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Imaging n-Dimensional Spaces Within m-Dimensional Spaces: An Extension of Hinton's Method

Louis M. Houston[1]

[1] Department of Innovation and Research, The University of Louisiana at Lafayette, LA, USA

Correspondence: Louis M. Houston, Department of Innovation and Research, The University of Louisiana at Lafayette, LA, USA.

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Abstract

We derive a Euclidean manifold that is virtually cyclic using a simple equation based on a Euclidean geometry and related to Hinton's method. The derived equation is simple to understand, but able to project n-dimensional spaces into m-dimensional spaces. In addition, the method produces exact images of rectangular cuboids as elements of a vector space, implying that information is a **vector** and not a scalar.

Keywords: manifold, topology, tesseract, Euclidean, countability, uncountability, information

1. Introduction

The general concept of Euclidean space (Howard, 1987) with any number of dimensions was fully developed by the Swiss mathematician Ludwig Schläfli (1901, 1853). Schläfli's work received little attention during his lifetime and was published only posthumously, in 1901, but meanwhile the fourth Euclidean dimension was rediscovered by others. In 1880 Charles Howard Hinton (popularized it in an essay, "What is the Fourth Dimension?", in which he explained the concept of a "four-dimensional cube" or "tesseract" with a step-by-step generalization of the properties of lines, squares, and cubes. The simplest form of Hinton's method is to draw two ordinary 3D cubes in 2D space, one encompassing the other, separated by an "unseen" distance, and then draw lines between their equivalent vertices. This is illustrated in Figure 1.

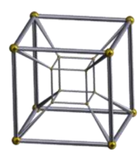


Figure 1. A four-dimensional cube known as a "tesseract"

In mathematics, a 3-manifold is a topological space that locally looks like a 3-dimensional Euclidean space (Johannson, 1979).

Topologists have known how to describe and classify all possible 2-manifolds for more than a century, but the systematic classification of all 3-manifolds has remained an unsolved problem due to the exceedingly complex forms to which some three-manifolds give rise. Our work directly addresses this problem and generalizes the solution for arbitrary dimensions.

What would be the value of a simple equation that projects informational structures from one dimension to any other dimension, enabling dimensional reduction or expansion? And what if it was determined that all such modifications could produce 3-dimensional images of rectangular cuboids? (Houston, 2023).

If it were determined that all Euclidian-dimensional projections produced 3-dimensional images of cuboids, this would provide a useful constraint on the equation's output. It would indicate that any dimensional projection of an informational structure could result in a three-dimensional cuboid shape, which could simplify analysis and visualization of the data. In addition, it would provide a more viable connection to physics than traditional information theory (Jaynes, 1957) because it so efficiently images physical structures from deterministic information rather than stochastic information. We easily surmise that our work can simplify higher dimensional analysis by answering the above questions.

2. The Projection Theorem

This Theorem derives the mathematics needed to expand a positive integer into a finite sum of a predetermined length.

If $n, k \in \mathbb{Z}^+ \cup 0 \; and \; m \in \mathbb{Z}^+$,

Then

$$n = \sum_{k=0}^{m-1} \left\lfloor \frac{n+k}{m} \right\rfloor$$

.

Proof:

Let:

$$S = \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n+1}{m} \right\rfloor + \left\lfloor \frac{n+2}{m} \right\rfloor + .. + \left\lfloor \frac{n+m-1}{m} \right\rfloor$$

Case (1): $n \le m$:

We can rewrite S as:

$$S = \left\lfloor 1 + \frac{n-m}{m} \right\rfloor + \left\lfloor 1 + \frac{n-m+1}{m} \right\rfloor + \left\lfloor 1 + \frac{n-m+2}{m} \right\rfloor + .. + \left\lfloor \frac{n}{m} \right\rfloor$$

If n=m, then S=m. If n = m-1, then S=m-1. If m=1, then S=1.

