Hamilton-Jacobi Treatment of Singular Systems Using Fractional Calculus

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Received: October 27, 2023 Accepted: March 22, 2024 Online Published: April 22, 2024

doi:10.5539/apr.v16n1p180 URL: https://doi.org/10.5539/apr.v16n1p180

Abstract
In this paper, the theory of fractional singular systems is investigated with second-order derivatives. The fractional Hamilton–Jacobi treatment of these systems is examined. The fractional Hamilton–Jacobi partial differential equations (FHJPDEs) are constructed. The (FHJPDEs) can be solved to obtain the fractional Hamilton–Jacobi function. By building the fractional Hamilton–Jacobi function, the equations of motion can be obtained.

Keywords: Fractional Hamilton-Jacobi Function, fractional calculus, fractional constrained systems

PACS numbers: 11. 10. Ef, 45. 20. –j, 45. 10. Hj

1. Introduction

The study of singular systems has been treated first by (Dirac, 1950) for solving mechanical systems to find the equations of motion for these systems. Following Dirac’s work, researchers studied the Hamilton-Jacobi treatment with more interest and extensively and used the canonical formalism for investigating these systems (Guler, 1992; Pimentel, R.G. and Teixeira 1996; Rabei et al., 2004, 2005; Hasan et al., 2004).

In this paper, we would like to develop our theory using fractional calculus to be applicable for these systems. Recently, the Hamilton-Jacobi formalism has been investigated for these systems within fractional derivatives with second order Lagrangian (Hasan, 2016; Hasan and Asad, 2017; Hasan, 2018; Hasan, 2020; 2023). In this formalism, researchers examined the Euler-Lagrange equations and Hamilton’s equations.

In this paper, we would like to extend the work for systems with the second order Lagrangian. We constructed a formalism for investigating fractional singular systems and Hamilton-Jacobi formalism for these systems. Besides, we obtained the fractional Hamilton-Jacobi function for these systems.

Now, the basic definitions of fractional derivatives are (Samko et al., 1993).

The left Riemann–Liouville

\[ {_aD^\alpha_t}f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_{a}^{t} (t-\tau)^{n-\alpha-1} f(\tau) d\tau . \]  

and right Riemann–Liouville.

\[ {_bD^\alpha_t}f(t) = \frac{1}{\Gamma(n-\alpha)} \left( -\frac{d}{dt} \right)^n \int_{\tau}^{b} (\tau-t)^{n-\alpha-1} f(\tau) d\tau . \]  

These derivatives have properties as follows:

\[ {_aD^\alpha_t}f(t) = \left( \frac{d}{dt} \right)^\alpha f(t) ; \]
\[ iD^\alpha_b f(t) = \left( -\frac{d^\alpha}{dt^\alpha} \right) f(t). \] (4)

**Remark**

The fractional derivatives \( D^{\alpha-1}q_i \) and \( D^\alpha q_i \) are treated as coordinates. So, the Poisson bracket can be defined as

\[
\{A, B\} = \frac{\partial A}{\partial D^{\alpha-1}q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial D^{\alpha-1}q_i} + \frac{\partial A}{\partial \pi_i} \frac{\partial B}{\partial D^\alpha q_i} - \frac{\partial A}{\partial D^\alpha q_i} \frac{\partial B}{\partial \pi_i}. \] (5)

Here, the generalized momenta \( p_i \) and \( \pi_i \) are conjugated to the generalized coordinates \( D^{\alpha-1}q_i \) and \( D^\alpha q_i \) respectively. Thus, the fundamental Poisson brackets for second-order fractional derivatives are written as,

\[
\{D^{\alpha-1}q_i, D^{\alpha-1}q_j\} = \{D^\alpha q_i, D^\alpha q_j\} = 0 = \{D^\alpha q_i, D^{\alpha-1}q_j\} = \{p_i, \pi_i\}. \]

\[
\{D^{\alpha-1}q_i, p_j\} = \delta_{ij} \ and \ \{D^\alpha q_i, \pi_j\} = \delta_{ij}, \ where \ i, j = 1, \ldots, N \] (6)

**2. Singular Lagrangian Using Fractional Calculus**

We start with a Lagrangian depends on the fractional derivatives (Hasan, 2017, 2018, 2023)

\[ L = L(D^{\alpha-1}q_i, D^\alpha q_i, D^{2\alpha} q_i, t). \] (7)

Thus, the fractional of the Hessian matrix is defined as

\[ W_{ij} = \frac{\partial^2 L}{\partial D^{2\alpha}q_i \partial D^{2\alpha}q_j} \ i, j = 1, 2, \ldots, N \] (8)

With property that the fractional Lagrangian is regular and its rank is \( N \), otherwise the fractional Lagrangian is singular \( N - R, \ R < N \), where \( R \) denotes to constraints.

