Applied Physics Research; Vol. 15, No. 2; 2023
ISSN 1916-9639 E-ISSN 1916-9647
Published by Canadian Center of Science and Education

Fractional Constrained Hamiltonian Systems

Ola A. Jarab'ah[1]

[1] Department of Applied Physics, Tafila Technical University, Tafila, Jordan

Correspondence: Ola A. Jarab'ah, Department of Applied Physics, Tafila Technical University, P. O. Box 179 Tafila 66110, Jordan. E-mail: oasj85@yahoo.com

Received: August 5, 2023 Accepted: August 30, 2023 Online Published: September 15, 2023

doi:10.5539/apr.v15n2p103 URL: https://doi.org/10.5539/apr.v15n2p103

Abstract

This work is concerned with fractional constrained Hamiltonian systems. Constrained systems with first class constraints and second class constraints are studied using fractional Lagrangian, then the Hamiltonian is found in fractional form, after that the conjugate momenta are obtained from the fractional Lagrangian, when $\alpha \to 1$; the results of fractional technique reduce to those obtained from classical technique. Two illustrative examples are used to explain this technique.

Keywords: Constrained Systems, Hamilton Jacobi Equation, Constraints, Fractional Derivatives, Momenta

1. Introduction

The motion of an object or any system is restricted by many factors. The limitations on the motion of the system are called constraints (Goldstein, 1980). The constrained systems have been treated by (Dirac, 1950; Dirac, 1964), then the constraints have been classified into kinds: first class constraints, and second class constraints. Also converting first class constraints into second class constraints was extended by (Faddeev, 1970; Hanson and Regge, 1976; Evans, 1991).

Canonical method has been presented to study the constrained systems by (Guler, 1992). Singular Lagrangian with first class constraints and second class constraints are investigated using Hamiltonian formalism (Goldberg, 1991; Rabei and Guler, 1992). Moreover, constrained systems with second order Lagrangian and higher order Lagrangian are quantized using path integral technique (Senjanovic, 1976; Rabei, 1996; Muslih, 2001; Rabei, 2000). The Hamilton Jacobi formalism has been developed to study constrained systems by (Nawafleh et al., 2004; Nawafleh et al., 2005), through this formalism the equations of motion are obtained and the action function is formulated using Hamilton Jacobi equation, this function able one to obtain the conjugate momentum.

After that, the higher order Lagrangian systems have been developed using Hamilton Jacobi equation, then the equations of motion have been obtained and the constrained systems have been quantized using the WKB approximation (Muslih, 2002; Rabei et al., 2002; Rabei et al., 2003; Hasan et al., 2004). Fractional calculus and fractional derivatives have been used in different areas of classical mechanics, electrodynamic, scaling phenomena, astrophysics, potential theory, optics, science, engineering and thermodynamics (Miller, 1993; Samko, 1993; Gorenflo, 1997).

The fractional calculus method for both conservative and nonconservative systems has been developed by Riewe (Riewe, 1996; Riewe, 1997). The classical calculus of variations was extended by (Agrawal, 2006) for systems containing Riemann- Liouville fractional derivatives. Recently, Euler Lagrange equations for holonomic constrained systems with regular Lagrangian have been presented by (Hasan, 2016) using the fractional variational problems.

More recently, as a continuation of the previous works, the fractional Euler Lagrange equations are used by (Jarab'ah, 2018; Jarab'ah et al., 2018) to obtain the equations of motion for first order irregular Lagrangian with holonomic constraints and second order Lagrangian for nonconservative systems. In this paper we hope to study constrained systems for both first and second class constraints using Lagrangians and Hamiltonian formalism, but in another technique which is fractional method and at a certain condition which is $\alpha \to 1$; the results of fractional technique reduce to those obtained from classical technique.

This paper is organized as follows. In section 2, the fractional derivatives are reviewed. In section 3, the fractional Euler Lagrange equation and fractional Hamiltonian are discussed. In section 4, this work is illustrated through two physical examples. Finally, in section 5, the work closes with some concluding remarks.