Case (2): n >m:

Let n=mp+R, R<m. Then S can be written as:

$$S = \left\lfloor 1 + \frac{mp+R-m}{m} \right\rfloor + \left\lfloor 1 + \frac{mp+R-m+1}{m} \right\rfloor + .. + \left\lfloor 1 + \frac{mp+R-1}{m} \right\rfloor$$

$$S = mp + \left\lfloor 1 + \frac{R-m}{m} \right\rfloor + \left\lfloor 1 + \frac{R-m+1}{m} \right\rfloor + .. + \left\lfloor 1 + \frac{R-1}{m} \right\rfloor$$

Which reduces to

S=mp+R ⊡

Corollary:

$$x^n = x^{\sum_{k=0}^{m-1} \left\lfloor \frac{n+k}{m} \right\rfloor}, x \in \mathbb{R}$$

Or more specifically,

$$x_m^n = x^{\left\lfloor \frac{n}{m} \right\rfloor} * x^{\left\lfloor \frac{n+1}{m} \right\rfloor} * x^{\left\lfloor \frac{n+2}{m} \right\rfloor} *..x^{\left\lfloor \frac{n+m-1}{m} \right\rfloor}. \quad (1)$$

This series projects an *n*-dimensional 'volume' into an *m*-dimensional 'volume'.

Consider the case for m=3:

$$x_3^n = x^{\left\lfloor \frac{n}{3} \right\rfloor} * x^{\left\lfloor \frac{n+1}{3} \right\rfloor} * x^{\left\lfloor \frac{n+3-1}{3} \right\rfloor}. \quad (2)$$

This equation specifically generates the 3 cuboid states: a cube, a long cuboid, and a wide cuboid. In this case, there is no difference between the input dimensions and the output dimensions. We can produce 3-D graphic images by treating each factor in equation (2) as elements of an n-dimensional volume within 3-dimensional space (i.e., length, width, and height). The unit cubes that comprise our images are called 'voxels'. Voxels are essentially 3-D pixels, but instead of being squares, they are perfect cubes of unit volume (Shchorova, 2015). In theory, voxels are the ideal modeling technique for replicating reality. After-all, our world is made of something akin to voxels (but they are much smaller, and we call them 'sub-atomic particles'). If we have a high enough density (or 'resolution') and the proper rendering techniques, we can use voxels to replicate real-world objects that would be impossible to differentiate from the real thing. Note that the unit cubic cells (i.e., boxes or voxels) shown in Figure (2) are contiguous, while the individual cuboids are spatially separated. x^n is an exponential set of uncountably infinite subsets, with the same cardinality as the real numbers, $\mathbb{R}$. This precludes distinct, real-valued measurements, required to acquire, or store countable packets of information. However, **geometry** enables nearly precise images of cuboids, as one way around this impediment, because the images consist of **countable voxels** within separate cuboids that can undergo continuous, invariant translation.

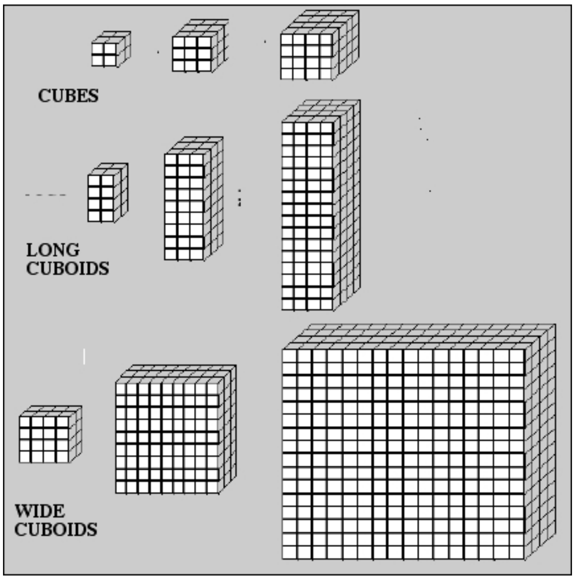


Figure 2. Some examples of cuboids (cubes, long cuboids, and wide cuboids) produced by the finite series given in (1). The phases evolve vertically, from top to bottom.