Now, the generalized momenta \( \pi_i \) conjugate to the generalized coordinates \( D^\alpha q_i \) as (Hasan, 2017, 2018, 2023):

\[ \pi_a = \frac{\partial L}{\partial D^{2\alpha}q_a}; \] (9)

\[ \pi_\mu = \frac{\partial L}{\partial D^{2\alpha}q_\mu}. \ a = 1, 2, \ldots, N - R; \ \mu = 1, \ldots, R \] (10)

We solve Eq. (9) to obtain N-R accelerations \( D^{2\alpha}q_a \) in terms of \( D^{\alpha-1}q_i, D^\alpha q_i, \pi_a \) and \( D^{2\alpha}q_\mu \) as follows:
\[ D^{2\alpha} q_a = w_a \left( D^{\alpha-1} q_i, D^\alpha q_i, \pi_a, D^{2\alpha} q_\mu \right). \]  

We substitute Eq. (11) in Eq. (10) to obtain the following equation.

\[ \pi_\mu = -H_\mu^p \left( D^{\alpha-1} q_i, D^\alpha q_i, p_\mu, \pi_a \right) \]  

(12)

A similar expression for the momenta \( p_\mu \) can be obtained as:

\[ p_\mu = -H_\mu^p \left( D^{\alpha-1} q_i, D^\alpha q_i, p_\mu, \pi_a \right) \]  

(13)

and the generalized momenta \( p_i \) conjugate to the generalized coordinate \( D^{\alpha-1} q_i \) can be written as:

\[ p_a = \frac{\partial L}{\partial D^\alpha q_a} - \frac{d}{dt} \left( \frac{\partial L}{\partial D^{2\alpha} q_a} \right); \]  

(14a)

\[ p_\mu = \frac{\partial L}{\partial D^\alpha q_\mu} - \frac{d}{dt} \left( \frac{\partial L}{\partial D^{2\alpha} q_\mu} \right). \]  

(14b)

Equations (12) and (13) can be written as

\[ H_\mu^p \left( D^{\alpha-1} q_i, D^\alpha q_i, p_i, \pi_i \right) = p_\mu + H_\mu^p = 0 \]  

(15a)

\[ H_\mu^\pi \left( D^{\alpha-1} q_i, D^\alpha q_i, p_i, \pi_i \right) = \pi_\mu + H_\mu^\pi = 0 \]  

(15b)

Thus, equations (15) represent primary constraints (Dirac, 1950).

The theory for singular systems states that the number of degrees of freedom can be reduced due to the constraints (Dirac, 1950).

Thus, the fractional Hamiltonian \( H_\omega \) can be defined as

\[ H_\omega = -L(D^{\alpha-1} q_i, D^\alpha q_i, W_a) + p_a D^\alpha q_a + \pi_a D^{2\alpha} q_a - D^\alpha q_\mu H_\mu^p - D^{2\alpha} q_\mu H_\mu^\pi. \]  

(16)

\[ \mu = 1, \ldots, R; \quad a = R + 1, \ldots, N. \]

Here, the generalized momenta \( p_\mu \) and \( \pi_\mu \) are not independent of \( p_a \) and \( \pi_a \). Thus, the set of the fractional Hamilton-Jacobi partial differential equations can be written as

\[ H_\mu^p \left( D^{\alpha-1} q_i, D^\alpha q_i, \frac{\partial S}{\partial D^{\alpha-1} q_a}, \frac{\partial S}{\partial D^\alpha q_a}, \frac{\partial S}{\partial D^{2\alpha} q_a}, \frac{\partial S}{\partial D^\alpha q_v} \right) = p_\mu + H_\mu^p = 0; \]  

