Fractional Constrained Hamiltonian Systems

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Abstract
This work is concerned with fractional constrained Hamiltonian systems. Constrained systems with first class constraints and second class constraints are studied using fractional Lagrangian, then the Hamiltonian is found in fractional form, after that the conjugate momenta are obtained from the fractional Lagrangian, when $\alpha \to 1$; the results of fractional technique reduce to those obtained from classical technique. Two illustrative examples are used to explain this technique.

Keywords: Constrained Systems, Hamilton Jacobi Equation, Constraints, Fractional Derivatives, Momenta

1. Introduction

The motion of an object or any system is restricted by many factors. The limitations on the motion of the system are called constraints (Goldstein, 1980). The constrained systems have been treated by (Dirac, 1950; Dirac, 1964), then the constraints have been classified into kinds: first class constraints, and second class constraints. Also converting first class constraints into second class constraints was extended by (Faddeev, 1970; Hanson and Regge, 1976; Evans, 1991).

Canonical method has been presented to study the constrained systems by (Guler, 1992). Singular Lagrangian with first class constraints and second class constraints are investigated using Hamiltonian formalism (Goldberg, 1991; Rabei and Guler, 1992). Moreover, constrained systems with second order Lagrangian and higher order Lagrangian are quantized using path integral technique (Senjanovic, 1976; Rabei, 1996; Muslih, 2001; Rabei, 2000). The Hamilton Jacobi formalism has been developed to study constrained systems by (Nawafleh et al., 2004; Nawafleh et al., 2005), through this formalism the equations of motion are obtained and the action function is formulated using Hamilton Jacobi equation, this function able one to obtain the conjugate momentum.

After that, the higher order Lagrangian systems have been developed using Hamilton Jacobi equation, then the equations of motion have been obtained and the constrained systems have been quantized using the WKB approximation (Muslih, 2002; Rabei et al., 2002; Rabei et al., 2003; Hasan et al., 2004). Fractional calculus and fractional derivatives have been used in different areas of classical mechanics, electrodynamics, scaling phenomena, astrophysics, potential theory, optics, science, engineering and thermodynamics (Miller, 1993; Samko, 1993; Gorenflo, 1997).

The fractional calculus method for both conservative and nonconservative systems has been developed by Riewe (Riewe, 1996; Riewe, 1997). The classical calculus of variations was extended by (Agrawal, 2006) for systems containing Riemann-Liouville fractional derivatives. Recently, Euler Lagrange equations for holonomic constrained systems with regular Lagrangian have been presented by (Hasan, 2016) using the fractional variational problems.

More recently, as a continuation of the previous works, the fractional Euler Lagrange equations are used by (Jarab'ah, 2018; Jarab'ah et al., 2018) to obtain the equations of motion for first order irregular Lagrangian with holonomic constraints and second order Lagrangian for nonconservative systems. In this paper we hope to study constrained systems for both first and second class constraints using Lagrangians and Hamiltonian formalism, but in another technique which is fractional method and at a certain condition which is $\alpha \to 1$; the results of fractional technique reduce to those obtained from classical technique.
This paper is organized as follows. In section 2, the fractional derivatives are reviewed. In section 3, the fractional Euler Lagrange equation and fractional Hamiltonian are discussed. In section 4, this work is illustrated through two physical examples. Finally, in section 5, the work closes with some concluding remarks.