2. Fractional Derivatives

From Agrawal's work (Agrawal, 2002; Podlubny, 1999)

The left Riemann – Liouville fractional derivative written as:

$$ {}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha-1} f(\tau) d\tau \quad (1)$$

which is defined as the LRLFD, and the right Riemann – Liouville fractional derivative written as:

$$ {}_xD_b^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_x^b (\tau-x)^{n-\alpha-1} f(\tau) d\tau \quad (2)$$

which is defined as the RRLFD.

where Γ represents the Euler's gamma function and α is the order of the derivative such that $n-1 \le \alpha < n$, and is not equal to zero. If α is an integer, these derivatives are written as:

$$ {}_aD_x^{\alpha} f(x) = \left(\frac{d}{dx}\right)^{\alpha} f(x) \quad (3)$$

$$ {}_xD_b^{\alpha} f(x) = \left(-\frac{d}{dx}\right)^{\alpha} f(x) \quad (4)$$

The fractional operator, ${}_aD_x^{\alpha} f(x)$ can be written as (Igor, 2002).

$$ {}_aD_x^{\alpha} = \frac{d^n}{dx^n} {}_aD_x^{\alpha-n} \quad (5)$$

$\alpha = 1,2,...$

and has the following properties:

$$ 1) \quad {}_aD_x^{\alpha} = \frac{d^{\alpha}}{dx^{\alpha}}, \ \text{Re}(\alpha) > 0 \quad (6)$$

$$ 2) \quad {}_aD_x^{\alpha} = 1, \ \text{Re}(\alpha) = 0 \quad (7)$$

$$ 3) \ {}_aD_x^{\alpha} = \int_a^x (d\tau)^{-\alpha}, \quad \text{Re}(\alpha) < 0 \quad (8)$$

Theorem: Let f and g be two continuous functions on $[a,b]$. Then, for all $x \in [a,b]$, the following properties hold:

$$ 1) \ \text{For } m > 0, \ {}_aD_x^{m} = [f(x) + g(x)] = {}_aD_x^{m} f(x) + {}_aD_x^{m} g(x) \quad (9)$$

$$ 2) \ \text{For } m \ge n \ge 0, {}_aD_x^{m} ({}_aD_x^{-n} f(x)) = {}_aD_x^{m-n} f(x) \quad (10)$$

3) For m >0, ${}_aD_x^m({}_aD_x^{-m}f(x)) = f(x)$ (11)

4) For m >0, $\int_a^b ({}_aD_x^m f(x))g(x)dx = \int_a^b f(x)({}_xD_b^m g(x))dx$ (12)

3. Fractional Hamiltonian and Fractional Euler Lagrange Equation

Using Lagrangian in the following fractional form

$$L = L({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, {}_aD_t^{\alpha}q, {}_tD_b^{\beta}q, t) \tag{13}$$

Recalling that, action function for all $x \in [a,b]$, can be defined as follows:

$$S = \int_a^b L({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, {}_aD_t^{\alpha}q, {}_tD_b^{\beta}q, t)dt \tag{14}$$

The fractional Euler Lagrange equation (Agrawal, 2002) is given by:

$$\frac{\partial L}{\partial q} + {}_tD_b^{\alpha}\frac{\partial L}{\partial {}_aD_t^{\alpha}q} + {}_aD_t^{\beta}\frac{\partial L}{\partial {}_tD_b^{\beta}q} = 0 \tag{15}$$

Thus, if $\alpha = \beta = 1$, we find that:

$${}_tD_b^{\alpha} = -\frac{d}{dt} \tag{16}$$

and

$${}_aD_t^{\alpha} = \frac{d}{dt} \tag{17}$$

And the fractional Euler Lagrange equation reduce to the classical form of the Euler Lagrange equation.

Where the generalized momenta can be obtained from:

$$p_{\alpha} = \frac{\partial L}{\partial {}_aD_t^{\alpha}q} \tag{18}$$

and

$$p_{\beta} = \frac{\partial L}{\partial {}_tD_b^{\beta}q} \tag{19}$$

Thus, the generalized coordinate q in fractional form is defined as:

$$q_{\alpha} = {}_aD_t^{\alpha-1}q \tag{20}$$

Also,

$$q_{\beta} = {}_tD_b^{\beta-1}q \tag{21}$$

Making use that,

$$D_t^0 = 1 \tag{22}$$

and

$$D_t^1 = \frac{d}{dt} \tag{23}$$

The Hamiltonian depending on the fractional derivatives is written as:

$$H({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, p_\alpha, p_\beta) = p_\alpha\, {}_aD_t^\alpha q + p_\beta\, {}_tD_b^\beta q - L({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, {}_aD_t^\alpha q, {}_tD_b^\beta q, t) \tag{24}$$

