Applied Physics Research; Vol. 15, No. 1; 2023
ISSN 1916-9639 E-ISSN 1916-9647
Published by Canadian Center of Science and Education

Fractional Lagrangian Formulation of Classical Fields With Second-Order Derivatives

Eyad Hasan Hasan[1]

[1] Applied Physics Department, Faculty of Science, Tafila Technical University, Tafila, Jordan

Correspondence: Eyad Hasan Hasan, Applied Physics Department, Faculty of Science, Tafila Technical University, P. O. Box 179, Tafila 66110, Jordan. E-mail: iyad973@yahoo.com

Received: March 28, 2022 Accepted: December 28, 2022 Online Published: March 1, 2023

doi:10.5539/apr.v15n1p14 URL: https://doi.org/10.5539/apr.v15n1p14

Abstract

This paper examined the Lagrangian theory of a classical scalar field from the perspective of fractional calculus. Fractional Lagrangian formulation is extended to classical fields systems with second-order derivatives. This formulation is examined within Riemann-Liouville fractional derivatives and examined for one classical field example.

Keywords: fractional calculus, fractional Euler-Lagrange equation, second-order lagrangian, classical fields

1. Introduction

The study of fractional calculus has gained a considerable important and played a significant role in physical systems, engineering and applied mathematics, (Samko et al., 1993), starting from Riewe's work of the Hamiltonian formalism (Riewe, 1996). Riewe has developed a formalism using fractional calculus for Lagrangian and Hamiltonian formulation. In Riewe's formalism the Euler's and Hamilton's equations of motion can be applied for both conservative and non-conservative systems. The generalization of Riewe's work of Lagrangian and Hamiltonian mechanics has been investigated in details in references (Agrawal, 2002; Baleanu and Muslih, 2005; Hasan, 2016).

The fractional variational problem of Lagrange formalism has been investigated by Agrawal approach (Agrawal, 2002). Agrawal has investigated the Lagrangian formulation using fractional variational problems. Agrawal has arrived that the resulting equations are similar to those for variational problems containing integral order derivatives. This approach is extended for classical fields with fractional derivatives (Baleanu and Muslih, 2005) for obtaining the fractional formalism for first-order Lagrangians. Recently, Agrawal formalism has been extended for Lagrangian systems with higher derivatives (Hasan and Asad, 2017; Hasan, 2018), authors have investigated the Lagrangian and Hamiltonian analysis for higher-order derivatives systems for discrete regular systems with the generalization of Ostrogradski's formulation within the fractional calculus.

Now, we will present the basic definitions of fractional derivatives (Agrawal, 2002).

(i) The left Riemann–Liouville fractional derivative

$$ {}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha-1} f(\tau)d\tau . \tag{1a}$$

(ii) The right Riemann–Liouville fractional derivative

$$ {}_xD_b^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b (\tau-x)^{n-\alpha-1} f(\tau)d\tau . \tag{1b}$$

Where $n \in N$, $n-1 \le \alpha < n$ and Γ is the Euler gamma function.

Remark If α is an integer, we can define these derivatives as follows:

$$ {}_aD_x^{\alpha} f(t) = \left(\frac{d}{dx}\right)^{\alpha} f(x)\,,\quad {}_xD_b^{\alpha} f(x) = \left(-\frac{d}{dx}\right)^{\alpha} f(x)\,, \tag{2} $$

Now, we define a function f depending on n variables, x_1, x_2, …, x_n and differentiable. We can consider the fractional derivative of order α_k, 0< α_k <1 in the k-th variable (Samko et al., 1993)

$$ (D_{a_{k^+}}^{\alpha_k} f)(x) = \frac{1}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int_{a_k}^{x_k} (x_k - u)^{-\alpha_k} f(x_1,...,x_{k-1},u,x_{k+1},...,x_n)du\,, \tag{3} $$

$$ (D_{a_{k^-}}^{\alpha_k} f)(x) = \frac{1}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int_{x_k}^{a_k} (-x_k + u)^{-\alpha_k} f(x_1,...,x_{k-1},u,x_{k+1},...,x_n)du\,, \tag{4} $$

$$ \begin{aligned} (D_{a_{k^+}}^{\alpha_k} f)(x) &= \frac{1}{\Gamma(1-\alpha_k)}[(x_k - a_k)^{-\alpha_k} f(x_1,...,x_{k-1},a_k,x_{k+1},...,x_n) + \\ &\int_{a_k}^{x_k} (x_k - u)^{-\alpha_k} \frac{\partial f}{\partial u}(x_1,...,x_{k-1},u,x_{k+1},...,x_n)]du \end{aligned} \quad . \tag{5} $$

We can note that the second term in Eq. (5) represents the Caputo derivative Caputo (1967).

