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# Navier–Stokes Equation's Existence and the Smoothness Established by Using Globular Protein at the Casimir Temperature

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**Abstract**

The widely discussed result of derivation for the Casimir temperature has been published by the authors of this paper. Recently we noticed that the unsolved problem of Navier–Stokes equation's existence and the smoothness can be thought by the idea of the globular protein (Einstein's colloidal particles (1905)) placed in liquid $^3$He at the Casimir temperature. Although the globular protein is not dissolved in liquid $^3$He and is as the cusp but it is exactly due to such nature of cusp in liquid $^3$He, leads to an interesting phenomenon if it is differentiable in the Navier–Stokes equation in a peculiar case, one can understand the existence and the smoothness are established as to a promising candidate answer.

**Keywords:** Casimir temperature, Navier–Stokes equation, globular protein, liquid 3-He, existence, smoothness

## 1. Introduction

An excerpt from Einstein's article describes the basic principles of stochastic models:

Einstein: "It must be explicitly assumed that the movement performed by each single particle is independent of the movements of all other particles; it will also be considered that one action and the same particle at different time intervals are independent processes, as long as these time intervals not very small".

Einstein said: "We introduce a time interval $\tau$ considering that, this is relatively small, but the time interval we can observe is still too large, between two consecutive time intervals $\tau$ proteins, the actions performed by particles can be thought of as independent events". And then in 1950s, relativistic superfluidity is a fascinating area in condensed matter physics, exploring the behavior of superfluids under relativistic conditions. Several prominent scholars have made significant contributions to this field. For instance, the work published by Volovik in superfluid 3He and the Casimir effect (Volovik, G. E., 1996) and exhibits a form of the Casimir effect. The study is significant because it connects concepts from quantum field theory, such as the Casimir effect, with condensed matter physics, specifically the behavior of the interface between different superfluid phases. Volovik demonstrates that the vacuum energy of the quantum fields can lead to observable forces in the AB interface, which provides a deeper understanding of both superfluidity and quantum vacuum effects in macroscopic systems.

This excerpt outlines Einstein's fundamental principles of stochastic models, which emphasize the independence of particle movements both in relation to one another and across different time intervals, as long as these intervals are not too small. In the context of stochastic processes, this assumption of independence is a core feature that enables a simplified mathematical treatment of random particle motions.

Einstein's early work on stochastic models laid the foundation for many modern theories in statistical physics. His emphasis on the independence of particle movements in different time intervals forms the basis of Brownian motion theory, which describes the random motion of particles suspended in a fluid. The introduction of a time interval $\tau$ in which the particle movements are independent is key for understanding stochastic models in physics, leading to modern applications like the Langevin equation and diffusion processes.

In the 1950s, the study of relativistic superfluidity became an important topic in condensed matter physics. One of the major contributions to this field was made by Volovik, whose work explored the behavior of superfluid $^3$He, especially in relation to the Casimir effect. The Casimir effect, traditionally associated with quantum field theory and vacuum energy, was applied to superfluid interfaces, revealing how quantum vacuum fluctuations could manifest as observable forces at the macroscopic level in the interface between different superfluid phases, specifically the AB interface in 3He.

Volovik's work established a connection between quantum vacuum effects, such as the Casimir effect, and condensed matter physics. This provided a richer understanding of the vacuum energy's role in macroscopic systems, furthering the exploration of quantum field theory applications in condensed matter phenomena like superfluidity.

One interesting story (Su, H. & Lee, P., 2024) proposes an intriguing extension of Curie's law to cosmic scales, specifically applying it to Higgs fields within the context of the Casimir effect. In the realm of superfluid, we encounter complex patterns, such as fractional vortices in the Larkin-Ovchinnikov states (Radzihovsky, L. & Vishwanath, 2009) (Radzihovsky, L., 2011), and topological superfluid (Nissinen, J. & Volovik, G. E., 2020) (Eltsov, V. B. et al., 2019). Additionally, research on relativistic superfluidity (Collett, C. A. et al., 2013) (Xiong, C. et al., 2014) (Alford, M. G. et al., 2012) draws parallels between superfluidity and cosmology, providing deep insights into the quantum fluids' nature. Ultra-low temperature investigations, particularly on quantum liquids and solids like liquid and solid Helium-3, are also discussed. Boris Svistunov, a professor at the University of Massachusetts Amherst, is recognized for his pioneering work on superfluidity, supersolidity, and superfluid turbulence. He co-authored the book *Superfluid States of Matter*, offering a comprehensive overview of these phenomena.