4. Illustrative Examples (Nawafleh, 1998)

An Example with First Class Constraints:

Let us consider the following Lagrangian

$$L = \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_3^2\right) + \dot{q}_1\dot{q}_3 + q_2\dot{q}_2 - q_1 - q_3 \tag{25}$$

Using fractional derivatives equation (25) becomes

$$L = \frac{1}{2}({}_0D_t^\alpha q_1)^2 + \frac{1}{2}({}_0D_t^\alpha q_3)^2 + ({}_0D_t^\alpha q_1)({}_0D_t^\alpha q_3) + ({}_0D_t^{\alpha-1} q_2)({}_0D_t^\alpha q_2) - ({}_0D_t^{\alpha-1} q_1) - ({}_0D_t^{\alpha-1} q_3) \tag{26}$$

From equation (26) and by using of equation (18) the conjugate momenta are

$$p_1 = \frac{\partial L}{\partial\, {}_0D_t^\alpha q_1} = {}_0D_t^\alpha q_1 + {}_0D_t^\alpha q_3 \tag{27}$$

Also,

$$p_2 = \frac{\partial L}{\partial\, {}_0D_t^\alpha q_2} = ({}_0D_t^{\alpha-1} q_2) \tag{28}$$

Then,

$$p_3 = \frac{\partial L}{\partial\, {}_0D_t^\alpha q_3} = {}_0D_t^\alpha q_3 + {}_0D_t^\alpha q_1 \tag{29}$$

Thus, in this example we find that, $p_3 = p_1$

The fractional Hamiltonian has the standard form using equation (24)

$$H = p_1({}_0D_t^\alpha q_1) + p_2({}_0D_t^\alpha q_2) + p_3({}_0D_t^\alpha q_3) - L({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, {}_aD_t^\alpha q, {}_tD_b^\beta q, t) \tag{30}$$

Now, substituting equations. (26), (27), (28) and (29) into equation (30), Hamiltonian reads:

$$H = \frac{1}{2}({}_0D_t^\alpha q_1)^2 + \frac{1}{2}({}_0D_t^\alpha q_3)^2 + ({}_0D_t^\alpha q_1)({}_0D_t^\alpha q_3) + ({}_0D_t^{\alpha-1} q_1) + ({}_0D_t^{\alpha-1} q_3) \tag{31}$$

This equation can readily be solved to give

$$H = \frac{1}{2} p_1^2 + ({}_0D_t^{\alpha-1} q_1) + ({}_0D_t^{\alpha-1} q_3) \tag{32}$$

An Example with Primary and Secondary Constraints of the Second Class:

As a second example, let us take the following Lagrangian (Rabei and Guler, 1992):

$$L = \frac{1}{2}\dot{q}_1^2 - \frac{1}{4}(\dot{q}_2 - \dot{q}_3)^2 + (q_1 + q_3)\dot{q}_2 - (q_1 + q_2 + q_3^2) \tag{33}$$

The corresponding fractional Lagrangian is:

$$\begin{aligned} L = &\frac{1}{2}({}_0D_t^\alpha q_1)^2 - \frac{1}{4}\left[({}_0D_t^\alpha q_2) - ({}_0D_t^\alpha q_3)\right]^2 + \left[({}_0D_t^{\alpha-1} q_1) + ({}_0D_t^{\alpha-1} q_3)\right]({}_0D_t^\alpha q_2) \\ &- \left[({}_0D_t^{\alpha-1} q_1) + ({}_0D_t^{\alpha-1} q_2) + ({}_0D_t^{\alpha-1} q_3)^2\right] \end{aligned} \tag{34}$$

Making use of equation (34) and equation (18), the momenta can be computed as

$$p_1 = \frac{\partial L}{\partial {}_0D_t^\alpha q_1} = {}_0D_t^\alpha q_1 \tag{35}$$

Then,

$$p_2 = \frac{\partial L}{\partial {}_0D_t^\alpha q_2} = -\frac{1}{2}\left[({}_0D_t^\alpha q_2) - ({}_0D_t^\alpha q_3)\right] + ({}_0D_t^{\alpha-1} q_1) + ({}_0D_t^{\alpha-1} q_3) \tag{36}$$