(17a)

\[ H_\mu^p \left( D^{\alpha-1} q_i, D^\alpha q_i, \frac{\partial S}{\partial D^{\alpha-1} q_a}, \frac{\partial S}{\partial D^\alpha q_a}, \frac{\partial S}{\partial D^{2\alpha} q_a}, \frac{\partial S}{\partial D^\alpha q_v} \right) = p_\mu + H_\mu^p = 0; \]  

(17b)

\[ H_\mu^\pi \left( D^{\alpha-1} q_i, D^\alpha q_i, \frac{\partial S}{\partial D^{\alpha-1} q_a}, \frac{\partial S}{\partial D^\alpha q_a}, \frac{\partial S}{\partial D^{2\alpha} q_a}, \frac{\partial S}{\partial D^\alpha q_v} \right) = \pi_\mu + H_\mu^\pi = 0. \]  

(17c)

Here, the fractional Hamilton-Jacobi function is written as
\[ S = S(D^{\alpha-1}q_a, D^{\alpha-1}q_\mu, D^\alpha q_a, D^\alpha q_\mu, t), \]  

and the generalized momenta can be defined as

\[ p_a = \frac{\partial S}{\partial D^{\alpha-1}q_a}, \quad p_\mu = \frac{\partial S}{\partial D^{\alpha-1}q_\mu}, \quad \pi_a = \frac{\partial S}{\partial D^\alpha q_a}, \quad \pi_\mu = \frac{\partial S}{\partial D^\alpha q_\mu} \]

Thus, we can write the fractional equations of motion as total differential equations as follows:

\[ dD^{\alpha-1}q_a = \frac{\partial H'}{\partial p_a} dt + \frac{\partial H'}{\partial q_a} dD^{\alpha-1}q_a + \frac{\partial H'}{\partial \pi_a} dD^\alpha q_a, \]  

\[ dD^\alpha q_a = \frac{\partial H'}{\partial \pi_a} dt + \frac{\partial H'}{\partial p_a} dD^{\alpha-1}q_a + \frac{\partial H'}{\partial q_a} dD^\alpha q_a, \]  

\[ -dp_i = \frac{\partial H'}{\partial D^{\alpha-1}q_i} dt + \frac{\partial H'}{\partial D^{\alpha-1}q_i} dD^{\alpha-1}q_i + \frac{\partial H'}{\partial \pi_i} dD^\alpha q_i, \]  

\[ -d\pi_i = \frac{\partial H'}{\partial D^\alpha q_i} dt + \frac{\partial H'}{\partial p_i} dD^{\alpha-1}q_i + \frac{\partial H'}{\partial q_i} dD^\alpha q_i, \]

If the total derivative of equation (17) is zero (Guler, 1992).

\[ dH' = 0; \quad dH'^p = 0; \quad dH'^\pi = 0. \]  

This means that equations (19) are integrable, and the rank of Hessian matrix is \( N - R \) and the constraints reduce the canonical phase space coordinates from \( \{D^{\alpha-1}q_i, p_i, D^\alpha q_i, \pi_i\} \) to \( \{D^{\alpha-1}q_a, p_a, D^\alpha q_a, \pi_a\} \).

Thus, we can write Eqs. (17) as follows

\[ \frac{\partial S}{\partial t} + H(D^{\alpha-1}q, D^\alpha q, p, \pi) = 0; \]  

\[ \frac{\partial S}{\partial D^{\alpha-1}q_\mu} + H_\mu(D^{\alpha-1}q, D^\alpha q, p, \pi) = 0; \]  

\[ \frac{\partial S}{\partial D^\alpha q_\mu} + H^\pi(D^{\alpha-1}q, D^\alpha q, p, \pi) = 0. \]

### 3. Fractional Hamilton-Jacobi Function

Following refs (Hasan et al., 2004; Hasan, 2023), we guess a general solution for Eqs. (21) that can be written in a separable form. Thus, the fractional Hamilton-Jacobi function can be written as

\[ S(D^{\alpha-1}q_a, D^{\alpha-1}q_\mu, D^\alpha q_a, D^\alpha q_\mu, t) = f(t) + W_a(D^{\alpha-1}q_a, E_a) + W_\mu(D^\alpha q_a, E_a, E'_a) + f_\mu(D^{\alpha-1}q_\mu), \]

Here, the functions \( W_a(D^{\alpha-1}q_a, E_a) \), \( W'_a(D^\alpha q_a, E_a, E'_a) \), \( f_\mu(D^{\alpha-1}q_\mu) \), \( f'_\mu(D^\alpha q_\mu) \) are the time-independent Hamilton-Jacobi function, \( D^{\alpha-1}q_a \) and \( D^\alpha q_\mu \) are treated as independent variables.
Finally, the fractional Hamilton-Jacobi function $S$ enable us to obtain the equations of motion by using the canonical transformations (Goldstein, 1980) as follows:

$$
\eta_a = \frac{\partial S}{\partial E_a}; \lambda_a = \frac{\partial S}{\partial q_a}; p_i = \frac{\partial S}{\partial D^{a^{-1}}q_i}; \pi_i = \frac{\partial S}{\partial D^a q_i}.
$$

(23)

Here, $\eta_a$ and $\lambda_a$ are constants.

4. Example

Consider the fractional singular Lagrangian

$$
L = \frac{1}{2} \left( (D^{2\alpha}q)^2 + (D^{2\alpha}q_3)^2 \right) + D^a q_3 D^{\alpha}q_3 + D^\alpha q_3 D^{\alpha^{-1}}q_3 + D^{\alpha^{-1}}q_2 D^a q_2.
$$

(24)

The generalized momenta can be obtained from Eqs. (9,10) and (14)

$$
p_1 = -D^{3\alpha}q_1;
$$

(25a)

$$
p_2 = D^{\alpha^{-1}}q_2 - D^{3\alpha}q_2;
$$

(25b)

$$
p_3 = D^{\alpha^{-1}}q_3 = -H^p_3;
$$

(25c)

$$
\pi_1 = D^{2\alpha}q_1;
$$

(25d)

$$
\pi_2 = D^{2\alpha}q_2;
$$

(25e)

$$
\pi_3 = D^\alpha q_3 = -H^\pi_3.
$$

(25f)

Here, rewrite equations (25c) and (25f) as

$$
H^p_3 = p_3 - D^{\alpha^{-1}}q_3 = 0;
$$

(26a)

$$
H^\pi_3 = \pi_3 - D^\alpha q_3 = 0.
$$

(26b)

and represent as primary constraints (Dirac, 1950).

The Hamiltonian $H_0$ is calculated as

$$
H_0 = p_1 D^\alpha q_1 + (p_2 - D^{\alpha^{-1}}q_2) D^\alpha q_2 + \frac{1}{2} (\pi_1^2 + \pi_2^2).
$$

(27)

The corresponding set of fractional HJPDEs, Eqs. (17), reads

$$
H'_0 = p_0 + H_0 = p_1 D^\alpha q_1 + (p_2 - D^{\alpha^{-1}}q_2) D^\alpha q_2 + \frac{1}{2} (\pi_1^2 + \pi_2^2).
$$

(28a)

$$
H'^p_3 = p_3 - D^{\alpha^{-1}}q_3 = 0;
$$

(28b)

$$
H'^\pi_3 = \pi_3 - D^\alpha q_3 = 0.
$$

(28c)
Here, we have first class constraints (Dirac, 1992). The Poisson brackets
\[ \{ H_3^{\rho}, H_3^\alpha \} = 0, \quad \{ H_3^{\sigma}, H_3^\tau \} = 0, \]
and there are no secondary constraints.

The corresponding set of fractional HJPDEs, Eqs. (21), reads.

\[ H_3^p = \frac{\partial S}{\partial t} + D^\alpha q_1 + D^\alpha q_2\left( \frac{\partial S}{\partial D^{\alpha-1} q_2} - D^\alpha q_2 \right) + 2\left( \frac{\partial S}{\partial D^\alpha q_1} \right)^2 \frac{1}{2} \left( \frac{\partial S}{\partial D^\alpha q_2} \right)^2 = 0; \quad (29a) \]

\[ H_3^{\rho} = \frac{\partial S}{\partial D^{\alpha-1} q_3} - D^{\alpha-1} q_3 = 0; \quad (29b) \]

\[ H_3^{\sigma} = \frac{\partial S}{\partial D^\alpha q_3} - D^\alpha q_3 = 0. \quad (29c) \]

We can write the Hamilton-Jacobi function \( S \) Eq. (22) as

\[ S(D^{\alpha-1} q_1, D^{\alpha-1} q_2, D^{\alpha-1} q_3, D^\alpha q_1, D^\alpha q_2, D^{\alpha-1} q_4, t) = f(t) + W_1(D^{\alpha-1} q_1, E_1) + W_2(D^{\alpha-1} q_2, E_2) + W_3(D^\alpha q_3, E_3) + f_3(D^\alpha q_3) + A \quad (30) \]

The coordinates can be treated \( D^{\alpha-1} q_3 \) and \( D^\alpha q_3 \) as independent variables, because the Hamiltonian \( H_0 \) is time-independent. So, we can write.

\[ f(t) = -(E_1' + E_2')t. \]

Substituting \( S \) into Eq. (29a), we get

\[ -E_1' + D^\alpha q_1 \frac{\partial W_1}{\partial D^{\alpha-1} q_1} + \frac{1}{2} \left( \frac{\partial W_1}{\partial D^\alpha q_1} \right)^2 - E_2' + D^\alpha q_2 \left( \frac{\partial W_2}{\partial D^{\alpha-1} q_2} - D^\alpha q_2 \right) + \frac{1}{2} \left( \frac{\partial W_2}{\partial D^\alpha q_2} \right)^2 = 0. \quad (31) \]

We note that \( W_1 \) depends only on \( D^{\alpha-1} q_1 \). We can write it as

\[ \frac{\partial W_1}{\partial D^{\alpha-1} q_1} = E_1; \]

so that

\[ W_1 = D^{\alpha-1} q_1 E_1. \quad (32a) \]

Similarly, for \( W_2 \) depends only on \( D^{\alpha-1} q_2 \)

and

\[ \frac{\partial W_2}{\partial D^{\alpha-1} q_2} - D^{\alpha-1} q_2 = E_2; \]

so that
Substituting Eqs. (32) into (31), we get

\[-E' + D^a q_1 E_1 + \frac{1}{2} \left( \frac{\partial W_1'}{\partial D^a q_1} \right)^2 = -E' + D^a q_1 E_2 + \frac{1}{2} \left( \frac{\partial W_2'}{\partial D^a q_2} \right)^2 = 0.\]  

(33)

Using separation of variables in Eq. (33) yields

\[\frac{1}{2} \left( \frac{\partial W_1'}{\partial D^a q_1} \right)^2 + D^a q_1 E_1 - E'_1 = 0;\]  

(34a)

\[\frac{1}{2} \left( \frac{\partial W_2'}{\partial D^a q_2} \right)^2 + D^a q_2 E_2 - E'_2 = 0.\]  

(34b)

And the solution of Eqs. (34) can be determined as

\[W_1'(D^a q_1, E_1, E'_1) = \int \sqrt{2E'_1 - 2D^a q_1 E_1} dD^a q_1;\]  

(35a)

\[W_2'(D^a q_2, E_2, E'_2) = \int \sqrt{2E'_2 - 2D^a q_2 E_2} dD^a q_2.\]  

(35b)

Using Eqs. (29b,c), we find \( f_3(D^{-1}q_3) = \frac{1}{2} (D^{-1}q_3)^2 \) and \( f_3(D^a q_3) = \frac{1}{2} (D^a q_3)^2 \).

Thus, the Hamilton-Jacobi function becomes.

\[S(D^{-1}q_1, D^{-1}q_2, D^{-1}q_3, D^a q_1 D^a q_2, D^a q_3, t) = (-E'_1 - E'_2)t + D^{-1}q_1 E_1 + \frac{1}{2} (D^{-1}q_3)^2 + \int \sqrt{2E'_1 - 2D^{-1}q_1 E_1} dD^{-1}q_1 + \int \sqrt{2E'_2 - 2D^{-1}q_2 E_2} dD^{-1}q_2 + \frac{1}{2} (D^{-1}q_3)^2 + \frac{1}{2} (D^a q_3)^2 + A.\]  

(36)

We use the transformations Eqs. (23) to obtain the solutions for the generalized coordinates as follows:

\[\eta_1 = \frac{\partial S}{\partial E'_1} = -t + \int \frac{dD^a q_1}{\sqrt{2E'_1 - 2D^a q_1 E_1}};\]  

(37a)

\[\eta_2 = \frac{\partial S}{\partial E'_2} = -t + \int \frac{dD^a q_2}{\sqrt{2E'_2 - 2D^a q_2 E_2}};\]  

(37b)

\[\lambda_1 = \frac{\partial S}{\partial E_1} = D^{-1}q_1 + \int \frac{D^a q_1}{\sqrt{2E'_1 - 2D^a q_1 E_1}} dD^a q_1;\]  

(37c)

\[\lambda_2 = \frac{\partial S}{\partial E_2} = D^{-1}q_2 + \int \frac{D^a q_2}{\sqrt{2E'_2 - 2D^a q_2 E_2}} dD^a q_2.\]  

(37d)

We can solve these four equations, we obtain.

\[D^a q_1 = \frac{E'_1}{E_1} - \frac{E_1}{2} (\eta_1 + t)^2;\]  

(38a)
\[ D^\alpha q_2 = \frac{E'_2}{E_2} - \frac{E_2}{2}(\eta_2 + t)^2; \]  
(38b)

\[ D^{\alpha-1}q_1 = \lambda_1 + \frac{E'_1}{E_1}(\eta_1 + t) - \frac{E_1}{6}(\eta_1 + t)^3; \]  
(38c)

\[ D^{\alpha-1}q_2 = \lambda_2 + \frac{E'_2}{E_2}(\eta_2 + t) - \frac{E_2}{6}(\eta_2 + t)^3. \]  
(38d)

Also, we use Eqs. (23) to find the other half of the equations of motion

\[ p_1 = \frac{\partial S}{\partial D^{\alpha-1}q_1} = E_1; \]  
(39a)

\[ p_2 = \frac{\partial S}{\partial D^{\alpha-1}q_2} = E_2 + D^{\alpha-1}q_2 = E_2 + \lambda_2 + \frac{E'_2}{E_2}(\eta_2 + t) - \frac{E_2}{6}(\eta_2 + t)^3; \]  
(39b)

\[ p_3 = \frac{\partial S}{\partial D^{\alpha-1}q_3} = D^{\alpha-1}q_3; \]  
(39c)

\[ \pi_1 = \frac{\partial S}{\partial D^\alpha q_1} = \sqrt{2E'_1 - 2D^\alpha q_1E_1} = -E_1(\eta_1 + t); \]  
(39d)

\[ \pi_2 = \frac{\partial S}{\partial D^\alpha q_2} = \sqrt{2E'_2 - 2D^\alpha q_2E_2} = -E_2(\eta_2 + t); \]  
(39e)

\[ \pi_3 = \frac{\partial S}{\partial D^\alpha q_3} = D^\alpha q_3, \]  
(39f)

where \( D^{\alpha-1}q_3 \) and \( D^\alpha q_3 \) are arbitrary parameters.

5. Conclusion
The theory of fractional singular systems is extended for second order Lagrangian. We wrote the Hamilton-Jacobi function and the equations of motion within fractional calculus as total fractional partial differential equations. The set of (FHJPDs) for these systems satisfies the integrability conditions and separable. The fractional Hamilton-Jacobi function \( S \) is determined to obtain the solutions of the equations of motion.

Acknowledgments
Not available

Authors’ contributions
I have the manuscript (one author): Eyad Hasan Hasan

Funding
No Fundding, not available.
Competing interests
The author declares no conflict of interest with regards to the publication of this paper.

Informed consent
Obtained.

Ethics approval
The Publication Ethics Committee of the Canadian Center of Science and Education. The journal and publisher adhere to the Core Practices established by the Committee on Publication Ethics (COPE).

Provenance and peer review
Not commissioned; externally double-blind peer reviewed.

Data availability statement
The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

Data sharing statement
No additional data are available.

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References
