Applied Physics Research; Vol. 15, No. 1; 2023
ISSN 1916-9639 E-ISSN 1916-9647
Published by Canadian Center of Science and Education

Space-time Connections With Inertial Masses of Ordinary and Novel Particles and Deducing Some Limiting Velocities From Maximal Particle Velocities

Josip Šoln[1]

[1] JZS Phys-Tech, Vienna, USA
Correspondence: Josip Šoln, JZS Phys-Tech, Vienna, Virginia 22182, USA. E-mail: jurasoln@yahoo.com

Received: October 17, 2022 Accepted: November 16, 2022 Online Published: March 1, 2023
doi:10.5539/apr.v15n1p1 URL: https://doi.org/10.5539/apr.v15n1p1

Abstract
The increasing number of physical phenomena, such as the Dark Matter, as well as the difficulties to understand the enormous distances in the Universe, encourages one to formulate matter description that goes beyond the Standard Model. Here we present the description of ordinary and novel (some as Dark Matter) particles through the bicubic equation limiting velocity solutions, globally denoted as, c_1, c_2 and c_3, (primary, secondary and tertiary). These solutions depend on the congruent parameter $z = 3\sqrt{3}mv^2/2E$ which connect them to m, v, and E , respectively being particle mass, velocity and energy. When the bicubic equation discriminant D and z satisfy $D \leq 0$, $z^2 \leq 1$, the limiting velocities describe ordinary particles (electron, neutrino, etc.) and when $D \succeq 0$, $z^2 \succeq 1$ limiting velocities describe the novel (some as Dark Matter), yet to be directly observed particles. At $z = 1$ ordinary particles with c_1, c_2 and c_3 (primary, secondary and tertiary) transit from, $z^2 \leq 1$ into novel (some as Dark Matter) patrticles, $z^2 \succeq 1$ with the same values at $z = 1$ for novel particle limiting velocities Rc_1, Rc_2 and c_3 (primary, secondary and tertiary).The ordinary tertiary particle limiting velocity c_3 and novel primary plus secondary particle limiting velocities Rc_1 and Rc_2 are convenient to be deduced from maximum particle velocities. The velocity of the neutrino with $v = c$ is a good example for $c_3 = c$,while, with the assumption that a novel particle maximum velocity is, say, $v = c^\bullet$, which leads to Rc_1 and $Rc_2 = c^\bullet$. Hopefully, it may turn out that also $c^\bullet = c$. An example of a lethargic low energy novel particle appears to be a good candidate for gravitational Dark Matter particle.

Keywords: limiting velocity, real energy, congruent parameter, novel particle, DM particle

1. Introduction

Just the difficulty to explain the distances in the Universe, lead some people to the need that possibly one should introduce new kind of particles and interactions (K. Dawson and W. Percival, 2021). This particularly so since many cosmological observations, such as Dark Matter (DM), indicate that description of matter goes beyond the Standard Model. As a consequence, may new ideas flourished about existence of new particles in the Universe. Although they are definitely there in the Universe, nobody as yet has identified any of them. Many new theories are still flourishing about their existance. Here we wish to present description of ordinary and novel (some as DM) particles, through their limiting velocities. Hopefully, if not presently, this theory eventually may yield desired results.

By upgrading the usual relativistic kinematics (Šoln, J., 2014-2021), one ends up with the bicubic equation for particle limiting velocity c

$$\left(\frac{c^2}{v^2}\right)^3 - \left(\frac{E}{mv^2}\right)^2\left(\frac{c^2}{v^2}\right) + \left(\frac{E}{mv^2}\right)^2 = 0. \tag{1}$$

where E, m and v^2 were respectively real particle energy, constant (real) mass and real velocity squared. One can simplify equation (1) with introducing real congruent parameter z $(z^* = z)$,

$$z = \frac{3\sqrt{3}mv^2}{2E}, E = \frac{3\sqrt{3}mv^2}{2z} \tag{2}$$

The three limiting velocity solution squares can be cast symbolically into two categories, all real $c_i^{2*} = c_i^2$ or some complex $c_i^{2*} \neq c_i^2$ $(i = 1, 2, 3)$. On general grounds, the three solutions are also categorized by the discriminant D through the energy E together with particle mass m and velocity squared v^2 or simply the congruent parameter z :

$$
\begin{aligned}
D &= \frac{1}{4}\left(\frac{E}{mv^2}\right)^4\left[1-\frac{4}{27}\left(\frac{E}{mv^2}\right)^2\right] \\
&= \left(\frac{27}{8}\right)^2\frac{1}{z^4}\left(1-\frac{1}{z^2}\right),
\end{aligned} \tag{3}
$$