This paper further explores the behavior of weakly interacting Bose gases near the Bose-Einstein condensation transition, which is crucial for understanding superfluidity (Halperin, W. P. et al., 1974) (Lee, Y. et al., 1999) (Prokof'ev, N. & Svistunov, B., 2005). Quantum physics continues to unravel the mysteries of matter at extremely low temperatures, with recent studies shedding light on fascinating phenomena in superfluid Helium-3 and chiral superconductors. These discoveries not only enhance our understanding of quantum states but also hold potential applications in cosmology, quantum computing, and condensed matter physics.

**Superfluid Analogies of Cosmological Phenomena**

G.E. Volovik's work on superfluid Helium-3 establishes a connection between quantum fluids and cosmological models. In his 2001 study, Volovik examines how superfluid Helium-3 can serve as an analog system for investigating cosmological phenomena, such as event horizons and black holes. By drawing parallels between the behavior of quasiparticles in superfluid systems and particles in the early universe, this research enables the experimental exploration of cosmological theories under controlled laboratory conditions. It provides valuable insights into phenomena that would otherwise be inaccessible due to the vast scale of the universe (Volovik, G. E., 2001).

**Fermion Zero Modes on Vortices in Chiral Superconductors**

In a related area of study, Volovik's 1999 paper investigates fermion zero modes in chiral superconductors. These modes, which emerge on the vortices within superconductors, are of significant interest for both theoretical physics and practical applications. The findings enhance our understanding of topological quantum states, with important implications for developing fault-tolerant quantum computing systems. The robustness of these zero modes against local disturbances makes them promising candidates for quantum computation (Volovik, G. E., 1999).

**Nuclear Magnetic Order in Solid Helium-3**

A landmark study by Halperin and colleagues in 1974 made significant strides in low-temperature physics with the observation of nuclear magnetic order in solid Helium-3. This discovery revealed how nuclear spins in solid Helium-3 can arrange into an ordered state, similar to that found in magnetic materials. The study showed that quantum effects dominate the behavior of solid Helium-3, offering valuable insights into magnetism in quantum systems and the role of spin interactions in quantum fluids. William Halperin, a professor at Northwestern University, has extensively researched ultra-low temperature quantum liquids and solids, with a particular focus on liquid and solid Helium-3 (Halperin, W. P. et al., 1974).

**Acoustic Faraday Effect in Superfluid Helium-3-B**

A 1999 study by Lee, Haard, Halperin, and Sauls advanced our understanding of quantum fluids by uncovering the acoustic Faraday effect in superfluid Helium-3-B. This effect is analogous to the electromagnetic Faraday effect, in which the polarization of light is rotated under the influence of a magnetic field. In the acoustic Faraday effect, sound waves in the superfluid exhibit similar rotational behavior. This discovery not only introduced a novel phenomenon in quantum fluids but also provided new experimental methods for probing the unique properties of superfluid Helium-3. The acoustic Faraday effect offers insights into the rotational symmetries and quantum mechanical properties of superfluid (Lee, Y. et al., 1999).

Additionally, this work contributes to the ongoing research into supersolidity by disproving the existence of a direct superfluid-to-supersolid transition in certain commensurate lattice systems, thus enhancing the understanding of supersolidity (Pollet, L. et al., 2010). Interestingly, this research also addresses the existence and smoothness of the famous Navier-Stokes equation, leveraging the physical phenomena observed in superfluid Helium-3 as the basis for discussions in fluid mechanics. Considering the practical physical implications of the Navier-Stokes equation, the solution under these conditions must be a smooth function that does not exhibit rapid increases, offering a potential solution to one of the key mathematical problems.

1. $v(x,t) \in \left[C^{\infty}\left(R^3 \times [0,\infty)\right)\right]^3, p(x,t) \in C^{\infty}\left(R^3 \times [0,\infty)\right)$
2. $\exists \text{const } E \in (0,\infty), \int_{R^3} \left|v(x,t)^2\right| dx < E, \forall t \geq 0$

**Condition 1** indicates that the function is smooth and globally defined, while **Condition 2** implies that the kinetic energy of the solution is bounded both from above and below, globally.

## 2. Methods and Discussion

Given that the famous expression of the scalar Navier–Stokes equation as shown as

$$\frac{\partial v_i}{\partial t} + \sum_{j=1}^{3} \frac{\partial v_i}{\partial x_j} v_j = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \sum_{j=1}^{3} \frac{\partial^2 v_i}{\partial x_j^2} v_j + f_i(x,t) \tag{1}$$

In expression of *Einstein's relationships*:

$$D = \mu_p K_B T \tag{2}$$

Consider a globular protein as a cusp (i.e., a local singularity) that remains undissolved in $^3$He at a specific temperature with $T_{C'} = 0.00206K \approx 0.002K$ . Therefore, the diffusion of the globular protein can be expressed as:

$$\begin{gathered} D \approx \mu_p K_B T_{C'} = \mu_p \cdot const = 2.8 \times 10^{-26} J \cdot \mu_p, \\ D \approx \left[ \underbrace{2.85 \times 10^{-26} J}_{=E_{plate}^{Casimir}} - \underbrace{0.05 \times 10^{-26} J}_{=\Delta flct.} \right] \cdot \mu_p \end{gathered} \tag{3}$$

Here $T_{C'}$ means the Casimir Temperature with values of 0.00206 K $(2.06\,mK)$. If $\mu_p$ can be denoted as

$$\mu_p = \lim_{v_T \to C} \frac{v_T}{F_{Casimir}} \tag{4}$$

Where $C$ represents the speed of light for massless bosons (or fermions, as discussed in the section titled "Massless Fermions" on pages 2-3 of Ref. (Volovik, G. E., 1996), in proximity to the indicated globular protein, within a Markov chain (random walk) framework

$$\mu_{p,boson} = C/F_{Casimir}, m_{boson} = m_{fermion} = 0 \text{ with } |x_i| \overset{(Mrk.)}{=} |x_j|, i \neq j \tag{5}$$

$$C \cdot \mu_{p,boson} \Delta x = \frac{C^2 \Delta x}{F_{Casimir}} = 0, t \geq 0, \qquad C \equiv \frac{\Delta x}{\tau} \tag{5.1}$$

Where $\Delta x \neq f(x)$ since $f(x_1, x_2, x_3, \cdots)$ belongs to memorization, does not fit the definition of a Markov chain. It is important to note that independent protein is denoted as C, and event-1 is independent of event-2. Notably, the above additional conditions are required. Therefore, due to the cosmological constant, the Casimir effect is regarded as (Note 1).

$$\mu_p = const, global, \qquad D = const \to 0 \,(\text{smooth implicit functions}) \tag{6}$$

With $v_i = v_j = v_T = const$ and then the scalar Navier–Stokes equation becomes the external force (Note 2):

$$f_i(x,t) = \frac{1}{\rho} \frac{\partial p}{\partial x_i}, local \tag{7}$$

Obviously, the term above reveals the gradient of the pressure in the fluid He. Based on the statistical function of the liquid boson gas:

$$p = \frac{U}{V} \propto m^{3/2} \int_0^\infty \frac{\varepsilon_i^{3/2}}{e^{\frac{\varepsilon_i - \mu}{kT}} - 1} d\varepsilon_i, \varepsilon_i - \mu = finite \tag{8}$$

If $T = T_C \approx T_{C'}$ accordingly, exactly use the approximation to obtain constant pressure (because of the integral terms become to the zeta function of $s = 5/2, \zeta(5/2)$ and is constant):

$$p \approx \frac{kT_{C'}}{V} = const \quad \text{with} \quad \varepsilon_i - \mu >> kT_{C'} \quad D.O.S. \quad \bar{n}_i(\varepsilon) \approx \frac{kT_{C'} g_i(\varepsilon)}{\varepsilon_i - \mu} \tag{9}$$

Where *D.O.S.* means the density of states. Substitute the above equation into the expression of the external force in the global region:

$$f_i(x,t) = \frac{1}{\rho} \frac{\partial}{\partial x_i} \frac{kT_{C'}}{x_i^3} = -3 \frac{kT_{C'}}{\rho} \frac{1}{x_i^4} < 0, \qquad f_i(x,t) = \frac{1}{\rho} \frac{\partial}{\partial x_i} \frac{kT_{C'}}{x_i^3} = -3 \frac{k}{\rho x_i} \left( \frac{T_{C'}}{x_i^3} \right) < 0 \tag{10}$$