The usual treatment gives the following momentum

$$p_3 = \frac{\partial L}{\partial {}_0D_t^\alpha q_3} = \frac{1}{2}\left[({}_0D_t^\alpha q_2) - ({}_0D_t^\alpha q_3)\right] \tag{37}$$

From equation (36) and equation (37) one can see that,

$$p_2 = ({}_0D_t^{\alpha-1} q_1) + ({}_0D_t^{\alpha-1} q_3) - p_3$$

The fractional Hamiltonian can be expressed as

$$H = p_1({}_0D_t^\alpha q_1) + p_2({}_0D_t^\alpha q_2) + p_3({}_0D_t^\alpha q_3) - L({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, {}_aD_t^\alpha q, {}_tD_b^\beta q, t) \tag{38}$$

As usual, putting equation. (34), (35), (36) and equation (37) into equation (38), the Hamiltonian takes this form

$$H = \frac{1}{2}({}_0D_t^\alpha q_1)^2 - \frac{1}{4}({}_0D_t^\alpha q_2)^2 - \frac{1}{4}({}_0D_t^\alpha q_3)^2 + \frac{1}{2}\left[({}_0D_t^\alpha q_3)({}_0D_t^\alpha q_2)\right] + ({}_0D_t^{\alpha-1} q_1) + ({}_0D_t^{\alpha-1} q_2) + ({}_0D_t^{\alpha-1} q_3)^2 \tag{39}$$

Simple manipulations yield

$$H = \frac{1}{2}(p_1^2 - 2p_3^2) + ({}_0D_t^{\alpha-1} q_1) + ({}_0D_t^{\alpha-1} q_2) + ({}_0D_t^{\alpha-1} q_3)^2 \tag{40}$$

5. Conclusions

In this work fractional constrained systems are studied using fractional Lagrangian and fractional Hamiltonian, also two types of constraints are studied: first class constraints and second class constraints. The Lagrangians are written in **fractional** form and then we find the fractional Hamiltonian, then the fractional conjugate momenta are obtained from our fractional Lagrangian, when $\alpha \to 1$; the results of **fractional technique** reduce to those obtained from **classical technique**. In this paper we find the ability of the first class constraints and second class constraints to studying using fractional technique. Some physical examples are considered to demonstrate the application of the formalism.

References

Agrawal, O. P. (2002). Formulation of Euler–Lagrange equations for fractional variational problems. *Journal of Mathematical Analysis and Applications*, *272*, 368-379. https://doi.org/10.1016/S0022-247X(02)00180-4

Agrawal, O. P. (2006). Fractional variational calculus and the transversality conditions. *Journal of Physics A: Mathematical and General, 39*(33). https://doi.org/10.1088/0305-4470/39/33/008

Dirac, P. A. M. (1950). Generalized Hamiltonian Dynamics. *Canadian Journal of Mathematical Physics, 2*, 129-148. https://doi.org/10.4153/CJM-1950-012-1

Dirac, P. A. M. (1964). *Lectures on Quantum Mechanics. Belfer Graduate School of Science.* Yeshiva University, New York.

Evans, J. M. (1991). On Dirac's Methods for Constrained Systems and Gauge Fixing Conditions with Explicit Time Dependence. *Physical Letters B., 256*(2), 245-250. https://doi.org/10.1016/0370-2693(91)90681-F

Faddeev, L.D. (1970). The Feynman integral for singular Lagrangians. *Theoretical and Mathematical Physics*, *1*, 1-13. https://doi.org/10.1007/BF01028566

Goldberg, J. (1991). On Hamiltonian Systems with First-Class Constraints. *Journal of Mathematical Physics, 32*(10), 2739-2743. https://doi.org/10.1063/1.529065

Goldstein, H. (1980). *Classical Mechanics* (2nd ed.). Addison-Wesley, Reading-Massachusetts.

Gorenflo, R., & Mainardi, F. (1997) *Fractional Calculus: Integral and Differential Equations of Fractional Orders, Fractals and Fractional Calculus in Continuums Mechanics.* Springer Verlag, Wien and New York, pp. 223-276. https://doi.org/10.1007/978-3-7091-2664-6_5

Guler, Y. (1992). Canonical Formulation of Singular Systems. *Il Nuovo Cimento, 107B,* 1389-1395. https://doi.org/10.1007/BF02722849

Hanson, A., Regge, T., & Teitelbon, C. (1976). *Constrained Hamiltonian Systems.* Accademia Nazionale Dei Lincei, Rome, pp. 30-94.