From (3) we have

$$
\begin{aligned}
Ordinary\ particles &: D(\preceq 0, z^2 \preceq 1, c_i^{2*} = c_i^2, \\
E &= \frac{3\sqrt{3}mv^2}{2z} \succeq \frac{3\sqrt{3}mv^2}{2} && (3.1) \\
Novel\ particles &: D(\succeq 0, z^2 \succeq 1, c_i^{2*} \neq c_i^2, \\
E &= \frac{3\sqrt{3}mv^2}{2z} \preceq \frac{3\sqrt{3}mv^2}{2} && (3.2)
\end{aligned}
$$

As we wish to establish space-time connections for limiting velocities from bicubic equation, in what follows we use global notations without detailed specifications. For instance acceptable real limiting velocity squares we denote simply c_i^2 without specifying i. For a particle with an exact mass m its inertial mass $\overline{m}$ is defined with

$$
\begin{aligned}
E(c_i) &= \overline{m}(c_i)\,(\pm c_i^2),\ c_i^{2*} = c_i^2 && (4.1) \\
\overline{m}(c_i) &= \gamma(c_i)m && (4.2)
\end{aligned}
$$

$\pm c_i^2$ is allowed by the bicubic equations and $\gamma(c_i)$ is the inertial scaling factor. For ordinary particles (4.1,2) are satisfied as written. For novel particles, however, with the help from the IRC Imaginary-Reality connection, that follows from the bicubic limiting velocity solutions, the complex limiting velocity squares are reduced to three real squares of limiting velocities Rc_1^2, Rc_2^2 and c_3^2. The γ inertial scaling factors can be derived for both the ordinary, $z \preceq 1$, and novel, $z \succeq 1$, particle cases from the limiting velocity solutions. It is customary to write the particle energy in terms of a proper mass m together with explicit expression of the inertial scaling factor γ factor, rather than in terms just of inertial mass $\overline{m}$.

As shown in (Šoln, J., 2021), on the global level, already from (1) we can deduce for ordinary particles with $z^2 \preceq 1$, $c_i^{2*} = c_i^2$ the scaling factors in Lorentzian forms by simply transforming (1) into

$$
\begin{aligned}
z^2 &\preceq 1,\ c_i^{2*} = c_i^2\ : m^2c_i^4 - E^2 + E^2\frac{v^2}{c_i^2} = 0, \\
E &= \frac{3\sqrt{3}mv^2}{2z} = \frac{mc_i^2}{(1-v^2/c^2)^{\frac{1}{2}}} = \gamma^{\div}(c_i)mc_i^2 = \overline{m}^{\div}(c_i)c_i^2, \\
\gamma^{\div}(c_i) &= (1-v^2/c^2)^{-\frac{1}{2}},\ z = \frac{3\sqrt{3}v^2}{2\gamma^{\div}(c_i)c_i^2} \preceq 1
\end{aligned} \tag{5}
$$

Where with the small $\div$ above the symbols of both the energy and the inertial scaling factors indicate that they are in the Lorentzian forms. Later on, numerically equivalent scaling factor expressions for ordinary particles, $z^2 \preceq 1$, will be given with the limiting velocity solutions, as derived in (Šoln, J., 2016). However, when $z^2 \succeq 1$, for novel particles the inertial scaling factors can be calculated only from bicubic equation,with the IRC Imaginary-Reality connection for now real limiting velocity squares Rc_1^2, Rc_2^2 and c_3^2.

As we see from (5), the congruent parameter z, particle velocity $\overrightarrow{v}$, and limiting velocity c_i are all interdependent. The definition of the inertial scaling factor came quite naturally for each limiting velocity c_i.For both ordinary, $z^2 \preceq 1$, and novel, $z^2 \succeq 1$, particles the knowledge of the inertial scaling factors γ allow the particle limiting velocities to establish space-time connection. Here we do that by adopting the formalism of Relativistic Dynamics as appearing in (Sard, R. D., 1970, Barut, A. D., 1964 and Griffiths, D., 1987).