3. Cuboid Vector Space

Figure 2 shows 2-D views of ***exact*** models of cuboids that occur, commonly, in certain natural minerals, such as diamonds (Pechnikov and Kaminsky, 2008) The pictures are also isomorphic to *solid* (cube), *liquid* (long cuboid), and gas (wide cuboid) phases of matter (Bridgman, 1937). Observe that the phase changes are what gives the vector its **direction**, while the number of voxels are what determines its **magnitude**. Cuboid geometric phases are also cyclic with distinctive phase transitions, based on geometry and shown in Figure 3.

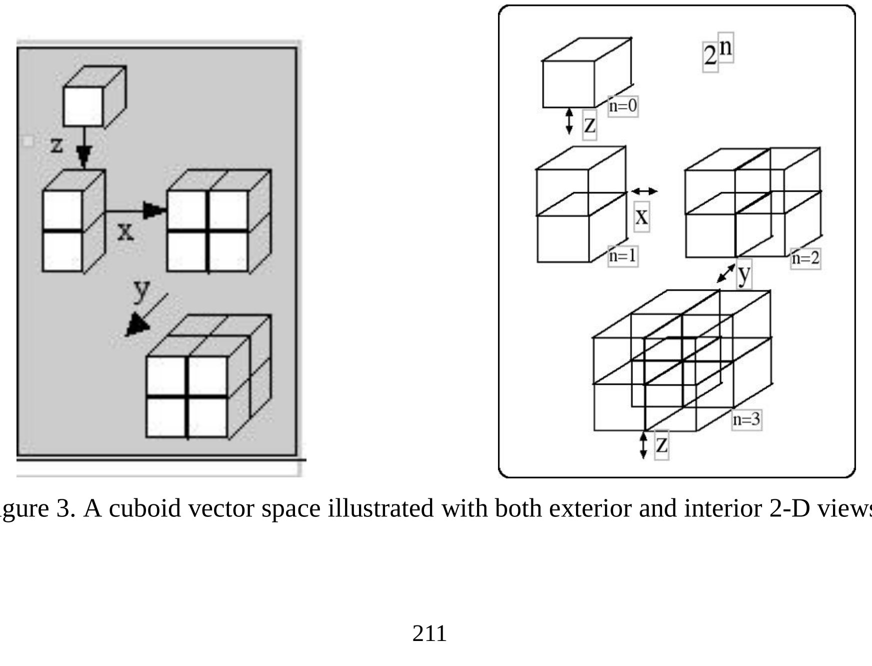
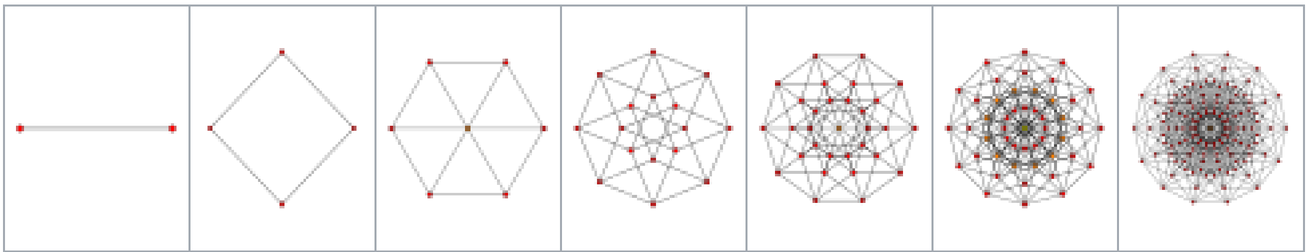


Figure 3. A cuboid vector space illustrated with both exterior and interior 2-D views

4. Visualization

In order to appreciate the level of complexity that our cuboid images **remove** from the more standard visualization methods, we show in Figure 4 some Petrie Polygons.



