Applied Physics Research; Vol. 16, No. 1; 2024
ISSN 1916-9639 E-ISSN 1916-9647
Published by Canadian Center of Science and Education

Hamilton-Jacobi Treatment of Singular Systems Using Fractional Calculus

Eyad Hasan Hasan[1]

[1] Applied Physics Department, Faculty of Science, Tafila Technical University, Tafila, Jordan

Correspondence: Eyad Hasan Hasan, Applied Physics Department, Faculty of Science, Tafila Technical University, P. O. Box 179, Tafila 66110, Jordan. E-mail: iyad973@yahoo.com, dr_eyad2004@ttu.edu.jo

Received: October 27, 2023 Accepted: March 22, 2024 Online Published: April 22, 2024

doi:10.5539/apr.v16n1p180 URL: https://doi.org/10.5539/apr.v16n1p180

Abstract

In this paper, the theory of fractional singular systems is investigated with second-order derivatives. The fractional Hamilton–Jacobi treatment of these systems is examined. The fractional Hamilton–Jacobi partial differential equations (FHJPDEs) are constructed. The (FHJPDEs) can be solved to obtain the fractional Hamilton–Jacobi function. By building the fractional Hamilton–Jacobi function, the equations of motion can be obtained.

Keywords: Fractional Hamilton-Jacobi Function, fractional calculus, fractional constrained systems

PACS numbers: 11. 10. Ef, 45. 20. –j, 45. 10. Hj

1. Introduction

The study of singular systems has been treated first by (Dirac, 1950) for solving mechanical systems to find the equations of motion for these systems. Following Dirac's work, researchers studied the Hamilton-Jacobi treatment with more interest and extensively and used the canonical formalism for investigating these systems (Guler, 1992; Pimentel, R.G. and Teixeira 1996; Rabei et al., 2004, 2005; Hasan et al., 2004).

In this paper, we would like to develop our theory using fractional calculus to be applicable for these systems. Recently, the Hamilton-Jacobi formalism has been investigated for these systems within fractional derivatives with second order Lagrangian (Hasan, 2016; Hasan and Asad, 2017; Hasan, 2018; Hasan, 2020; 2023). In this formalism, researchers examined the Euler-Lagrange equations and Hamilton's equations.

In this paper, we would like to extend the work for systems with the second order Lagrangian. We constructed a formalism for investigating fractional singular systems and Hamilton-Jacobi formalism for these systems. Besides, we obtained the fractional Hamilton-Jacobi function for these systems.

Now, the basic definitions of fractional derivatives are (Samko et al., 1993).

The left Riemann–Liouville

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau \,. \tag{1}$$

and right Riemann–Liouville.

$$ {}_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau)d\tau \,. \tag{2}$$

These derivatives have properties as follows:

$$ {}_aD_t^\alpha f(t) = \left(\frac{d}{dt}\right)^\alpha f(t)\,; \tag{3}$$

$$ {}_tD_b^{\alpha} f(t) = \left(-\frac{d}{dt}\right)^{\alpha} f(t) . \tag{4}$$

Remark

The fractional derivatives $D^{\alpha-1}q_i$ and $D^{\alpha}q_i$ are treated as coordinates. So, the Poisson bracket can be defined as

$$\{A,B\} \equiv \frac{\partial A}{\partial D^{\alpha-1}q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial D^{\alpha-1}q_i} + \frac{\partial A}{\partial D^{\alpha}q_i}\frac{\partial B}{\partial \pi_i} - \frac{\partial A}{\partial \pi_i}\frac{\partial B}{\partial D^{\alpha}q_i} . \tag{5}$$

Here, the generalized momenta p_i and π_i are conjugated to the generalized coordinates $D^{\alpha-1}q_i$ and $D^{\alpha}q_i$ respectively. Thus, the fundamental Poisson brackets for second-order fractional derivatives are written as.

$$\{D^{\alpha-1}q_i, D^{\alpha-1}q_j\} = \{D^{\alpha}q_i, D^{\alpha}q_j\} = 0 = \{D^{\alpha}q_i, D^{\alpha-1}q_j\} = \{p_i, \pi_i\},$$

$$\{D^{\alpha-1}q_i, p_j\} \equiv \delta_{ij} \text{ and } \{D^{\alpha}q_i, \pi_j\} \equiv \delta_{ij}, \text{ where } \quad i,j = 1,\dots,N \tag{6}$$

