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Action Integral of Regular Systems With Higher Order Lagrangian

Ola A. Jarab'ah[1]

[1] Applied Physics Department, Faculty of Science, Tafila Technical University, Tafila, Jordan

Correspondence: Ola A. Jarab'ah, Applied Physics Department, Faculty of Science, Tafila Technical University, P. O. Box 179, Tafila 66110, Jordan. E-mail: oasj85@yahoo.com

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Abstract

In this paper Euler Lagrange equation is studied to obtain the equations of motion of the regular systems with higher order Lagrangian. The solutions of the equations of motion help us to obtain the action integral by substituting the solutions in the given Lagrangian. Then, action integral is typically represented as an integral of the Lagrangian over time, taken along the path of the system between the initial time and the final time of the system. The regular systems with higher order Lagrangian are examined using illustrative example.

Keywords: regular systems, Euler Lagrange Equation, action integral, equations of motion

1. Introduction

There is a famous formulation of classical mechanics known as the Lagrangian mechanics. The Lagrangian function L for a system is defined to be the difference between the kinetic and potential energies expressed as a function of positions and velocities $L = K - U$ (Goldstein, 1980).

The Euler Lagrange equations form the basis of Lagrangian mechanics. In terms of Lagrangian L, the Euler Lagrange equation is defined as (Goldstein, 1980):

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

This Lagrangian is a function of the coordinate q and only its first time derivative. The Euler Lagrange equation of a first order Lagrangian is one of the most known and widely used variational equations in mathematics, mechanics and physics. Most physical systems can be described by regular Lagrangians that depend at most on the first derivatives of the generalized coordinate q (Dirac, 1950, 1964).

There are thus a wide range of applications that motivates to look further in the properties of higher derivative theories. Systems with higher order Lagrangian have been studied with increasing interest because they appear in many relevant physical problems and in many models in theoretical and mathematical physics, they also appear in some problems of fluid mechanics, electric networks and classical physics.

The treatment of theories with higher order derivatives has been first developed by (Ostrogradski, 1850; Pon, 1989) and allows writing the Euler Lagrange equations introduce conjugated momenta and develop Hamilton formalism for such systems. A Treatment of a higher order Singular Lagrangian as fields system was studied by (Farahat, 2002, 2006). There is another interest in the generalized dynamics; that is, the study of physical systems described by Lagrangians containing derivatives of order higher than the first (Pimentel et al., 1996, 1998; Muslih, 2000, 2004; Nawafleh, 2008, 2011). More recently higher order Lagrangians for classical mechanics and scalar fields were developed by (Harmanni, 2016).

Hamilton used the principle of least action to derive the Hamilton Jacobi relation $H(q, p, t) + \frac{\partial S}{\partial t} = 0$.

The generating function for solving the Hamilton Jacobi equation then equals the action function S. Since $L \equiv T - V$ is the physical Lagrangian of the system, it follows that H is the physical hamiltonian representing the system's total energy $H \equiv T + V$ (Santilli, 1981).

Thus, the resulting action S is

$$S = \int L(q, \dot{q}, t)dt = \int (p\dot{q} - H)dt.$$

Formulation of the action function has been investigated using Hamilton Jacobi equation (Arnold, 1989; Lanczos, 1986). Recently obtaining the action function for dissipative systems is investigated within the framework of the Hamilton Jacobi equation, this function is determined using the method of separation of variables (Jarab'ah et al.,2013, 2014; Nawafleh et al., 2004). In addition, action function formulation for irregular first order Lagrangian has been studied by (Jarab'ah, 2017). Moreover, action function has been obtained for regular systems with second order Lagrangian (Jarab'ah, 2018).

The aim of this paper is to study regular systems with higher order Lagrangians and then to formulate the action integral using Euler Lagrange equation by starting at third order Lagrangian.