2. Fractional Derivatives

From Agrawal’s work (Agrawal, 2002; Podlubny, 1999)

The left Riemann–Liouville fractional derivative written as:

\[ a D^\alpha_x f(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dx} \right)^n (x-a)^{n-\alpha-1} f(x) \tag{1} \]

which is defined as the LRLFD, and the right Riemann–Liouville fractional derivative written as:

\[ b D^\alpha_x f(x) = \frac{1}{\Gamma(n-\alpha)} \left( -\frac{d}{dx} \right)^n (\tau-x)^{n-\alpha-1} f(\tau) d\tau \tag{2} \]

which is defined as the RRLFD.

where \( \Gamma \) represents the Euler’s gamma function and \( \alpha \) is the order of the derivative such that \( n-1 \leq \alpha < n \), and is not equal to zero. If \( \alpha \) is an integer, these derivatives are written as:

\[ a D^\alpha_x f(x) = \left( \frac{d}{dx} \right)^\alpha f(x) \tag{3} \]

\[ b D^\alpha_x f(x) = \left( -\frac{d}{dx} \right)^\alpha f(x) \tag{4} \]

The fractional operator, \( a D^\alpha_x f(x) \) can be written as (Igor, 2002).

\[ a D^\alpha_x f(x) = \frac{d^\alpha}{dx^\alpha} a D^{\alpha-n}_x \]

\( \alpha = 1, 2, \ldots \)

and has the following properties:

1) \( a D^\alpha_x = \frac{d^\alpha}{dx^\alpha}, \) \( \text{Re} (\alpha) > 0 \) \tag{6}

2) \( a D^\alpha_x = 1, \) \( \text{Re} (\alpha) = 0 \) \tag{7}

3) \( x D^\alpha_x = \int_a^x (d\tau)^{-\alpha}, \) \( \text{Re} (\alpha) < 0 \) \tag{8}

Theorem: Let \( f \) and \( g \) be two continuous functions on \( [a, b] \). Then, for all \( x \in [a, b] \), the following properties hold:

1) For \( m > 0, \) \( a D^m_x = \left[ f(x) + g(x) \right] = a D^m_x f(x) + a D^m_x g(x) \) \tag{9}

2) For \( m \geq n \geq 0, \) \( a D^m_x (a D^n_x f(x)) = a D^{m-n}_x f(x) \) \tag{10}
3) For \( m > 0 \),
\[
\frac{D^m}{D_x^m} (a D_x^{-m} f(x)) = f(x)
\]  
(11)

4) For \( m > 0 \),
\[
\int_a^b \left( \frac{D^m}{D_x^m} f(x) \right) g(x) dx = \int_a^b f(x) (a D_x^{-m} g(x)) dx
\]  
(12)

3. Fractional Hamiltonian and Fractional Euler Lagrange Equation

Using Lagrangian in the following fractional form
\[
L = L(a D_t^{\alpha-1} q_t, D_b^\beta q_t, a D_t^\alpha q_t, D_b^\beta q_t, t)
\]  
(13)

Recalling that, action function for all \( x \in [a, b] \), can be defined as follows:
\[
S = \int_a^b L(a D_t^{\alpha-1} q_t, D_b^\beta q_t, a D_t^\alpha q_t, D_b^\beta q_t, t) dt
\]  
(14)

The fractional Euler Lagrange equation (Agrawal, 2002) is given by:
\[
\frac{\partial L}{\partial q} + D_{\alpha}^a \frac{\partial L}{\partial a D_t^\alpha q} + D_{\beta}^b \frac{\partial L}{\partial b D_b^\beta q} = 0
\]  
(15)

Thus, if \( \alpha = \beta = 1 \), we find that:
\[
d_{\alpha}^a D_t^\alpha = -\frac{d}{dt}
\]  
(16)

and
\[
d_{\beta}^b D_t^\beta = \frac{d}{dt}
\]  
(17)

And the fractional Euler Lagrange equation reduce to the classical form of the Euler Lagrange equation.