2. Singular Lagrangian Using Fractional Calculus

We start with a Lagrangian depends on the fractional derivatives (Hasan, 2017, 2018, 2023)

$$L = L(D^{\alpha-1}q_i, D^{\alpha}q_i, D^{2\alpha}q_i, t). \tag{7}$$

Thus, the fractional of the Hessian matrix is defined as

$$W_{ij} = \frac{\partial^2 L}{\partial D^{2\alpha}q_i \partial D^{2\alpha}q_j} \quad i,j = 1,2,\dots,N \tag{8}$$

With property that the fractional Lagrangian is regular and its rank is N, otherwise the fractional Lagrangian is singular $N-R,\ R<N$, where R denotes to constraints.

Now, the generalized momenta π_i conjugate to the generalized coordinates $D^{\alpha}q_i$ as (Hasan, 2017, 2018, 2023):

$$\pi_a = \frac{\partial L}{\partial D^{2\alpha}q_a}; \tag{9}$$

$$\pi_\mu = \frac{\partial L}{\partial D^{2\alpha}q_\mu}. \quad a = 1.2,\dots,N-R; \ \mu = 1,\dots,R \tag{10}$$

We solve Eq. (9) to obtain N-R accelerations $D^{2\alpha}q_a$ in terms of $D^{\alpha-1}q_i, D^{\alpha}q_i, \pi_a$ and $D^{2\alpha}q_\mu$ as follows:

$$D^{2\alpha} q_a = w_a (D^{\alpha-1} q_i, D^{\alpha} q_i, \pi_a, D^{2\alpha} q_\mu). \tag{11}$$

We substitute Eq. (11) in Eq. (10) to obtain the following equation.

$$\pi_\mu = -H^{\pi}_{\mu} (D^{\alpha-1} q_i, D^{\alpha} q_i, p_a, \pi_a) \tag{12}$$

A similar expression for the momenta p_μ can be obtained as:

$$p_\mu = -H^{p}_{\mu} (D^{\alpha-1} q_i, D^{\alpha} q_i, p_a, \pi_a) \tag{13}$$

and the generalized momenta p_i conjugate to the generalized coordinate $D^{\alpha-1} q_i$ can be written as:

$$p_a = \frac{\partial L}{\partial D^{\alpha} q_a} - \frac{d}{dt}\left(\frac{\partial L}{\partial D^{2\alpha} q_a} \right); \tag{14a}$$

$$p_\mu = \frac{\partial L}{\partial D^{\alpha} q_\mu} - \frac{d}{dt}\left(\frac{\partial L}{\partial D^{2\alpha} q_\mu} \right). \tag{14b}$$

Equations (12) and (13) can be written as

$$H'^{p}_{\mu} (D^{\alpha-1} q_i, D^{\alpha} q_i, p_i, \pi_i) = p_\mu + H^{p}_{\mu} = 0 \tag{15a}$$

$$H'^{\pi}_{\mu} (D^{\alpha-1} q_i, D^{\alpha} q_i, p_i, \pi_i) = \pi_\mu + H^{\pi}_{\mu} = 0 \tag{15b}$$

Thus, equations (15) represent primary constraints (Dirac, 1950).

The theory for singular systems states that the number of degrees of freedom can be reduced due to the constraints (Dirac, 1950).

Thus, the fractional Hamiltonian $H_\circ$ can be defined as

$$H_\circ = -L(D^{\alpha-1} q_i, D^{2\alpha} q_v, D^{\alpha} q_v, W_a) + p_a D^{\alpha} q_a + \pi_a D^{2\alpha} q_a - D^{\alpha} q_\mu H^{p}_{\mu} - D^{2\alpha} q_\mu H^{\pi}_{\mu}. \tag{16}$$

$$\mu = 1, \ldots\ldots, R; \qquad a = R+1, \ldots, N.;$$

Here, the generalized momenta p_μ and π_μ are not independent of p_a and π_a. Thus, the set of the fractional Hamilton-Jacobi partial differential equations can be written as

$$H'_\circ \left(D^{\alpha-1} q_i, D^{\alpha} q_i, \frac{\partial S}{\partial D^{\alpha-1} q_a}, \frac{\partial S}{\partial D^{\alpha-1} q_v}, \frac{\partial S}{\partial D^{\alpha} q_a}, \frac{\partial S}{\partial D^{\alpha} q_v} \right) = p_\circ + H_\circ = 0; \tag{17a}$$