This paper is organized as follows. In section 2, the formulation of the action integral for regular systems with higher order Lagrangian is discussed. In section 3, the regular systems with higher order Lagrangian are examined using illustrative example. Finally, in section 4, the work closes with some concluding remarks.

2. Formulation of the Action Integral for Regular Systems With Higher Order Lagrangian

When I speak of higher order Lagrangian, I will mainly focus on third and possibly fourth order.

The Lagrangian formulation of regular systems with third order Lagrangian is given by

$$L = L(q, \dot{q}, \ddot{q}, \dddot{q}, t) \tag{1}$$

Third order Lagrangian means the time derivative of acceleration $\dddot{q}$ (Harmanni, 2016).

The equations of motion corresponding to this Lagrangian are given by the Euler-Lagrange equation:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) - \frac{d^3}{dt^3}\left(\frac{\partial L}{\partial \dddot{q}_i}\right) = 0 \\ i = 1,2,3,... \tag{2}$$

which in general will give rise to equations of motion containing $(q, \dot{q}, \ddot{q}, \dddot{q}, q^{(4)}, q^{(5)}, q^{(6)})$, this means the solution of Euler Lagrange equation contains the sixth derivative of the generalized coordinate q (Harmanni, 2016).

In this formalism the equations that result from application of Euler Lagrange equation to a particular Lagrangian are known as the equations of motion, substituting the solution of these equations in the given Lagrangian. Then, action integral is typically represented as an integral of the Lagrangian over time, taken along the path of the system between the initial time and the final time of the system:

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, \ddot{q}, \dddot{q}, t)dt \tag{3}$$

This paper will be illustrated by the following example:

3. Example

Let us consider the following Lagrangian:

$$L = \frac{1}{2}\left(\dddot{q}^2 + \ddot{q}^2\right) \tag{4}$$

Using Euler Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{d^3}{dt^3}\left(\frac{\partial L}{\partial \dddot{q}}\right) = 0$$

The equation that result from application of Euler Lagrange equation to equation (4) is

$$\overset{\bullet\bullet\bullet\bullet}{q} - \overset{\bullet\bullet\bullet\bullet\bullet\bullet}{q} = 0 \tag{5}$$

Note that:

$\overset{\bullet\bullet\bullet\bullet}{q} \equiv q^{(4)} \equiv$ fourth derivative

$\overset{\bullet\bullet\bullet\bullet\bullet\bullet}{q} \equiv q^{(6)} \equiv$ sixth derivative

So that we can write equation (5) in this form

$$q^{(4)} - q^{(6)} = 0$$

After some calculations, we obtain the following solution for equation (5)

$$q(t) = c_1 e^t + c_2 e^{-t} + c_3 + tc_4 + t^2 c_5 + t^3 c_6 \tag{6}$$

Taking the second time derivative of equation (6), we obtain

$$\ddot{q}(t) = c_1 e^t + c_2 e^{-t} + 2c_5 + 6tc_6 \tag{7}$$

To simplify, these constants are used in equation (8) and equation (10)

$c_1^2 = A$, $2c_1c_2 = B$, $4c_1c_5 = C$, $12c_1c_6 = D$, $c_2^2 = E$, $4c_2c_5 = F$, $12c_2c_6 = G$, $4c_5^2 = H$, $24c_5c_6 = K$, and $36c_6^2 = N$.

The square of equation (7) is

$$\ddot{q}^2(t) = Ae^{2t} + B + Ce^t + tDe^t + Ee^{-2t} + Fe^{-t} + tGe^{-t} + H + tK + Nt^2 \tag{8}$$

The third time derivative of equation (6) is

$$\dddot{q}(t) = c_1 e^t - c_2 e^{-t} + 6c_6 \tag{9}$$

Squaring equation (9) we get

$$\dddot{q}^2(t) = Ae^{2t} - B + De^t + Ee^{-2t} - Ge^{-t} + N \tag{10}$$

By substituting equation (8) and equation (10) in equation (4), the corresponding Lagrangian is

$$L = \frac{1}{2}\left[2Ae^{2t} + De^t + 2Ee^{-2t} - Ge^{-t} + N + Ce^t + tDe^t + Fe^{-t} + tGe^{-t} + H + tK + Nt^2\right] \tag{11}$$

