrm{e,B}}\right)v_{p,n}^2. \tag{54}$$

Due to Formula (51), the following relationship holds between phase velocity and group velocity of the electron wave (Suto, 2022b).

$$v_{p,n}^2 = \left(\frac{m_n}{m_n + m_{\mathrm{e,B}}}\right)^2 v_{g,n}^2. \tag{55}$$

Finally, the energy $E_{\mathrm{tab},n}$ of an electron, including its latent photon energy $m_{\mathrm{e,B}}c^2$, was defined as follows.

$$E_{\mathrm{tab},n} = \left(m_n + m_{\mathrm{e,B}}\right)c^2. \tag{56}$$

The following relationship holds between the rest mass energy and relativistic energy of an electron in a hydrogen atom.

$$m_{\mathrm{e,A}}c^2 = m_n c^2 + E_{\mathrm{ph},n}. \tag{57}$$

Thus, Formula (57) can be expressed as follows if Formula (52) is taken into account.

$$m_{\mathrm{e,A}}c^2 = m_n c^2 + \left(m_{\mathrm{e,A}} - m_n\right)c^2. \tag{58}$$

Also, if the energy of the electron is taken to be $2m_{\mathrm{e}}c^2$ rather than $m_{\mathrm{e,A}}c^2$, then the following relationship holds.

$$\left(m_{\mathrm{e,A}} + m_{\mathrm{e,B}}\right)c^2 = E_{\mathrm{tab},n} + E_{\mathrm{ph},n} = \left(m_n + m_{\mathrm{e,B}}\right)c^2 + \left(m_{\mathrm{e,A}} - m_n\right)c^2. \tag{59}$$

This time, the second term of Formulas (58) and (59) is rewritten using Formula (54).

$$m_{\mathrm{e,A}}c^2 = m_n c^2 + \left(m_n + m_{\mathrm{e,B}}\right)v_{p,n}^2. \tag{60}$$

$$\left(m_{\mathrm{e,A}} + m_{\mathrm{e,B}}\right)c^2 = \left(m_n + m_{\mathrm{e,B}}\right)c^2 + \left(m_n + m_{\mathrm{e,B}}\right)v_{p,n}^2. \tag{61}$$

Next, Formulas (60) and (61) are rewritten as follows using Formula (55).

$$m_{\mathrm{e,A}}c^2 = m_n c^2 + \left(m_n + m_{\mathrm{e,B}}\right)\left(\frac{m_n}{m_n + m_{\mathrm{e,B}}}\right)^2 v_{g,n}^2. \tag{62}$$

$$\left(m_{\mathrm{e,A}} + m_{\mathrm{e,B}}\right)c^2 = \left(m_n + m_{\mathrm{e,B}}\right)c^2 + \left(m_n + m_{\mathrm{e,B}}\right)\left(\frac{m_n}{m_n + m_{\mathrm{e,B}}}\right)^2 v_{g,n}^2. \tag{63}$$

Rearranging formulas (62) and (63), the following formula can be derived in both cases.

$$m_{\mathrm{e,A}} \cdot m_{\mathrm{e,B}} c^2 = m_n^2 c^2 + p_{\mathrm{re},n}^2. \tag{64}$$

Next, when both sides are multiplied by c^2, an energy-momentum relationship applicable to an electron in a hydrogen atom can be derived. That is,

$$m_{\mathrm{e,A}} c^2 \cdot m_{\mathrm{e,B}} c^2 = \left(m_n c^2\right)^2 + c^2 p_{\mathrm{re},n}^2. \tag{65}$$

No matter which energy of the electron, $E_{\mathrm{ab},n}$ or $E_{\mathrm{tab},n}$ is used, the derived formula matches Formula (40).

Furthermore, even though Formula (57) does not include $m_{\mathrm{e,B}} c^2$, in the end Formula (65) incorporating $m_{\mathrm{e,B}} c^2$ was derived. That is a great result.

5. An Electron Energy-Momentum Relationship Applicable in Free Space

This section considers the energy-momentum relationship when the electron is in an isolated system in free space. The following relationship holds if the relativistic energy of an electron outside the atom is taken to be $m'_{\mathrm{e}} c^2$.

$$\left(m'_{\mathrm{e}} + m_{\mathrm{e,B}}\right) c^2 = \left(m_{\mathrm{e,A}} + m_{\mathrm{e,B}}\right) c^2 + \left(m'_{\mathrm{e}} - m_{\mathrm{e,A}}\right) c^2. \tag{66}$$

From this point, the method is used whereby Formula (65) was derived from Formula (58). The second term on the right side of Formula (66) becomes as follows if it is rewritten using the relationship in Formula (54).