This work is an intriguing generalization of standard calculus whose actual import in theoretical physics and gives new results in fractional calculus with second-order Lagrangians of classical fields systems; therefore, we aim to construct the formulation of Euler-Lagrange equations for these systems. Our article paper is organized as follows. In section 2, the fractional Euler-Lagrange equation of classical field with second-order derivatives is presented. In section 3, one classical field example is examined. The work closes with some concluding remarks in section 4.

2. Fractional Euler –Lagrange Equation of Continuous Systems (Classical Field)

In this section, we will discuss the Lagrangian theory of a classical scalar field from the perspective of Riemann-Liouville fractional derivatives for second-order Lagrangian.

We will start with the action

$$ S = \int \ell(\phi(x),(D_{a_k-}^{\alpha_k})\phi(x),(D_{a_k+}^{\alpha_k})\phi(x),(D_{a_k-}^{2\alpha_k})\phi(x),(D_{a_k+}^{2\alpha})\phi(x),x)d^3xdt\,. \tag{6} $$

The value of α_k is 0< α_k ≤ 1 and a_k represent x_1, x_2, x_3 and t. In our work we consider the integration limit are $-\infty$ and ∞ respectively.

Thus, $D_{a_k+}^{\alpha_k}$ changes to $D_{-\infty+}^{\alpha_k}$, $D_{a_k-}^{\alpha_k}$ changes to $D_{\infty-}^{\alpha_k}$. Similarly, $D_{a_k+}^{2\alpha_k}$ becomes $D_{-\infty+}^{2\alpha_k}$ and $D_{a_k-}^{2\alpha_k}$ becomes $D_{\infty-}^{2\alpha_k}$.

We can write the functional $S(\phi)$ in terms of $\in$ as finite variation, one can obtain

$$ \begin{aligned} \Delta_{\in} S(\phi) = \int &[\ell(x,\phi + \in \delta\phi,(D_{\infty-}^{\alpha_k})\phi(x) + \in (D_{\infty-}^{\alpha_k})\delta\phi,(D_{-\infty+}^{\alpha_k})\phi(x) + \in (D_{-\infty+}^{\alpha_k})\delta\phi, \\ &(D_{\infty-}^{2\alpha_k})\phi(x) + \in (D_{\infty-}^{2\alpha_k})\delta\phi,(D_{-\infty+}^{2\alpha_k})\phi(x) + \in (D_{-\infty+}^{2\alpha_k})\delta\phi) - \\ &\ell(x,\phi,(D_{\infty-}^{\alpha_k})\phi(x),(D_{-\infty+}^{\alpha_k})\phi(x) + (D_{\infty-}^{2\alpha_k})\phi(x) + (D_{-\infty+}^{2\alpha_k})\phi(x))]d^3xdt \end{aligned} \quad . \tag{7} $$

The first term in eq.7 is a function on $\in$. Using a Taylor series in $\in$. Thus, Eq. 7 becomes.