Hasan, E. H. (2016). Fractional Variational Problems of Euler-Lagrange Equations with Holonomic Constrained Systems. *Applied Physics Research*, *10*(5), 223-234. https://doi.org/10.5539/apr.v8n3p60

Hasan, E., Rabei, E. M., & Ghassib, H. B. (2004). Quantization of Higher-Order Constrained Lagrangian Systems Using the WKB Approximation. *International Journal of Theoretical Physics*, *43*(11), 2285-2298. https://doi.org/10.1023/B:IJTP.0000049027.45011.37

Igor, M., Sokolove, J. K., & Blumen, A. (2002). Fractional kinetics. *Physics Today*, *55*(11), 48-54. https://doi.org/10.1063/1.1535007

Jarab'ah, O. (2018). Fractional Euler Lagrange Equations for Irregular Lagrangian with Holonomic Constraints. *Journal of Modern Physics*, *9*, 1690-1696. https://doi.org/10.4236/jmp.2018.98105

Jarab'ah. O., & Nawafleh, K. (2018). Fractional Hamiltonian of Nonconservative Systems with Second Order Lagrangian. *American Journal of Physics and Applications*, *6*(4), 85-88. https://doi.org/10.11648/j.ajpa.20180604.12

Miller, K. S., & Ross, B. (1993). *An Introduction to the Fractional Integrals and Derivatives- Theory and Applications*. John Willey and Sons, New York, pp. 80-120.

Muslih, S. I. (2001). Path Integral Formulation of Constrained Systems with Singular Higher-Order Lagrangians. *Hadronic Journal*, *24*, 713-721.

Muslih, S. I. (2002). Quantization of Singular Systems with Second-Order Lagrangians. *Modern Physics Letters A*, *17*, 2383-2391. https://doi.org/10.1142/S0217732302009027

Nawafleh, K. (1998). Constrained Hamiltonian Systems: A Preliminary Study. *Unpublished M.Sc. Thesis,* The University of Jordan, Amman, Jordan.

Nawafleh, K. I., Rabei, E. M., & Ghassib, H. B. (2004). Hamilton-Jacobi Treatment of Constrained Systems. *International Journal of Modern Physics A*, *19*, 347-354. https://doi.org/10.1142/S0217751X04017719

Nawafleh, K. I., Rabei, E. M., & Ghassib, H. B. (2005). Quantization of Reparametrized Systems Using the WKB Method. *Turkish Journal of Physics*, *29*, 151-162.

Podlubny, I. (1999). *Fractional Differential Equations*. Academic Press, New York, pp. 97-104.

Rabei, E. M. (1996). On Hamiltonian Systems with Constraints. *Hadronic Journal*, *19*, 597-605.

Rabei, E. M. (2000). On the Quantization of Constrained Systems Using Path Integral Techniques. *Il Nuovo Cimento B*, *115*(10), 1159-1165.

Rabei, E. M., Nawafleh, K. I., & Ghassib, H. B. (2002). Quantization of Constrained Systems Using the WKB Approximation. *Physical Review A*, *66*, 024101. https://doi.org/10.1103/PhysRevA.66.024101

Rabei, E. N., Nawafleh, K. I., Abdelrahman, Y. S., & Rashed Omari, H. Y. (2003). Hamilton Jacobi Treatment of Lagrangian with Linear Velocities. *Modern Physics Letters A*, *18*(23), 1591-1596. https://doi.org/10.1142/S0217732303011277

Rabei, Eqab M., & Guler, Y. (1992). Hamilton-Jacobi Treatment of Second-Class Constraints. *Physical Review A.*, *46*(6), 3513-3515. https://doi.org/10.1103/PhysRevA.46.3513

Riewe, F. (1996). Nonconservative Lagrangian and Hamiltonian mechanics. *Physical Review E.*, *53*, 1890. https://doi.org/10.1103/PhysRevE.53.1890

Riewe, F. (1997). Mechanics with fractional derivatives. *Physical Review E.*, *55*, 3581. https://doi.org/10.1103/PhysRevE.55.3581

Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach Science Publishers, Amsterdam, pp. 33-40.

Senjanovic, P. (1976). Path Integral Quantization of Field Theories with Second-Class Constraints. *Annals of Physics, 100*, 227-261. https://doi.org/10.1016/0003-4916(76)90062-2