Line segment	Square	Cube	4-cube	5-cube	6-cube	7-cube

Figure 4. A more standard approach to multi-dimensional imaging (2-D views). Petrie Polygons courtesy of Wikipedia.

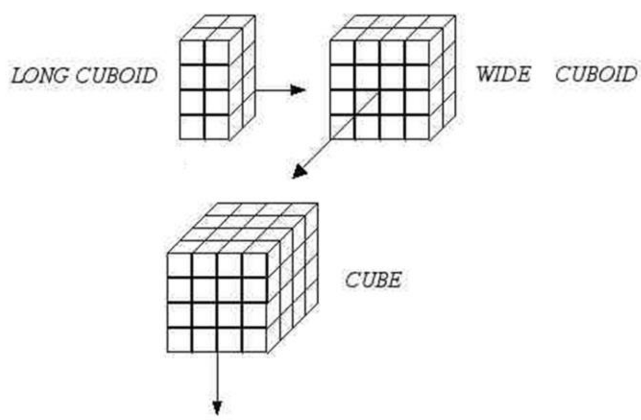


Figure 5. A Cuboid cycle: LONG CUBOID (4D) to WIDE CUBOID (5D) to CUBE (6D)

SIZE= 16 to 32 to 64 voxels. (2-D perspective view).

Based on visual comparison between Figures 4 and 5, a significant reduction in complexity is very clear. Note that Hinton's method of imaging a 4D cube started by coupling two cubes, one inside the other. In contrast, our approach starts with *externally* connecting two cuboids.

5. Manifolds

In brief, a (real) n-*dimensional manifold* is a topological space M for which every point $x \in M$ has a neighborhood homeomorphic to Euclidean space R^n. This definition clearly applies to cuboids and even to curved 3-D spaces like a torus:

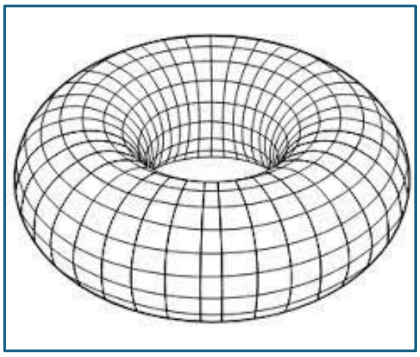


Figure 6. A 3-maniold with the shape of a torus. (Wikipedia)

Note the local Euclidean unit components.

Recall equation (2):

$$x_3^n = x^{\left\lfloor \frac{n}{3} \right\rfloor} * x^{\left\lfloor \frac{n+1}{3} \right\rfloor} * x^{\left\lfloor \frac{n+3-1}{3} \right\rfloor}.$$

In which x_3^n is the volume of the cuboid, a scalar equal to the determinant of the associated diagonal matrix, written as:

$$\vec{x}_3^n = \begin{pmatrix} x^{\left\lfloor \frac{n}{3} \right\rfloor} & 0 & 0 \\ 0 & x^{\left\lfloor \frac{n+1}{3} \right\rfloor} & 0 \\ 0 & 0 & x^{\left\lfloor \frac{n+3-1}{3} \right\rfloor} \end{pmatrix} \begin{pmatrix} i \\ j \\ k \end{pmatrix} \quad (3)$$

In which the unit vectors are cyclic and m=3 is simply replaced by m for a general representation: $\vec{x}_m^n$ or an m-manifold.

6. Conclusions

Based largely on a Theorem ("The Projection Theorem") for decomposing a positive integer into a finite series, we find volume components of multi-dimensional cuboids that geometrically comprise vectors with both magnitude and direction. Consequently, our cuboid space defines information as a vector. This result was illustrated by Houston (2023) and will be more thoroughly examined in future research.

The unit cubes or voxels support a discrete Euclidean space. It would be interesting to examine how well a discrete spherical or discrete cylindrical space would compare to our Euclidean method for multi-dimensional imaging. And is our imaging method sufficient for a c-manifold that describes the shape of the universe?

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