$$\Delta_\in S(\phi)=\int \begin{aligned}&[\ell(x,\phi,(D_{\infty-}^{\alpha_k})\phi(x),(D_{-\infty+}^{\alpha_k})\phi(x),(D_{\infty-}^{2\alpha_k})\phi(x),(D_{-\infty+}^{2\alpha_k})\phi(x))+(\frac{\partial\ell}{\partial\phi}\delta\phi)\in+\\ &\sum_k^{1,4}\frac{\partial\ell}{\partial(D_{\infty-}^{\alpha_k}\phi)}\delta(D_{\infty-}^{\alpha_k}\phi)\in+\sum_k^{1,4}\frac{\partial\ell}{\partial(D_{\infty+}^{\alpha_k}\phi)}\delta(D_{\infty+}^{\alpha_k}\phi)\in+\sum_k^{1,4}\frac{\partial\ell}{\partial(D_{\infty-}^{2\alpha_k}\phi)}\delta(D_{\infty-}^{2\alpha_k}\phi)\in\\ &+\sum_k^{1,4}\frac{\partial\ell}{\partial(D_{\infty+}^{2\alpha_K}\phi)}\delta(D_{\infty+}^{2\alpha_k}\phi)\in+O(\in)-\ell(x,\phi,(D_{\infty-}^{\alpha_k})\phi(x),(D_{-\infty+}^{\alpha_k})\phi(x)+(D_{\infty-}^{2\alpha_k})\phi(x)+\\ &(D_{-\infty+}^{2\alpha_k})\phi(x))]d^3xdt\end{aligned} \tag{8}$$

Considering Eq. 7, thus, Eq.8 becomes

$$\Delta_\in S(\phi)=\int \begin{aligned}&\in[(\frac{\partial\ell}{\partial\phi}\delta\phi)+\sum_k^{1,4}\frac{\partial\ell}{\partial(D_{\infty-}^{\alpha_k}\phi)}\delta(D_{\infty-}^{\alpha_k}\phi)+\sum_k^{1,4}\frac{\partial\ell}{\partial(D_{-\infty+}^{\alpha_k}\phi)}\delta(D_{\infty+}^{\alpha_k}\phi)+\sum_k^{1,4}\frac{\partial\ell}{\partial(D_{\infty-}^{2\alpha_k}\phi)}\delta(D_{\infty-}^{2\alpha_k}\phi)\\ &+\sum_k^{1,4}\frac{\partial\ell}{\partial(D_{-\infty+}^{2\alpha_K}\phi)}\delta(D_{\infty+}^{2\alpha_k}\phi)+O(\in)]d^3xdt\end{aligned} \tag{9}$$

Now, we integrate the second term in Eq. 9 by parts (Samko et al., 1993) and similarly for the third, fourth and fifth term.

$$\int_{-\infty}^{\infty} f(x)(D_{-\infty+}^{\alpha_k}g)(x)dx=\int_{-\infty}^{\infty} g(x)(D_{\infty-}^{\alpha_k}f)(x)dx\,; \tag{10a}$$

$$\int_{-\infty}^{\infty} f(x)(D_{-\infty+}^{2\alpha_k}g)(x)dx=\int_{-\infty}^{\infty} g(x)(D_{\infty-}^{2\alpha_k}f)(x)dx\,. \tag{10b}$$

Thus, we get

$$\Delta_\in S(\phi)=\int \begin{aligned}&\in[(\frac{\partial\ell}{\partial\phi}\delta\phi)+\sum_k^{1,4}\{(D_{-\infty+}^{\alpha_k})\frac{\partial\ell}{\partial(D_{\infty-}^{\alpha_k})\phi}\}\delta\phi+\sum_k^{1,4}\{(D_{\infty-}^{\alpha_k})\frac{\partial\ell}{\partial(D_{-\infty+}^{\alpha_k})\phi}\}\delta\phi+\sum_k^{1,4}\{(D_{-\infty+}^{2\alpha_k})\frac{\partial\ell}{\partial(D_{\infty-}^{2\alpha_k})\phi}\}\delta(\phi)\\ &+\sum_k^{1,4}\{(D_{\infty-}^{2\alpha_k})\frac{\partial\ell}{\partial(D_{-\infty+}^{2\alpha_K})\phi}\}\delta(\phi)]d^3xdt+\int O(\in)d^3xdt\end{aligned} \tag{11}$$

Taking $\lim_{\in\to0}\dfrac{\Delta_\in S(\phi)}{\in}$, one can obtain the fractional second-order Euler-Lagrange equations of the form.

$$\frac{\partial\ell}{\partial\phi}+\sum_k^{1,4}[(D_{-\infty+}^{\alpha_k})\frac{\partial\ell}{\partial(D_{\infty-}^{\alpha_k})\phi}+(D_{\infty-}^{\alpha_k})\frac{\partial\ell}{\partial(D_{-\infty+}^{\alpha_k})\phi}+(D_{-\infty+}^{2\alpha_k})\frac{\partial\ell}{\partial(D_{\infty-}^{2\alpha_k})\phi}+(D_{\infty-}^{2\alpha_k})\frac{\partial\ell}{\partial(D_{-\infty+}^{2\alpha_k})\phi}]=0\,. \tag{12}$$

The interesting point for $\alpha_k \to 1$, Eq. (12) gives the Lagrangian formalism of classical fields with second order (Pimentel, R.G. and Teixeira, 1996).