Where the generalized momenta can be obtained from:
\[
p_{\alpha} = \frac{\partial L}{\partial a D_t^\alpha q}
\]  
(18)

and
\[
p_{\beta} = \frac{\partial L}{\partial b D_b^\beta q}
\]  
(19)

Thus, the generalized coordinate \( q \) in fractional form is defined as:
\[
q_{\alpha} = a D_t^{\alpha-1} q
\]  
(20)

Also,
\[
q_{\beta} = D_b^{\beta-1} q
\]  
(21)

Making use that,
\[
D_t^0 = 1
\]  
(22)
and

$$D_t^1 = \frac{d}{dt}$$

The Hamiltonian depending on the fractional derivatives is written as:

$$H(\alpha_{\alpha_{\beta_1}}, \beta_{\beta_1}, q_t, p_{\alpha_{\beta_1}}, p_{\beta_1}) = p_{\alpha_{\beta_1}}D_t^{\alpha_{\beta_1}}q + p_{\beta_1}D_t^{\beta_1}q - L(\alpha_{\alpha_{\beta_1}}q_t, \beta_{\beta_1}q_t, \beta_{\beta_1}q_t, \alpha_{\alpha_{\beta_1}}q_t, \beta_{\beta_1}q_t, \alpha_{\alpha_{\beta_1}}q_t, \beta_{\beta_1}q_t)$$

(24)

4. Illustrative Examples (Nawafleh, 1998)

An Example with First Class Constraints:

Let us consider the following Lagrangian

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_3^2) + \dot{q}_1\dot{q}_3 + q_2\dot{q}_2 - q_1 - q_3$$

(25)

Using fractional derivatives equation (25) becomes

$$L = \frac{1}{2}(\alpha_{\alpha_{\beta_1}}q_1) + \frac{1}{2}(\beta_{\beta_1}q_3) + (\alpha_{\alpha_{\beta_1}}q_1)(\beta_{\beta_1}q_3) + (\beta_{\beta_1}q_1)(\alpha_{\alpha_{\beta_1}}q_3) - (\alpha_{\alpha_{\beta_1}}q_1) - (\beta_{\beta_1}q_3)$$

(26)

From equation (26) and by using of equation (18) the conjugate momenta are

$$p_1 = \frac{\partial L}{\partial \alpha_{\alpha_{\beta_1}}q_1} = (\alpha_{\alpha_{\beta_1}}q_1)$$

(27)

$$p_2 = \frac{\partial L}{\partial \beta_{\beta_1}q_2} = (\beta_{\beta_1}q_2)$$

(28)

Then,

$$p_3 = \frac{\partial L}{\partial \alpha_{\alpha_{\beta_1}}q_3} = (\alpha_{\alpha_{\beta_1}}q_3)$$

(29)

Thus, in this example we find that, $$p_3 = p_1$$

The fractional Hamiltonian has the standard form using equation (24)

$$H = p_1(\alpha_{\alpha_{\beta_1}}q_1) + p_2(\beta_{\beta_1}q_2) + p_3(\alpha_{\alpha_{\beta_1}}q_3) - L(\alpha_{\alpha_{\beta_1}}q_t, \beta_{\beta_1}q_t, \beta_{\beta_1}q_t, \alpha_{\alpha_{\beta_1}}q_t, \beta_{\beta_1}q_t, \alpha_{\alpha_{\beta_1}}q_t, \beta_{\beta_1}q_t)$$

(30)

Now, substituting equations. (26), (27), (28) and (29) into equation (30), Hamiltonian reads:

$$H = \frac{1}{2}(\alpha_{\alpha_{\beta_1}}q_1)^2 + \frac{1}{2}(\beta_{\beta_1}q_3)^2 + (\alpha_{\alpha_{\beta_1}}q_1)(\alpha_{\alpha_{\beta_1}}q_3) + (\beta_{\beta_1}q_1)(\beta_{\beta_1}q_3) + (\alpha_{\alpha_{\beta_1}}q_1) + (\beta_{\beta_1}q_3)$$

(31)

This equation can readily be solved to give

$$H = \frac{1}{2}p_1^2 + (\alpha_{\alpha_{\beta_1}}q_1) + (\beta_{\beta_1}q_3)$$

(32)

An Example with Primary and Secondary Constraints of the Second Class:

As a second example, let us take the following Lagrangian (Rabei and Guler, 1992):
\[ L = \frac{1}{2} \dot{q}_1^2 - \frac{1}{4} (\ddot{q}_2 - \dot{q}_3)^2 + (q_1 + q_3)\dot{q}_2 - (q_1 + q_2 + q_3^2) \] (33)

The corresponding fractional Lagrangian is:

\[ L = \frac{1}{2} (0 \alpha^a \dot{q}_1)^2 - \frac{1}{4} \left[ (0 \alpha^a \dot{q}_2) - (0 \alpha^a \dot{q}_3) \right]^2 + \left[ (0 \alpha^a \dot{q}_1) + (0 \alpha^a \dot{q}_3) \right] \left[ (0 \alpha^a \dot{q}_2) + (0 \alpha^a \dot{q}_3) \right] \] (34)

Making use of equation (34) and equation (18), the momenta can be computed as

\[ p_1 = \frac{\partial L}{\partial 0 \alpha^a \dot{q}_1} = 0 \alpha^a \dot{q}_1 \] (35)

Then,

\[ p_2 = \frac{\partial L}{\partial 0 \alpha^a \dot{q}_2} = -\frac{1}{2} \left[ (0 \alpha^a \dot{q}_2) - (0 \alpha^a \dot{q}_3) \right] + (0 \alpha^a \dot{q}_1) + (0 \alpha^a \dot{q}_3) \] (36)

The usual treatment gives the following momentum

\[ p_3 = \frac{\partial L}{\partial 0 \alpha^a \dot{q}_3} = \frac{1}{2} \left[ (0 \alpha^a \dot{q}_2) - (0 \alpha^a \dot{q}_3) \right] \] (37)

From equation (36) and equation (37) one can see that,

\[ p_2 = (0 \alpha^a \dot{q}_1) + (0 \alpha^a \dot{q}_3) - p_3 \]

The fractional Hamiltonian can be expressed as

\[ H = p_1 (0 \alpha^a \dot{q}_1) + p_2 (0 \alpha^a \dot{q}_2) + p_3 (0 \alpha^a \dot{q}_3) - L(0 \alpha^a \dot{q}_1,0 \alpha^a \dot{q}_2,0 \alpha^a \dot{q}_3,0 \beta^a \dot{q}_4,0 \beta^a \dot{q}_5,0 \beta^a \dot{q}_6) \] (38)

As usual, putting equation (34), (35), (36) and equation (37) into equation (38), the Hamiltonian takes this form

\[ H = \frac{1}{2} (0 \alpha^a \dot{q}_1)^2 - \frac{1}{4} (0 \alpha^a \dot{q}_2)^2 - \frac{1}{4} (0 \alpha^a \dot{q}_3)^2 + \frac{1}{2} \left[ (0 \alpha^a \dot{q}_1)(0 \alpha^a \dot{q}_2)(0 \alpha^a \dot{q}_3) \right] + (0 \alpha^a \dot{q}_1) + (0 \alpha^a \dot{q}_2) + (0 \alpha^a \dot{q}_3)^2 \] (39)

Simple manipulations yield

\[ H = \frac{1}{2} (p_1^2 - 2 p_2^2) + (0 \alpha^a \dot{q}_1) + (0 \alpha^a \dot{q}_2) + (0 \alpha^a \dot{q}_3)^2 \] (40)

5. Conclusions

In this work fractional constrained systems are studied using fractional Lagrangian and fractional Hamiltonian, also two types of constraints are studied: first class constraints and second class constraints. The Lagrangians are written in fractional form and then we find the fractional Hamiltonian, then the fractional conjugate momenta are obtained from our fractional Lagrangian, when \( \alpha \rightarrow 1 \); the results of fractional technique reduce to those obtained from classical technique. In this paper we find the ability of the first class constraints and second class constraints to studying using fractional technique. Some physical examples are considered to demonstrate the application of the formalism.

References


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