$$H'^{p}_{\mu} \left(D^{\alpha-1} q_i, D^{\alpha} q_i, \frac{\partial S}{\partial D^{\alpha-1} q_a}, \frac{\partial S}{\partial D^{\alpha-1} q_v}, \frac{\partial S}{\partial D^{\alpha} q_a}, \frac{\partial S}{\partial D^{\alpha} q_v} \right) = p_\mu + H^{p}_{\mu} = 0; \tag{17b}$$

$$H'^{p}_{\mu} \left(D^{\alpha-1} q_i, D^{\alpha} q_i, \frac{\partial S}{\partial D^{\alpha-1} q_a}, \frac{\partial S}{\partial D^{\alpha-1} q_v}, \frac{\partial S}{\partial D^{\alpha} q_a}, \frac{\partial S}{\partial D^{\alpha} q_v} \right) = \pi_\mu + H^{\pi}_{\mu} = 0. \tag{17c}$$

Here, the fractional Hamilton-Jacobi function is written as

$$S = S(D^{\alpha-1}q_a, D^{\alpha-1}q_\mu, D^\alpha q_a, D^\alpha q_\mu, t), \tag{18}$$

and the generalized momenta can be defined as

$$p_a = \frac{\partial S}{\partial D^{\alpha-1}q_a}, \quad p_\mu = \frac{\partial S}{\partial D^{\alpha-1}q_\mu}, \quad \pi_a = \frac{\partial S}{\partial D^\alpha q_a}, \quad \pi_\mu = \frac{\partial S}{\partial D^\alpha q_\mu} \quad \text{and} \quad p_\circ = \frac{\partial S}{\partial t}.$$

Thus, we can write the fractional equations of motion as total differential equations as follows:

$$dD^{\alpha-1}q_a = \frac{\partial H'_\circ}{\partial p_a}dt + \frac{\partial H'^{p}_\mu}{\partial p_a}dD^{\alpha-1}q_\mu + \frac{\partial H'^{\pi}_\mu}{\partial p_a}dD^\alpha q_\mu, \tag{19a}$$

$$dD^\alpha q_a = \frac{\partial H'_\circ}{\partial \pi_a}dt + \frac{\partial H'^{p}_\mu}{\partial \pi_a}dD^{\alpha-1}q_\mu + \frac{\partial H'^{\pi}_\mu}{\partial \pi_a}dD^\alpha q_\mu, \tag{19b}$$

$$-dp_i = \frac{\partial H'_\circ}{\partial D^{\alpha-1}q_i}dt + \frac{\partial H'^{p}_\mu}{\partial D^{\alpha-1}q_i}dD^{\alpha-1}q_\mu + \frac{\partial H'^{\pi}_\mu}{\partial D^{\alpha-1}q_i}dD^\alpha q_\mu, \tag{19c}$$

$$-d\pi_i = \frac{\partial H'_\circ}{\partial D^\alpha q_i}dt + \frac{\partial H'^{p}_\mu}{\partial D^\alpha q_i}dD^{\alpha-1}q_\mu + \frac{\partial H'^{\pi}_\mu}{\partial D^\alpha q_i}dD^\alpha q_\mu, \tag{19d}$$

If the total derivative of equation (17) is zero (Guler, 1992).

$$dH'_\circ = 0; \quad dH'^{p}_\mu = 0; \quad dH'^{\pi}_\mu = 0. \tag{20}$$

This means that equations (19) are integrable, and the rank of Hessian matrix is $N - R$ and the constraints reduce the canonical phase space coordinates from $\{D^{\alpha-1}q_i, p_i, D^\alpha q_i, \pi_i\}$ to $\{D^{\alpha-1}q_a, p_a, D^\alpha q_a, \pi_a\}$.