3. Example

We consider a mathematical model of Lagrangian density which has the form

$$\ell=\frac{1}{2}(D_{-\infty+}^{\alpha_t}\phi)^2-\frac{1}{2}(D_{-\infty+}^{2\alpha_x}\phi)^2-\frac{1}{2}(D_{-\infty+}^{\alpha_x}\varphi)^2-\frac{1}{2}m^2c^2(1-\cos\phi)\,. \tag{13}$$

The fractional Euler-Lagrange equation of classical field Eq. (12) becomes.

$$D_{\infty-}^{\alpha_t}(D_{-\infty+}^{\alpha_t})\phi - D_{\infty-}^{2\alpha_x}(D_{-\infty+}^{2\alpha_x})\phi - D_{\infty-}^{\alpha_x}(D_{-\infty+}^{\alpha_x})\phi - m^2c^2\sin\phi = 0 \quad (14)$$

One can discuss the fractional Lagrangian formalism of classical fields about possible concrete applications to actual physical systems with second-order derivatives.

4. Conclusion

We have extended the fractional Lagrangian formulation from discrete systems to continuous systems with second-order Lagrangian. The second-order derivatives of the usual Euler-Lagrange equations of motion for classical fields have been extended to the fractional Lagrangian of fields. For $\alpha_k \to 1$, we obtained the Lagrangian formulation for classical fields.

References

Agrawal, O. P. (2002). Formulation of Euler-Lagrange equations for fractional variational problems. *Journal of Mathematical Analysis and Application, 272*, 368-379. https://doi.org/10.1016/S0022-247X(02)00180-4

Baleanu, D., & Muslih, S. (2005). Lagrangian formulation of classical fields within Riemann Liouville fractional derivatives. *Physica. Scripta, 27*(105). https://doi.org/10.1115/DETC2005-84390

Caputo, M. (1967, November). Linear Models of Dissipation whose Q is almost Frequency Independent—II. *Geophysical Journal International*, *13*(5), 529-539. https://doi.org/10.1111/j.1365-246X.1967.tb02303.x

Hasan, E. H. (2016). Fractional Quantization of Holonomic Constrained Systems Using Fractional WKB Approximation. *Advanced Studies in Theoretical Physics, 10*(5), 223-234. https://doi.org/10.12988/astp.2016.6313

Hasan, E. H. (2016). Fractional Variational Problems of Euler-Lagrange Equations with Holonomic Constrained Systems. *Applied Physics Research*, *8*(3), 60-65. https://doi.org/10.5539/apr.v8n3p60

Hasan, E. H. (2018). On Fractional Solution of Euler-Lagrange Equations with Second-Order Linear Lagrangians**.** *Journal of Advanced Physics, 7*(1), 110-113. https://doi.org/10.1166/jap.2018.1388

Hasan, E. H., & Asad, J. H. (2017). Remarks on Fractional Hamilton-Jacobi Formalism with second-order Discrete Lagrangian Systems. *Journal of Advanced Physics, 6*(3), 430-433. https://doi.org/10.1166/jap.2017.1335

Pimentel, R. G., & Teixeira. (1996). Hamilton-Jacobi Formulation for Singular Systems with Second-Order Lagrangians. *Il Nuovo Ciemento B, 111*, 841-854. https://doi.org/10.1007/BF02749015

Riewe, F. (1996). Nonconservative Lagrangian and Hamiltonian mechanics. *Physical Review E, 53*(2), 1890-1899. https://doi.org/10.1103/PhysRevE.53.1890

Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). *Fractional Integrals and Derivatives- Theory and Applications*. Gordon and Breach.