$$\left(m'_{\mathrm{e}} + m_{\mathrm{e,B}}\right) c^2 = \left(m_{\mathrm{e,A}} + m_{\mathrm{e,B}}\right) c^2 + \left(m'_{\mathrm{e}} + m_{\mathrm{e,B}}\right) v_p^2. \tag{67}$$

Next, when Formula (67) is changed into a formula containing v_g by using Formula (55), the result is as follows.

$$\left(m'_{\mathrm{e}} + m_{\mathrm{e,B}}\right) c^2 = \left(m_{\mathrm{e,A}} + m_{\mathrm{e,B}}\right) c^2 + \left(m'_{\mathrm{e}} + m_{\mathrm{e,B}}\right) \left(\frac{m'_{\mathrm{e}}}{m'_{\mathrm{e}} + m_{\mathrm{e,B}}}\right)^2 v_g^2. \tag{68}$$

Rearranging this, we can derive the following equation.

$$m'^2_{\mathrm{e}} c^2 = m_{\mathrm{e,A}} \cdot m_{\mathrm{e,B}} c^2 + p_{\mathrm{re}}^2. \tag{69}$$

The following is the result when both sides of this equation are multiplied by c^2.

$$\left(m'_{\mathrm{e}} c^2\right)^2 = m_{\mathrm{e,A}} c^2 \cdot m_{\mathrm{e,B}} c^2 + c^2 p_{\mathrm{re}}^2. \tag{70}$$

Also, Formula (70) can be written as follows.

$$\left(m'_{\mathrm{e}} c^2\right)^2 = \left(m_{\mathrm{e}} c^2\right)^2 + c^2 p_{\mathrm{re}}^2. \tag{71}$$

Formula (70) is another expression of Einstein's energy-momentum relationship (71), and thus it is evident that $m_{\mathrm{e,B}} c^2$ is also incorporated into Formula (71).

6. Conclusion

A. The author has previously derived the following relationship, applicable to an electron in a hydrogen atom, using three methods.

$$\left(m_{\mathrm{e}}c^2\right)^2=\left(m_n c^2\right)^2+c^2 p_{\mathrm{re},n}^2. \tag{72}$$

This paper has further derived Formula (72) using two methods. As a result, it was found that the proper formula for Formula (72) is the following.

$$m_{\mathrm{e,A}}c^2\cdot m_{\mathrm{e,B}}c^2=\left(m_n c^2\right)^2+c^2 p_{\mathrm{re},n}^2. \tag{73}$$

Formula (73) incorporate an unknown latent energy $m_{\mathrm{e,B}}c^2$ of the electron.

According to the STR, an electron with rest mass m_{e} has a rest mass energy of $m_{\mathrm{e}}c^2$. However, the conclusion derived in this paper differs from the predictions of the STR. An electron with rest mass m_{e} has a photon energy of $2m_{\mathrm{e}}c^2$ and a negative energy of $-m_{\mathrm{e}}c^2$. $2m_{\mathrm{e}}c^2$ is the sum of the electron rest mass energy $m_{\mathrm{e,A}}c^2$ and the photon energy $m_{\mathrm{e,B}}c^2$ whose existence has not been confirmed in modern physics.

In section 4, Formula (73) including $m_{\mathrm{e,B}}c^2$ was derived by taking Formula (57), which doesn't contain $m_{\mathrm{e,B}}c^2$, as a departure point. If the existence of $m_{\mathrm{e,B}}c^2$ can be proven, that will simultaneously prove the existence of negative energy $-m_{\mathrm{e}}c^2$ which cancels it out. An electron with rest mass energy $m_{\mathrm{e}}c^2$ can emit an energy of $2m_{\mathrm{e}}c^2$. This can be regarded as a revolutionary discovery for physics.

B. Based on the discussion in this paper, there was found to be similarity between the formula for the kinetic energy of an electron in a hydrogen atom, and the formula for the photon energy of an electron. That is,

$$K_{\mathrm{re},n}=\left(m_n+m_{\mathrm{e,B}}\right)v_{p,n}^2. \tag{74}$$

$$E_{\mathrm{tab},n}=\left(m_n+m_{\mathrm{e,B}}\right)c^2. \tag{75}$$

The energy of a photon is found as the product of the photon's momentum and the speed of light. The kinetic energy of an electron, in contrast, is determined by the product of the electron's momentum and its phase velocity.

It was also found that the following relationship holds due to Formulas (74) and (75).

$$\frac{K_{\mathrm{re},n}}{E_{\mathrm{tab},n}}=-\frac{E_{\mathrm{re},n}}{E_{\mathrm{tab},n}}=\frac{E_{\mathrm{ph},n}}{E_{\mathrm{tab},n}}=\frac{v_{p,n}^2}{c^2}. \tag{76}$$

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