Thus, we can write Eqs. (17) as follows

$$\frac{\partial S}{\partial t} + H_\circ(D^{\alpha-1}q_i, D^\alpha q_i, p_a, \pi_a) = 0; \tag{21a}$$

$$\frac{\partial S}{\partial D^{\alpha-1}q_\mu} + H^{p}_\mu(D^{\alpha-1}q_i, D^\alpha q_i, p_a, \pi_a) = 0; \tag{21b}$$

$$\frac{\partial S}{\partial D^\alpha q_\mu} + H^{\pi}_\mu(D^{\alpha-1}q_i, D^\alpha q_i, p_a, \pi_a) = 0. \tag{21c}$$

3. Fractional Hamilton-Jacobi Function

Following refs (Hasan et al., 2004; Hasan, 2023), we guess a general solution for Eqs. (21) that can be written in a separable form. Thus, the fractional Hamilton-Jacobi function can be written as

$$\begin{aligned} S(D^{\alpha-1}q_a, D^{\alpha-1}q_\mu, D^\alpha q_a, D^\alpha q_\mu, t) = f(t) + W_a(D^{\alpha-1}q_a, E_a) + W'_a(D^\alpha q_a, E_a, E'_a) + \\ f_\mu(D^{\alpha-1}q_\mu) + f'_\mu(D^\alpha q_\mu) + A \end{aligned}. \tag{22}$$

Here, the functions $W_a(D^{\alpha-1}q_a, E_a)$, $W'_a(D^\alpha q_a, E_a, E'_a)$, $f_\mu(D^{\alpha-1}q_\mu)$, $f'_\mu(D^\alpha q_\mu)$ are the time-independent Hamilton-Jacobi function, $D^{\alpha-1}q_\mu$ and $D^\alpha q_\mu$ are treated as independent variables.

Finally, the fractional Hamilton-Jacobi function S enable us to obtain the equations of motion by using the canonical transformations (Goldstein, 1980) as follows:

$$\eta_a = \frac{\partial S}{\partial E'_a}\,;\, \lambda_a = \frac{\partial S}{\partial E_a}\,;\, p_i = \frac{\partial S}{\partial D^{\alpha-1} q_i}\,;\, \pi_i = \frac{\partial S}{\partial D^{\alpha} q_i}\,. \tag{23}$$

Here, η_a and λ_a are constants.

4. Example

Consider the fractional singular Lagrangian

$$L = \frac{1}{2}\Big((D^{2\alpha} q)^2 + (D^{2\alpha} q_2)^2\Big) + D^{\alpha} q_3 D^{2\alpha} q_3 + D^{\alpha} q_3 D^{\alpha-1} q_3 + D^{\alpha-1} q_2 D^{\alpha} q_2. \tag{24}$$

The generalized momenta can be obtained from Eqs. (9,10) and (14)

$$p_1 = -D^{3\alpha} q_1\,; \tag{25a}$$

$$p_2 = D^{\alpha-1} q_2 - D^{3\alpha} q_2\,; \tag{25b}$$

$$p_3 = D^{\alpha-1} q_3 = -H_3^{p}\,; \tag{25c}$$

$$\pi_1 = D^{2\alpha} q_1\,; \tag{25d}$$

$$\pi_2 = D^{2\alpha} q_2\,; \tag{25e}$$

$$\pi_3 = D^{\alpha} q_3 = -H_3^{\pi}\,. \tag{25f}$$

Here, rewrite equations (25c) and (25f) as

$$H_3'^{\,p} = p_3 - D^{\alpha-1} q_3 = 0\,; \tag{26a}$$

$$H_3'^{\,\pi} = \pi_3 - D^{\alpha} q_3 = 0\,. \tag{26b}$$

and represent as primary constraints (Dirac, 1950).

The Hamiltonian H_0 is calculated as

$$H_{\circ} = p_1 D^{\alpha} q_1 + (p_2 - D^{\alpha-1} q_2) D^{\alpha} q_2 + \frac{1}{2}(\pi_1^2 + \pi_2^2). \tag{27}$$

The corresponding set of fractional HJPDEs, Eqs. (17), reads

$$H'_{\circ} = p_{\circ} + H_{\circ} = p_1 D^{\alpha} q_1 + (p_2 - D^{\alpha-1} q_2) D^{\alpha} q_2 + \frac{1}{2}(\pi_1^2 + \pi_2^2). \tag{28a}$$

$$H_3'^{\,p} = p_3 - D^{\alpha-1} q_3 = 0\,; \tag{28b}$$

$$H_3'^{\,\pi} = \pi_3 - D^{\alpha} q_3 = 0\,. \tag{28c}$$

Here, we have first class constraints (Dirac, 1992). The Poisson brackets $\left\{H_3'^{p}, H_\circ'\right\} = 0$, $\left\{H_3'^{\pi}, H_\circ'\right\} = 0$, $\left\{H_3'^{p}, H_3'^{\pi}\right\} = 0$ and there are no secondary constraints.