To continue, we notice that a particle with exact (proper) mass m in the inertial time t moves with the velocity $\overrightarrow{v} = d\overrightarrow{x}/dt$. The inertial time interval $\triangle\ t$ and the proper time interval $\triangle\ \tau$ are related as $\triangle\ t =\gamma\ \triangle\ \tau$. Then the proper velocity $\overrightarrow{\eta} = d\overrightarrow{x}/d\tau$ is related to particle velocity as $\overrightarrow{\eta} = \gamma\overrightarrow{v}$; this can be generalized for each particle associated with a limiting velocity c_i into a four-vector form via the inertial factor $\gamma(c_i)$, $\eta^{\mu}(c_i) = dx^{\mu}/d\tau = \gamma(c_i)(c_i, v_x, v_y, v_z) = \gamma(c_i)(c_i, v_1, v_2, v_3) = \gamma(c_i)(c_i, \overrightarrow{v})$, with $x^0(c_i) = c_i t$ and we wrote $\eta^0(c_i) = dx^0(c_i)d\tau = \gamma(c_i)\ dx^0(c_i)/dt = \gamma(c_i)c_i$.Already here we wrote some expressions that utilize the 4 by 4 Minkowski metric whose diagonal values are $g_{\mu\nu} = (1, -1, -1, -1)$: $g_{00} = -g_{ii} = 1, g_{\mu\nu} = 0, \mu \neq \nu$, $x_{\mu} = (x_0, x_1, x_2, x_3) = (x^0, -x^1, -x^2, -x3)$, $x_{\mu} = g_{\mu\upsilon}x^{\nu}$.; $g^{\mu\nu} = g^{\mu\rho}g_{\rho\sigma}g^{\sigma\nu}$, $g^{\mu}_{\sigma} = \delta^{\mu}_{\sigma}$; $xy = x^{\mu}g_{\mu\nu}y^{\nu} = x_{\mu}g^{\mu\nu}y_{\nu} = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3$. Specifically, $\eta^{\mu}(c_i)\eta_{\mu}(c_i) = \eta^{02}(c_i) - \overrightarrow{\eta}^2(c_i) = \gamma^2(c_i)(c_i^2 - \overrightarrow{v}^2)$, which according to (5) suggests that the energy of an ordinary particle, is expressible in the Lorentzian form. This, as we shall see is consistent with the form derivable directly from the bicubic equation (Šoln, J., 2016). As long as $c_i^2 - \overrightarrow{v}^2 \succ 0$, $\eta^{\mu}(c_i)$ is a time-like four-vector. A four-vector x^{μ} is respectively: $x^2 \succ 0$, is time-like, $x^2 = 0$, is light-like or null vector, $x^2 \prec 0$, is space-like.

At this point we give some general characteristics for limiting velocity solutions. For ordinary particles, $z \preceq 1$, the limiting velocities satisfy: $c_1^2 \succeq 0$, $c_2^2 \leq 0$ and $c_3^2 \succeq 0$. For novel particles, $z \succ 1$ with associated congruent angle $\alpha \preceq \frac{\pi}{2}$,the limiting velocities satisfy: $Rc_1^2 \succeq 0, Rc_2^2 \succeq 0$ and $c_3^2 \leq 0$. And the inertial masses with inertial scaling factor are respectively:

$\overline{m}(\ c_1^2, c_2^2, c_3^2) = \gamma(\ c_1^2, c_2^2, c_3^2)m$ and $\overline{m}(Rc_1^2, Rc_2^2, c_3^2) = \gamma(Rc_1^2, Rc_2^2, c_3^2)m$ and for both cases are derivable with he help of limiting velocity solutions.