And,

$$f_i(x,t) = \frac{3kT_{C'}}{m_{[^3He]}(-x_i)} = const,$$
$$f_i(x,t) = \frac{-3gkT_{C'}}{E_{[^3He]}} = const, E > 0, E \in (0,\infty), \quad (11)$$
$$\frac{C^2\Delta x}{const \cdot F_{Casimir}} < E, const \to 0, const \neq 0$$

Sequentially (Note 3),

$$F_{Casimir} \cdot E > \upsilon(\text{ground}), \quad \upsilon \equiv \frac{C^2\Delta x}{const} \quad \text{such as Fermat's principle}$$
$$F_{Casimir}\left(-\frac{GM}{x}\right) > \upsilon, \quad (12)$$
$$F_{Casimir} < -\frac{\upsilon x}{GM}$$

This shows that the external force acting on $^3$He, $f_i(x,t)$, involves terms related to the Casimir force, $T_{C'}/x_i^3$, as referenced in (Volovik, G. E., 1996), where $-x_i \equiv h_o$. See Fig. 1. If $-x_j \equiv h$, the **"AB interface"**—this serves as the complete proof for the existence and the smoothness of the Navier–Stokes equation.

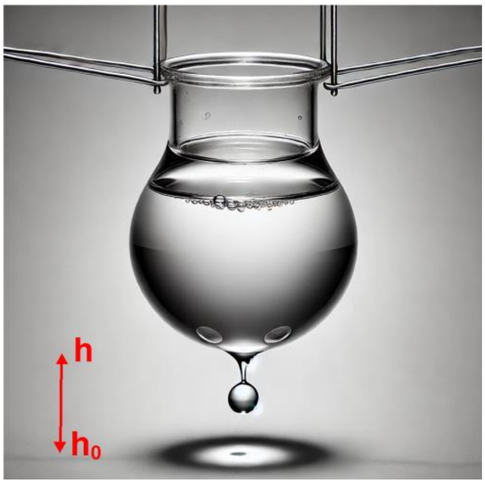


Figure 1. $^3$He process of the superfluid. In this case, $|x| := x = [h_0, h]$

By using the differential method, here is another respect based on Eq. (10),

$$f_i^{''}(x,t) = (-3)(-4)(-5)\frac{kT_{C'}}{\rho}\frac{1}{x_i^6} < 0 \quad (13)$$

Obviously, it corresponds to a repulsive force (note the negative sign). We can construct the general formula for Eq. (13) as follows

$$f_i^{(n/3)}(x,t) = (-1)^{n/2} \frac{(n-1)!}{2!} \frac{kT_{C'}}{\rho} \frac{1}{x_i^n} < 0, 5 < n \le 6 \tag{14}$$

This fits the requirement of Eq. (7), i.e., the localization (see Fig. 1, which illustrates the liquid ³He condensates). In Eq. (7), the force is considered as "anti-gravity", this is an internal force system. When viscosity is taken into account, the equation becomes:

$$f_i(x,t) = \frac{1}{\rho} \frac{\partial p}{\partial x_i}, global \tag{15}$$

And Eq. (12) becomes

$$F_{Casimir} < -\frac{\upsilon x}{GM} \tag{16}$$

Alternately and use the approximation (Note 4):

$$\begin{aligned} &F_{Casimir} \approx -\frac{C^2 x \Delta x}{const} \frac{1}{M}, \Delta x \overset{(Mrk.)}{\equiv} \left| x_i - x_j \right| \ne f(x_1, x_2, x_3, \cdots) \\ &\lim_{h\to\infty} \int_{x=h}^{h_0} F_{Casimir} dx \approx -\frac{C^2 \Delta x}{const} \int_{x=r_{\min}}^{r} \frac{x dx}{M}, \\ &\lim_{h\to\infty} E(h) \approx \frac{C^2 \Delta x}{const \cdot M} \frac{r^2 - r_{\min}{}^2}{2} + E(h_0) \end{aligned} \tag{17}$$

In the process of cooling liquid ³He, reaching the temperature $\mathrm{T} \approx 0.00333\,\mathrm{K}$ ³He leads to thermal noise, causing the entire ³He molecules to behave as if in a "boiling" state. As a result, the gap in random walks vanishes, as described in **(Garcia, R. & Chan, M. H. W., 2002)**

$$\begin{aligned} &\lim_{h\to\infty} E(h) \approx \frac{C^4}{const \cdot U_{earth}} \frac{r^2 - r_{\min}{}^2}{2} + E(h_0) = const, r > r_{\min}, \\ &E(h) = mgh \end{aligned} \tag{18}$$