The corresponding set of fractional HJPDEs, Eqs. (21), reads.

$$H_\circ' = p_\circ + H_\circ = \frac{\partial S}{\partial t} + D^\alpha q_1 \frac{\partial S}{\partial D^{\alpha-1} q_1} + D^\alpha q_2 \left(\frac{\partial S}{\partial D^{\alpha-1} q_2} - D^{\alpha-1} q_2 \right) + \frac{1}{2}\left(\frac{\partial S}{\partial D^\alpha q_1} \right)^2 \frac{1}{2}\left(\frac{\partial S}{\partial D^\alpha q_2} \right)^2 = 0; \quad (29a)$$

$$H_3'^{p} = \frac{\partial S}{\partial D^{\alpha-1} q_3} - D^{\alpha-1} q_3 = 0; \quad (29b)$$

$$H_3'^{\pi} = \frac{\partial S}{\partial D^\alpha q_3} - D^\alpha q_3 = 0. \quad (29c)$$

We can write the Hamilton-Jacobi function S Eq. (22) as

$$\begin{aligned} &S(D^{\alpha-1}q_1, D^{\alpha-1}q_2, D^{\alpha-1}q_3, D^\alpha q_1 D^\alpha q_2, D^\alpha q_3, t) = f(t) + W_1(D^{\alpha-1}q_1, E_1) + W_2(D^{\alpha-1}q_2, E_2) + \\ &W_1'(D^\alpha q_1, E_1, E_1') + W_2'(D^\alpha q_2, E_2, E_2') + f_3(D^{\alpha-1}q_3) + f_3'(D^\alpha q_3) + A \end{aligned} \quad (30)$$

The coordinates can be treated $D^{\alpha-1}q_3$ and $D^\alpha q_3$ as independent variables, because the Hamiltonian H_0 is time-independent. So, we can write.

$$f(t) = -(E_1' + E_2')t.$$

Substituting S into Eq. (29a), we get

$$-E_1' + D^\alpha q_1 \frac{\partial W_1}{\partial D^{\alpha-1} q_1} + \frac{1}{2}\left(\frac{\partial W_1'}{\partial D^\alpha q_1} \right)^2 - E_2' + D^\alpha q_2 \left(\frac{\partial W_2}{\partial D^{\alpha-1} q_2} - D^\alpha q_2 \right) + \frac{1}{2}\left(\frac{\partial W_2'}{\partial D^\alpha q_2} \right)^2 = 0. \quad (31)$$

We note that W_1 depends only on $D^{\alpha-1}q_1$. We can write it as

$$\frac{\partial W_1}{\partial D^{\alpha-1} q_1} = E_1;$$

so that

$$W_1 = D^{\alpha-1} q_1 E_1. \quad (32a)$$

Similarly, for W_2 depends only on $D^{\alpha-1}q_2$

and

$$\frac{\partial W_2}{\partial D^{\alpha-1} q_2} - D^{\alpha-1} q_2 = E_2;$$

so that

$$W_2 = D^{\alpha-1}q_2 E_2 + \frac{1}{2}(D^{\alpha-1}q_2)^2 . \tag{32b}$$

Substituting Eqs. (32) into (31), we get

$$-E_1' + D^{\alpha}q_1 E_1 + \frac{1}{2}\left(\frac{\partial W_1'}{\partial D^{\alpha}q_1}\right)^2 - E_2' + D^{\alpha}q_2 E_2 + \frac{1}{2}\left(\frac{\partial W_2'}{\partial D^{\alpha}q_2}\right)^2 = 0 . \tag{33}$$

Using separation of variables in Eq. (33) yields

$$\frac{1}{2}\left(\frac{\partial W_1'}{\partial D^{\alpha}q_1}\right)^2 + D^{\alpha}q_1 E_1 - E_1' = 0 ; \tag{34a}$$

$$\frac{1}{2}\left(\frac{\partial W_2'}{\partial D^{\alpha}q_2}\right)^2 + D^{\alpha}q_2 E_2 - E_2' = 0 . \tag{34b}$$

And the solution of Eqs. (34) can be determined as

$$W_1'(D^{\alpha}q_1, E_1, E_1') = \int \sqrt{2E_1' - 2D^{\alpha}q_1 E_1}\, dD^{\alpha}q_1 ; \tag{35a}$$

$$W_2'(D^{\alpha}q_2, E_2, E_2') = \int \sqrt{2E_2' - 2D^{\alpha}q_2 E_2}\, dD^{\alpha}q_2 . \tag{35b}$$

Using Eqs. (29b,c), we find $f_3(D^{\alpha-1}q_3) = \frac{1}{2}(D^{\alpha-1}q_3)^2$ and $f_3(D^{\alpha}q_3) = \frac{1}{2}(D^{\alpha}q_3)^2$.