Now we are ready to find the action integral S by using equation (3) and put $(t_1 = 0, t_2 = t)$

So that the integration becomes

$$S = \int_0^t \frac{1}{2}\left[2Ae^{2t} + De^t + 2Ee^{-2t} - Ge^{-t} + N + Ce^t + tDe^t + Fe^{-t} + tGe^{-t} + H + tK + Nt^2\right]dt \tag{12}$$

Further, the action integral takes the form

$$S=\frac{1}{2}\left[Ae^{2t}-A-Ee^{-2t}+E+Nt+Ce^{t}-C+Dte^{t}-Fe^{-t}+F-tGe^{-t}+Ht+\frac{t^{2}}{2}K+N\frac{t^{3}}{3}\right] \quad (13)$$

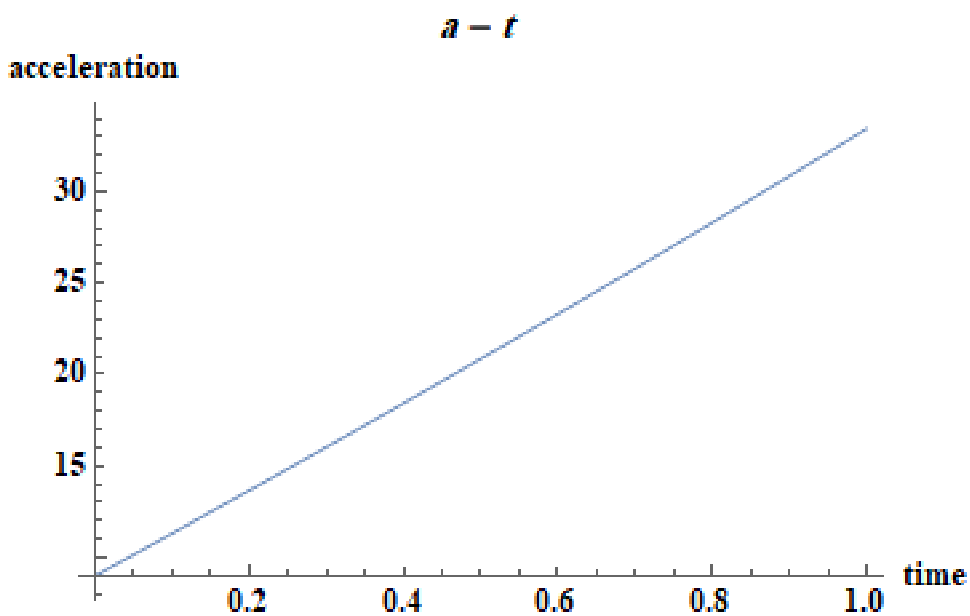
The conjugate acceleration as a function of time is

$$\ddot{q}(t)=c_1e^{t}+c_2e^{-t}+2c_5+6tc_6 \quad (14)$$

The following graph shows the variation of the acceleration through time interval equals one second.

Using equation (14) and putting $c_1=1, c_2=2, c_5=3, c_6=4$, we obtain

Plot [e^{t}+2e^{-t}+6+24t, {t,0,1}]



4. Conclusion

Our purpose from this work is to prepare a new way to obtain the action integral function **S** for regular systems using Euler Lagrange equation rather than Hamilton Jacobi equation. Through the Lagrangian treatment the action integral can be obtained by found the equations of motion using Euler Lagrange equation, then substituting the solutions of the equations of motion in our Lagrangian. Finally, integrate the Lagrangian through the given time interval. Also, the action integral is constructed for third order Lagrangian. This result has been obtained through illustrative example.

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