As noted in (Šoln, J, 2021) by utilizing the Cardano's formula (Rade, L., Westegren, B., 1999) on page 56, one deduces that, with the help of the congruent parameter z, limiting velocities can be expressed globally in forms of Limiting Velocity Algebras:

$$\begin{aligned} c_1^2 + c_2^2 + c_3^2 &= 0, \\ Rc_1^2 + Rc_2^2 + c_3^2 &= 0. \end{aligned} \tag{5.1}$$

$$\begin{aligned} c_1^2c_2^2c_3^2 &= -\left(\frac{Ev}{m}\right)^2 = -\frac{27v^6}{4z^2}{}' \\ z^2 &= -\frac{27v^6}{4c_1^2c_2^2c_3^2}. \end{aligned} \tag{5.2}$$

$$\begin{aligned} c_1^2c_2^2 + c_1^2c_3^2 + c_2^2c_3^2 &= -\left(\frac{E}{m}\right)^2 = -\frac{27v^4}{4z^2}, \\ z^2 &= \frac{-27v^4}{4\left(c_1^2c_2^2 + c_1^2c_3^2 + c_2^2c_3^2\right)}. \end{aligned} \tag{5.3}$$

$$\begin{aligned} c_1^4 + c_2^4 + c_3^4 &= 2\left(\frac{E}{m}\right)^2 = \frac{27v^4}{2z^2}, \\ z^2 &= \frac{27v^4}{2\left(c_1^4 + c_2^4 + c_3^4\right)}. \end{aligned} \tag{5.4}$$

The ordinary particle limiting velocity c_3 and novel (which may be DM) particle limiting velocities Rc_1 and Rc_1, defined as $Rc_{1,2} \equiv +\sqrt{Rc_{1,2}^2}$, are the only ones to be deduced with, respectively, maximum particle velocities, say $v = c$ and $v = c^{\bullet}$, yielding the maximum values of unity 1 , respectively for, v/c_3 and $v/\ Rc_{1,2}$, as is done in the Numerology Section. The ordinary limiting velocity c_3 is known to be c from the neutrino maximum velocity of $v = c$, as argued in (Šoln, J., 2014, 2021). It is likely that $c^{\bullet}$ of Rc_1 and Rc_2 is also c. As to other limiting velocities, one can resort to Limiting Velocity Algebras (5.1, 2, 3, 4) to find their possible values and roles in the theory.

2. Ordinary Particle Inertial Masses From Their Limiting Velocities and Space-time Connection

As indicated in the introduction, the ordinary particle limiting velocitiy solutions of (1) with the discriminant $D \preceq 0$ and the congruent parameter $z \preceq 1$ yield three squares of real particle limiting velocity solutions (Šoln, J., 2014-2021):

$$\frac{c_1^2}{v^2} = \frac{3}{z}\sin\left[\frac{1}{3}\left(\pi - \sin^{-1} z\right)\right] \succ 0; \tag{6.1}$$

$$\frac{c_2^2}{v^2} = -\frac{3}{z}\sin\left[\frac{1}{3}(\pi + \sin^{-1} z)\right] \prec 0; \tag{6.2}$$

$$\frac{c_3^2}{v^2} = \frac{3}{z}\sin\left[\frac{1}{3}\sin^{-1}(z)\right] \succ 0. \tag{6.3}$$

The fundamental energy expression in (2) suggests to evaluate from (6.1, 2, 3) v^2/z for every c_1^2, c_2^2 and c_3^2 :

$$c_1^2 \quad : \quad \frac{v^2}{z} = \frac{c_1^2}{3\sin\left[\frac{1}{3}\left(\pi - \sin^{-1} z\right)\right]},\ \frac{v^2}{c_1^2} = \frac{z}{3\sin\left[\frac{1}{3}\left(\pi - \sin^{-1} z\right)\right]} \tag{7.1}$$

$$c_2^2 \quad : \quad \frac{v^2}{z} = \frac{-c_2^2}{3\sin\left[\frac{1}{3}(\pi + \sin^{-1} z)\right]},\ \frac{v^2}{-c_2^2} = \frac{z}{3\sin\left[\frac{1}{3}(\pi + \sin^{-1} z)\right]} \tag{7.2}$$

$$c_3^2 \quad : \quad \frac{v^2}{z} = \frac{c_3^2}{3\sin\left[\frac{1}{3}\sin^{-1}(z)\right]},\ \frac{v^2}{c_3^2} = \frac{z}{3\sin\left[\frac{1}{3}\sin^{-1}(z)\right]} \tag{7.3}$$