Thus, the Hamilton-Jacobi function becomes.

$$\begin{gathered} S(D^{\alpha-1}q_1, D^{\alpha-1}q_2, D^{\alpha-1}q_3, D^{\alpha}q_1 D^{\alpha}q_2, D^{\alpha}q_3, t) = (-E_1' - E_2')t + D^{\alpha-1}q_1 E_1 + \frac{1}{2}(D^{\alpha-1}q_2)^2 + \\ \int \sqrt{2E_1' - 2D^{\alpha}q_1 E_1}\, dD^{\alpha}q_1 + \int \sqrt{2E_2' - 2D^{\alpha}q_2 E_2}\, dD^{\alpha}q_2 + \frac{1}{2}(D^{\alpha-1}q_3)^2 + \frac{1}{2}(D^{\alpha}q_3)^2 + A \end{gathered} . \tag{36}$$

We use the transformations Eqs. (23) to obtain the solutions for the generalized coordinates as follows:

$$\eta_1 = \frac{\partial S}{\partial E_1'} = -t + \int \frac{dD^{\alpha}q_1}{\sqrt{2E_1' - 2D^{\alpha}q_1 E_1}} ; \tag{37a}$$

$$\eta_2 = \frac{\partial S}{\partial E_2'} = -t + \int \frac{dD^{\alpha}q_2}{\sqrt{2E_2' - 2D^{\alpha}q_2 E_2}} ; \tag{37b}$$

$$\lambda_1 = \frac{\partial S}{\partial E_1} = D^{\alpha-1}q_1 + \int \frac{D^{\alpha}q_1}{\sqrt{2E_1' - 2D^{\alpha}q_1 E_1}} dD^{\alpha}q_1 ; \tag{37c}$$

$$\lambda_2 = \frac{\partial S}{\partial E_2} = D^{\alpha-1}q_2 + \int \frac{D^{\alpha}q_2}{\sqrt{2E_2' - 2D^{\alpha}q_2 E_2}} dD^{\alpha}q_2 . \tag{37d}$$

We can solve these four equations, we obtain.

$$D^{\alpha}q_1 = \frac{E_1'}{E_1} - \frac{E_1}{2}(\eta_1 + t)^2 ; \tag{38a}$$

$$D^{\alpha} q_2 = \frac{E_2'}{E_2} - \frac{E_2}{2}(\eta_2 + t)^2 ; \tag{38b}$$

$$D^{\alpha-1} q_1 = \lambda_1 + \frac{E_1'}{E_1}(\eta_1 + t) - \frac{E_1}{6}(\eta_1 + t)^3 ; \tag{38c}$$

$$D^{\alpha-1} q_2 = \lambda_2 + \frac{E_2'}{E_2}(\eta_2 + t) - \frac{E_2}{6}(\eta_2 + t)^3 . \tag{38d}$$

Also, we use Eqs. (23) to find the other half of the equations of motion

$$p_1 = \frac{\partial S}{\partial D^{\alpha-1} q_1} = E_1 ; \tag{39a}$$

$$p_2 = \frac{\partial S}{\partial D^{\alpha-1} q_2} = E_2 + D^{\alpha-1} q_2 = E_2 + \lambda_2 + \frac{E_2'}{E_2}(\eta_2 + t) - \frac{E_2}{6}(\eta_2 + t)^3 ; \tag{39b}$$

$$p_3 = \frac{\partial S}{\partial D^{\alpha-1} q_3} = D^{\alpha-1} q_3 ; \tag{39c}$$

$$\pi_1 = \frac{\partial S}{\partial D^{\alpha} q_1} = \sqrt{2E_1' - 2D^{\alpha} q_1 E_1} = -E_1(\eta_1 + t) ; \tag{39d}$$

$$\pi_2 = \frac{\partial S}{\partial D^{\alpha} q_2} = \sqrt{2E_2' - 2D^{\alpha} q_2 E_2} = -E_2(\eta_2 + t) ; \tag{39e}$$

$$\pi_3 = \frac{\partial S}{\partial D^{\alpha} q_3} = D^{\alpha} q_3 , \tag{39f}$$

where $D^{\alpha-1} q_3$ and $D^{\alpha} q_3$ are arbitrary parameters.