(Enclosed on atoms/molecules)

In permission of approximation: $T_{C'} = 0.00206\,K \approx 0.002\,K$, and by using Eqs. (3) and (6). If we let $h \to \infty$ ( $x \to \infty$ ), $const \to 0$ with fixed *C* and the Casimir temperature $T_{C'} \approx 0.00206K$ , this corresponds to a solid object (e.g., ³He clusters) possessing the conditions to move into infinite space-time, far from Earth's Ricci curvature. Notably, in this context, the picture of $U_{earth} = m_{earth} C^2 \equiv MC^2$ offers the advantage of facilitating comparison between the escaped ³He system's Hamiltonian distance $\left(\Delta h = \left| h - h_0 \right|\right)$ from Earth. In physical terms, $\exists \mathrm{const\ E} \in (0,\infty), \int_{R^3} \left| v(x,t)^2 \right| dx < E, \forall t \ge 0$ in Eq. (18) which has an upper limit h and a lower limit $h_0$ in global regions.

Table 1. One of the Millennium Prize Problems has been successfully solved

| **The Constants by This Paper's Statements** | **The Official Statement** |
|---|---|
| The existence:<br>$\text{const} \to 0$<br>$\lim_{h\to\infty} E(h) \to \infty, x \to \infty$ | $\exists \text{const E} \in (0,\infty), \int_{R^3} \left|v(x,t)^2\right| dx < E, \forall t \geq 0$ |
| Conditions:<br>Eq.(11) and (17) | |

Note: Above all *const* is denoted as $C^\infty \in \left(R^n \times [0,\infty)\right)$.

### 3. Conclusion

This paper ultimately resolves the famous Navier-Stokes equation at low temperature (i.e., the Casimir temperature $T_{C'} = 2.06\, mK$ ). By utilizing the nature of a cusp in the globular protein, as indicated by Einstein, undissolved in liquid $^3He$ under specific conditions, once differentiable, the existence and smoothness of the Navier-Stokes equation can be established. This is attributed to the superfluidity of liquid $^3He$, which possesses an AB interface. Through the deduction process, we employed the variation method $\upsilon \equiv c_0 = 0$, which fully embodies the AB interface's principles. When one considers $\upsilon > 0$, $\left|x_i\right| > 0, \left|x_j\right| > 0, i \neq j$ [i.e., the constant pressure (smooth functions) appearing in a fluid layer, or what is commonly known as the "Markov chain tree theorem", where the transition probability from state *i* to state *j* is defined], the official statement of the problem, as posed by Charles L. Fefferman, requires a solution. In his PDF titled "Existence and Smoothness of the Navier-Stokes Equation", $\upsilon \equiv c_0 = 0$ actually leads to the vanishing of Equation (1) from the literature [referring to the statement on incompressible superfluid (a system of BEC strong interactions) at constant pressure, as given by Euler's equation], allowing one to recover the Navier-Stokes equation. This conclusion cannot be obtained from other studies by using $H_2O$ at 300 K as the fluid for solving problems of Navier-Stokes equation, since that the globular protein is originally dissolved in $H_2O$. We suggest the further research of this theme must aim at the direction of the superfluidity at low temperatures and is suitable.

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**Authors' contributions**

P.-H. Lee and H.-T. H. Su were responsible for the study design and manuscript revision. Both authors contributed to data collection and collaborated on drafting the manuscript, with P.-H. Lee overseeing the final revisions. All authors read and approved the final manuscript. Additionally, P.-H. Lee and H.-T. H. Su contributed equally to this study.

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**Competing interests**

The authors declare that they have no known competing financial interests or personal relationships that could have influenced the work reported in this paper.

**Informed consent**

Obtained.

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**Data availability statement**

The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

**Data sharing statement**

No additional data are available.



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**Notes**

Note 1. The smooth implicit function implies that even if the cusp can be differentiable in case of this paper.

Note 2. In the Markov process, $f_j(x,t) = \frac{1}{\rho}\frac{\partial p}{\partial x_j}$ has a 50% probability.

Note 3. The “ground” $\upsilon$ origins from the idea of the variation method.

Note 4. In Eq. (16), we apply the concept of recurrence (Gambler's ruin), where $\Delta x$ is finite and $\left|\lim_{h\to\infty}\int_{x=h}^{h_0} F_{Casimir}dx\right|$ is infinite, with $f(x_1, x_2, x_3, \cdots)$ being individually numbered or memorized.