where we indicated in (7.1,2,3) the connection between every v^2/z with the inverted velocity solution (5.1, 2, 3). Now to get inertial masses with inertial scaling factors for ordinary particles, $z \le 1$, from limiting velocity solutions, we evaluate particle energy according to (2) with the help from (7.1,2,3) and at the same time with the energy expression in Lorentzian form from (5):

$$\begin{aligned} z \quad &\le \quad 1 : E(c_1) = \frac{m\sqrt{3}c_1^2}{2\sin\left[\frac{1}{3}(\pi - \sin^{-1} z)\right]} = m\gamma(c_1)c_1^2 = \overline{m}(c_1)c_1^2 \\ &= \quad \frac{mc_i^2}{(1 - v^2/c_1^2)^{\frac{1}{2}}} = m\gamma^{\div}(c_1)c_1^2 = \overline{m}^{\div}(c_1)c_1^2 \end{aligned} \tag{8.1}$$

$$\begin{aligned} z \quad &\le \quad 1 : E(c_2) = \frac{m\sqrt{3}\left(-c_2^2\right)}{2\sin\left[\frac{1}{3}(\sin^{-1}(z(m) + \frac{\pi}{3})\right]} = m\gamma(c_2)(-c_2^2) = \overline{m}(c_2)(-c_2^2) \\ &= \quad \frac{m(-c_2^2)}{(1 - v^2/c_2^2)^{\frac{1}{2}}} = m\gamma^{\div}(c_2)(-c_2^2) = \overline{m}^{\div}(c_{2)})(-c_2^2) \end{aligned} \tag{8.2}$$

$$\begin{aligned} z \quad &\le \quad 1 : E(c_3) = \frac{\sqrt{3}mc_3^2}{2\sin\left[\frac{1}{3}\sin^{-1} z\right]} = m\gamma(c_{3)})c_3^2 = \overline{m}(c_3)c_3^2 \\ . \quad &= \quad \frac{mc_3^2}{(1 - v^2/c_3^2)^{\frac{1}{2}}} = m\gamma^{\div}(c_3)c_3^2 = \overline{m}^{\div}(c_3)c_3^2 \end{aligned} \tag{8.3}$$

From relations (8.1, 2, 3) we can read deduce explicit expressions for ordinary particles, $z \preceq 1$, inertial masses and inertial scaling γ factors.

$$\begin{aligned} \gamma(c_1) \quad &= \quad \frac{\sqrt{3}}{2\sin\left[\frac{1}{3}(\pi - \sin^{-1} z)\right]}, \overline{m}(c_1) = \gamma(c_1)m, \\ \gamma^{\bullet}(c_1) \quad &= \quad \frac{1}{(1 - v^2/c_1^2)^{\frac{1}{2}}},\ \overline{m}^{\bullet}(c_1) = \gamma^{\bullet}(c_1)m \end{aligned} \tag{8.4}$$

$$\begin{aligned}\gamma(c_2) &= \frac{\sqrt{3}}{2\sin\left[\frac{1}{3}(\sin^{-1}(z(m)+\frac{\pi}{3})\right]}, \overline{m}(c_2)=\gamma(c_2)m,\\ \gamma^{\bullet}(c_2) &= \frac{1}{(1-v^2/c_2^2)^{\frac{1}{2}}}, \overline{m}^{\bullet}(c_2)=\gamma^{\bullet}(c_2)m\end{aligned} \tag{8.5}$$

$$\begin{aligned}\gamma(c_3) &= \frac{\sqrt{3}}{2\sin\left[\frac{1}{3}\sin^{-1}z\right]}, \overline{m}(c_3)=\gamma(c_3)m,\\ \gamma^{\div}(c_3) &= \frac{1}{(1-v^2/c_3^2)^{\frac{1}{2}}}, \overline{m}^{\div}(c_3)=\gamma^{\div}(c_3)m\end{aligned} \tag{8.6}$$