5. Conclusion

The theory of fractional singular systems is extended for second order Lagrangian. We wrote the Hamilton-Jacobi function and the equations of motion within fractional calculus as total fractional partial differential equations. The set of (FHJPDs) for these systems satisfies the integrability conditions and separable. The fractional Hamilton-Jacobi function S is determined to obtain the solutions of the equations of motion.

Acknowledgments

Not available

Authors' contributions

I have the manuscript (one author): Eyad Hasan Hasan

Funding

No Fundding, not available.

Competing interests

The author declares no conflict of interest with regards to the publication of this paper

Informed consent

Obtained.

Ethics approval

The Publication Ethics Committee of the Canadian Center of Science and Education.

The journal and publisher adhere to the Core Practices established by the Committee on Publication Ethics (COPE).

Provenance and peer review

Not commissioned; externally double-blind peer reviewed.

Data availability statement

The data that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

Data sharing statement

No additional data are available.

Open access

This is an open-access article distributed under the terms and conditions of the Creative Commons Attribution license (http://creativecommons.org/licenses/by/4.0/).

Copyrights

Copyright for this article is retained by the author(s), with first publication rights granted to the journal.

References

Dirac, P. A. M. (1950). Generalized Hamiltonian Dynamics. *Canadian Journal of Mathematical Physics, 2,* 129-148. https://doi.org/10.4153/CJM-1950-012-1

Goldstein, H. (1980). *Classical MechanBreach*. Addison-Wesley, Reading- Massachusetts.

Guler, Y. (1992b). Canonical Formulation of Singular Systems. *IL Nuovo Cimento B, 107*(10), 1143-1149. https://doi.org/10.1007/BF02727199

Hasan, E. H. (2016). Fractional Variational Problems of Euler-Lagrange Equations with Holonomic Constrained Systems. *Applied Physics Research, 8*(3), 60-65. https://doi.org/10.5539/apr.v8n3p60

Hasan, E. H. (2018). On Fractional Solution of Euler-Lagrange Equations with Second-Order Linear Lagrangians. *Journal of Advanced Physics, 7*(1), 110-113. https://doi.org/10.1166/jap.2018.1388

Hasan, E. H. (2020). Path Integral Quantization of Singular Lagrangians using Fractional Derivatives. *International Journal of Theoretical Physics, 59*, 1157-1164. https://doi.org/10.1007/s10773-020-04395-3

Hasan, E. H. (2023). Fractional Quantization of Singular Lagrangian systems with Second-Order Derivatives Using WKB Approximation. *arXiv*. https://doi.org/10.48550/arXiv.2301.08133

Hasan, E. H., &Asad, J. H. (2017). Remarks on Fractional Hamilton-Jacobi Formalism with second-order Discrete Lagrangian Systems. *Journal of Advanced Physics, 6*(3), 430-433. https://doi.org/10.1166/jap.2017.1335

Hasan, E. H., Rabei, E. M., & Ghassib, H. B. (2004). Quantization of Higher-Order Constrained Lagrangian Systems Using the WKB Approximation. *International Journal of Theoretical Physics, 43*(11), 2285-2298. https://doi.org/10.1023/B:IJTP.0000049027.45011.37

Pimentel, R.G. and Teixeira 1996. Hamilton-Jacobi Formulation for Singular Systems with Second Order Lagrangians. *IL Nuovo Ciemento B, 111*, 841-854. https://doi.org/10.1007/BF02749015

Rabei, E. M., Hasan, E. H., & Ghassib, H. B (2004). Hamilton-Jacobi Treatment of Constrained Systems with Second Order Lagrangians. *International Journal of Theoretical Physics*, *43*(4), 1073-1096. https://doi.org/10.1023/B:IJTP.0000048601.92005.fe

Rabei, E. M., Hasan, E. H., & Ghassib, H. B. (2005). Quantization of Second-Order Constrained Lagrangian Systems Using the WKB Approximation. *International Journal of Geometric methods in Modern Physics*, *2*, 485-504. https://doi.org/10.1142/S0219887805000661

Samko, S. G. Kilbas, A. A., & Marichev, O. I. (1993). *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach.