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Special Relativity, Mass, and Energy; $E = mc^2$

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Abstract

The purpose of this paper to provide a proof of $E = mc^2$ using a derivation completely consistent with Special Relativity. We base our proof on the calculation of the relativistic kinetic energy, K, and the invariance of the momentum 4-vector in Minkowski space. This proof provides a useful pedagogical method for teaching the momentum 4-vector through an excellent example to undergraduate physics students.

Keywords: $E = mc^2$, mass and energy, special relativity, the momentum 4-vector, Minkowski space

1. Introduction

A century of experiments has consistently shown that $E = mc^2$. Einstein predicted that relationship in 1905 (Einstein). However, a recent article (Rothman) pointed out that Einstein's proof was non-relativistic. We know that $E = mc^2$ and the purpose of this paper to provide a proof using a derivation completely consistent with Special Relativity (SR).

One of Albert Einstein's main goals when he formulated his theory of Special Relativity in 1905 was to consider what observers of an event would measure as seen from two different frames of reference. These frames, S and S', would be moving relative to each other at a constant velocity, v. The two postulates that he used were that 1) The laws of physics are the same in both frames, and 2) The speed of light, c, would be measured to be the same value in both inertial frames.

The classical method of comparing observations in two different frames has had a long history and was treated by transforming coordinates between the two frames. The transformation technique that was used in the past was developed by Galileo and worked quite well as long as v was small compared to c. However, it did not work for Maxwell's theory of electricity and magnetism and was inconsistent with Einstein's postulates. Hendrik Lorentz had developed a transformation consistent with Maxwell's theory and the Lorentz transform is the heart of SR. It is consistent with all experimental tests, but its consequence is that space and time are not absolute. Furthermore, the concept of simultaneity is impossible between the two frames.

The history of the discovery of the mass and energy relationship, and the concept and the description of $E = mc^2$ had been discussed in numerous articles (Fadner), (Okun; 1989; 2005; 2008), (Hecht), (Riggs), (Wilson). The derivation of $E = mc^2$ using different methods has been investigated in many papers (Feigenbaum & Mermin), (Rohrlich), (D'Abramo), (Redzic). The history of discovery of the mass and energy relationship including pedagogical methods would be useful in education of the physics students. Here authors provide a method. This method is based on the calculation of the relativistic kinetic energy, K, and the invariance of the momentum 4-vector in Minkowski space. This would be useful for undergraduate physics students to learn and practice these topics through the use of this proof of $E = mc^2$ as an excellent example.

2. The Invariance of the Momentum 4-Vector in Minkowski Space

Many of the ideas of SR were also developed by Henri Poincaré and Lorentz. Hermann Minkowski (Minkowski; 1908; 1908-1909) took the basic work of Einstein, Lorentz, (Lorentz; 1899; 1904; 1916) and Poincaré (Poincaré) introduced the specific four-dimensional approach which is mathematically treated in the 4D Minkowski Spacetime. An event in one frame, S, has the vector coordinates, $\vec{R} = (ct, x, y, z)$. The Lorentz transformation calculates the coordinates in S', $\vec{R}' = (ct', x', y', z')$. Note that there are several conventions for these

Minkowski spaces and this is our choice of convention. With this convention we can calculate $\vec{R} \cdot \vec{R}$ and $\vec{R}' \cdot \vec{R}'$ in Minkowski Space:

$$\vec{R} \cdot \vec{R} = (ct)^2 - x^2 - y^2 - z^2 \tag{1}$$

and

$$\vec{R}' \cdot \vec{R}' = (ct')^2 - x'^2 - y'^2 - z'^2. \tag{2}$$

A consequence of SR is that

$$\Delta\vec{R} \cdot \Delta\vec{R} = \Delta\vec{R}' \cdot \Delta\vec{R}' . \tag{3}$$

where $\Delta\vec{R}$ and $\Delta\vec{R}'$ are intervals in S and S' frams. In other words, $\Delta\vec{R} \cdot \Delta\vec{R}$ remains the same value as viewed by any two frames, S and S'. If $\Delta\vec{R} \cdot \Delta\vec{R} > 0$ the interval between two events is time-like. If $\Delta\vec{R} \cdot \Delta\vec{R} = 0$ then the interval between two events is light-like. And if $\Delta\vec{R} \cdot \Delta\vec{R} < 0$ the interval between two events is space-like.

The following two dimensionless parameters are very helpful when doing calculations in SR.

$$\beta = \frac{v}{c} \tag{4}$$

and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{5}$$

The expression for β in terms of γ:

$$\beta = \frac{\sqrt{\gamma^2-1}}{\gamma} \tag{6}$$

The values of β are $0 \leq \beta < 1$.

Equation (4) yields $v = \beta c$. When β is near 0, the velocity is then considered to be non-relativistic and calculations can be done using the methods of Galileo and Newton. As β closely approaches 1 the regime becomes highly relativistic and there are approximations that simplify calculations. For intermediate values of β one must use the full-blown relativistic equations. For all objects or particles with mass, β can never reach the value of 1. Particle velocities can approach arbitrarily close to c but not reach it because the relativistic momentum and kinetic energy would have to increase without limit.

The values of γ are $\gamma \geq 1$. If $\gamma = 1$ the object or particle is at rest ($\beta = 0$). The larger the value of γ the closer the velocity is to c, and γ can increase without bound. An example of the use of γ besides its use in the Lorentz transformation, is the apparent increase of particle masses when the particle is in motion. This increase was experimentally observed with electrons prior to Einstein's 1905 paper (Thomson), (Kaufmann; 1901a; 1901b; 1902a; 1902b). For example, the particle of mass, m, is at rest in frame S' which is moving with velocity, v, relative to frame S. The mass measured in S' will be m since the particle is at rest in its own frame. An observer in frame S will measure this same particle to have a mass equal to γm. It is clear that the increase in mass will affect the momentum and kinetic energy of a particle with relativistic velocity.

3. Using Relativistic Kinetic Energy and the Invariance of the Momentum 4-Vector in Minkowski Space for the Proof of $E = mc^2$

Einstein's 1905 proof started with special relativity then at the end, he used the Newtonian limit and he did not include this effect (Einstein). Our proof does include the effect of special relativity. We base our proof on the calculation of the relativistic kinetic energy, K, and the invariance of the momentum 4-vector in Minkowski space.

The coordinate system can be chosen so that the relative velocities lie along the x-axis and only a single space dimension is necessary to do calculations. In this case we can find the momentum observed in frame S when a particle of mass, m, is at rest in S'. (One often refers to m as the "rest mass"). S' is moving at a velocity, v, along the positive x-axis relative to S. In this case the linear momentum is given by

$$p = \gamma m v = \gamma m c \beta = mc\sqrt{\gamma^2 - 1}. \tag{7}$$

Deriving the kinetic energy, K, is not as simple as the derivation for the momentum. We start with the principle that the work done on a particle is equal to its change in kinetic energy. If we look at the rate of change in the kinetic energy which is equal to the power provided, we have

$$\frac{dK}{dt} = \vec{F} \cdot \vec{v} = \vec{v} \cdot \frac{d\vec{p}}{dt}. \tag{8}$$

This can be rewritten as

$$dK = c\vec{\beta} \cdot d\vec{p} = (mc^2\beta)\, d\left(\sqrt{\gamma^2 - 1}\right) = mc^2\beta \frac{\gamma}{\sqrt{\gamma^2-1)}} d\gamma. \tag{9}$$

Substituting β from (6) into (9) we get that

$$dK = mc^2 d\gamma. \tag{10}$$

Integrating this very simple equation, we get

$$K = \gamma mc^2 + constant. \tag{11}$$

We know that $K = 0$ when $\gamma = 1$. Thus the integration constant is:

$$constant = -mc^2 \tag{12}$$

and

$$K = (\gamma - 1)mc^2. \tag{13}$$

The following, therefore, is a proof of $E = mc^2$ that is fully relativistic.

The total energy, E, is the sum of the kinetic energy, K, plus some energy present when the particle is at rest. This energy might even be zero. Let's call this rest energy Q.

Then

$$E = K + Q. \tag{14}$$

In the rest frame of the particle the momentum 4-vector,

$$\vec{p} = (\frac{Q}{c}, 0,0,0), \tag{15}$$

and

$$\vec{p} \cdot \vec{p} = \left(\frac{Q}{c}\right)^2. \tag{16}$$

Thus, the value of $\vec{p} \cdot \vec{p} = \left(\frac{Q}{c}\right)^2$ in all frames because of the invariance of the momentum 4-vector.

For the frame where the particle has a momentum, p_x, in the x-direction and a total energy, $E = K + Q$, the momentum 4-vector,

$$p = (\frac{E}{c}, p_x, 0,0), \tag{17}$$

and

$$\vec{p} \cdot \vec{p} = \left(\frac{E}{c}\right)^2 - p_x^2 = \left(\frac{Q}{c}\right)^2. \tag{18}$$

Use equation (13), $K = (\gamma - 1)mc^2$, in equation (14) to calculate

$$\left(\frac{E}{c}\right)^2 = \frac{(K+Q)^2}{c^2} = (\gamma mc - mc + \frac{Q}{c})^2 = (\gamma^2 - 2\gamma + 1)m^2c^2 + 2(\gamma - 1)mQ + \left(\frac{Q}{c}\right)^2. \tag{19}$$

Next using equation (7), $p_x = \gamma\beta mc = \sqrt{\gamma^2 - 1}\, mc$, we obtain

$$p_x^2 = (\gamma^2 - 1)m^2c^2. \tag{20}$$

Now equations (18), (19) and (20) yield

$$\left(\frac{Q}{c}\right)^2 = \left(\frac{Q}{c}\right)^2 + 2(\gamma - 1)(mQ - m^2c^2). \tag{21}$$

This yields the equation

$$2(\gamma - 1)(mQ - m^2c^2) = 0. \tag{22}$$

The solution to this equation is

$$Q = mc^2. \tag{23}$$

The rest energy is the rest mass energy, $E = mc^2$. This result has been achieved by using SR with no non-relativistic approximations needed.

4. Conclusion

In this paper, we present a proof of $E = mc^2$ using a derivation fully consistent with Special Relativity. Our proof is based on the calculation of relativistic kinetic energy and the invariance of the momentum 4-vector in Minkowski space. This approach offers a valuable pedagogical method for teaching the concept of the momentum 4-vector, serving as an excellent example for undergraduate physics students.

Authors' contributions

All authors contributed to the study and participated in drafting and revising this manuscript. All authors have read and approved the final version of the manuscript.

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