At this point, we indicate that energy relations from (8.1,2,3), since they depend on respective limiting velocity squared, positive or negative, also tells us how much energy are carried way by the particle involved with specific limiting velocity, c_1, c_2 andc_3. Some details of these questions will be given in the Numerology Section. As it was already mentioned, in (Šoln, J., 2016) it was shown rigorously that numerically there are complete equalities between inertial scaling factors, $\gamma(c_i)=\gamma^{\div}(c_i)$, as well as between inertial masses, $\overline{m}(c_i)=\overline{m}^{\div}(c_i)$, respectively in the limiting velocity solutions form or in the Lorentzian form. To this end, for ordinary particles, $z\preceq 1$, the kinetic energy involving a limiting velocity c_i is simply:

$$T(c_i)=mc_i^2(\gamma(c_i)-1)=mc_i^2(\gamma^{\div}(c_i)-1), i=1,2,3 \tag{8.7}$$

Formally, the space like proper velocity vector component plus its time component, for ordinary particle, $z\preceq 1$, involving the limiting velocity c_i, $i=1,2,3$, are in both forms

$$\overrightarrow{\eta}(c_i)=\gamma(c_i)\overrightarrow{v},\ \overrightarrow{\eta}^{\div}(c_i)=\gamma^{\div}(c_i)\overrightarrow{v}\ ;\eta^0(c_i)=\gamma(c_i)c_i, \eta^{\div 0}(c_i)=\gamma^{\div}(c_i)c_i \tag{9.1}$$

and, by utilizing 4 by 4 Minkkowski metric, as discussed in the Introduction, they are also in four-vector forms, together with their scalar products,

$$\eta^{\mu}(c_i) = \gamma(c_i)(c_i,\overrightarrow{v});\eta^{\div\mu}(c_i)=\gamma^{\div}(c_i)(c_i,\overrightarrow{v}) \tag{9.2}$$

$$\eta^{\mu}(c_i)\eta_{\mu}(c_i) = \gamma^2(c_i)(c_i^2-\overrightarrow{v}^2);\ \eta^{\div\mu}(c_i)\eta^{\div}_{\mu}(c_i)=\gamma^{\div 2}(c_i)(c_i^2-\overrightarrow{v}^2) \tag{9.3}$$

Regardless of limiting velocity form or Lorentzian form, the relevant expression have the same value (Šoln, J., 2016). For instance, $\eta(c_i)^2=\eta^{\div}(c_i)^2$, since $\gamma^2(c_i)=\gamma^{\div 2}(c_i)$, etc.

Next, we talk about space-like, time-like and four-momentum forms of ordinary particles, $z\preceq 1$, by implicitly acknowledging the equality of limiting velocity forms and Lorentzian forms of relevant expressions. Hence, we have in succession,

$$\overrightarrow{p}(c_i)=m\gamma(c_i)\overrightarrow{v}=\overline{m}(c_i)\overrightarrow{v}=m\gamma^{\div}(c_i)\overrightarrow{v}=\overline{m}^{\div}(c_i)\overrightarrow{v}=m\overrightarrow{\eta}(c_i), i=1,2,3 \tag{10.1}$$

$$p^0(c_i)=m\gamma(c_i)c_i=\overline{m}(c_i)c_i=m\gamma^{\div}(c_i)c_i=\overline{m}^{\div}(c_i)c_i=m\eta^0(c_i), i=1,2,3 \tag{10.2}$$

$$p^{\mu}(c_i) = m\eta^{\mu}(c_i)=m\gamma(c_i)(c_i,\overrightarrow{v})=m\gamma^{\div}(c_i)(c_i,\overrightarrow{v}), \tag{10.3}$$

$$p(c_i)^2 = p_{\mu}(c_i)p^{\mu}(c_i)=p^0(c_i)^2-\overrightarrow{p}(c_i)^2=m^2c_i^2, \tag{10.4}$$

$$E(c_i) = p^0(c_i)c_i=\overline{m}(c_i^2)c_i^2=\overline{m}^{\div}(c_i)c_i^2.i=1,2,3 \tag{10.5}$$

3. Novel (Some as DM) Particle Inertial Masses From Their Limiting Velocities and Space-time Connection

Following the particle limiting velocity bicubic equation (1), we arrive also at solutions for novel (some as DM) particles with the discriminant $D(m)\succeq 0$ and congruent parameter , $z\succeq 1$ with the associated congruent angle $\alpha\preceq\pi/2$. Here, unlike with the ordinary particles, $z\preceq 1$, we also have complex quadratic limiting velocity solutions. Because of the IRC connecting imaginary and real quadratic limiting velocity values, one can express the numerically equal energy with either a real or imaginary quadratic limiting velocity solution.

For $z \succeq 1$, the novel particle quadratic limiting velocity solutions are given with, $z \succeq 1$ parameter as,

$$\begin{aligned}
\frac{c_{1,2}^2}{v^2} &= \frac{3}{2z}\csc 2\tan^{-1}\left(\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1}{z}\right)\right)\right)^{\frac{1}{3}} \\
&\pm i\frac{3\sqrt{3}}{2z} ctn\, 2\tan^{-1}\left(\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1}{z}\right)\right)\right)^{\frac{1}{3}}, \\
\frac{c_3^2}{v^2} &= -\frac{3}{z}\csc 2\tan^{-1}\left(\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1}{z}\right)\right)\right)^{\frac{1}{3}}. \qquad (11)
\end{aligned}$$

To proceed with $z \gtrsim 1$ novel particle quadratic limiting velocity solutions from bicubic equation (11), it is convenient to express the congruent parameter $z \gtrsim 1$ in terms of the congruent angle $\alpha \preceq \pi/2$:

$$\begin{aligned}
\frac{1}{z} &= \sin\left[2\tan^{-1}\left(\tan\left(\frac{\alpha}{2}\right)\right)^3\right]. \\
\alpha &= 2\tan^{-1}\left(\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1}{z}\right)\right)\right)^{\frac{1}{3}}, \qquad (12.1) \\
z &= \frac{4-3\sin^2\alpha}{\sin^3\alpha},\ \ z^2 \succeq 1,\ 0 \prec \alpha \preceq \pi/2. \qquad (12.2)
\end{aligned}$$

where (12.1) relations are revers of each other, while single relation (12.2) is convenient to just relate z to α. With relation (12.1) applied to solutions (11) the squares of limiting velocities for novel particles, with $Rc_{1,2}^2 \equiv \mathrm{Re}\, c_{1,2}^2$ and $Ic_{1,2}^2 \equiv \mathrm{Im}\, c_{1,2}^2$, become,

$$\frac{c_{1,2}^2}{v^2} = \frac{Rc_{1,2}^2}{v^2} + i\frac{Ic_{1,2}^2}{v^2} \qquad (13)$$

$$\frac{Rc_{1,2}^2}{v^2} = \frac{3}{2z\sin\alpha}, \frac{Ic_{1,2}^2}{v^2} = \pm\frac{3\sqrt{3}}{2z} ctn\alpha,\ \frac{c_3^2}{v^2} = -\frac{3}{z\sin\alpha}, \qquad (14)$$

$$Rc_1^2 + Rc_2^2 + c_3^2 = 0,\ Ic_1^2 + Ic_2^2 = 0 \qquad (15)$$

From solution expressions (14) one deduces very important relation, connecting Imaginary and Real limiting velocities:

$$IRC : Ic_{1,2}^2 = \pm\sqrt{3}\cos\alpha\, Rc_{1,2}^2 \qquad (16)$$

If allowed by the IRC from (16), for instance, the energy can be expressed in convenient combination of $Ic_{1,2}^2$ and $Rc_{1,2}^2$.

In order to get inertial masses with inertial connecting γ factors for novel particles, $z \succeq 1$, we have to first calculate the energy according to (2) using the limiting velocity solutions (13) and(14). We start with $c_{1,2}^2$ by calculating v^2/z and putting it into $E = 3\sqrt{3}mv^2/2z$ and then with the help of the IRC from (16) arrive at:

$$\begin{aligned}
c_{1,2}^2 \quad : \quad & \frac{v^2}{z} = c_{1,2}^2\frac{2\sin\alpha}{3}\left(1 + i(\pm)\sqrt{3}\cos\alpha\right)^{-1} \\
&= \left(Rc_{1,2}^2 + iIc_{1,2}^2\right)\frac{2\sin\alpha}{3}\frac{\left(1 - i(\pm)\sqrt{3}\cos\alpha\right)}{\left(1+3\cos^2\alpha\right)} \\
&= \frac{2\sin\alpha}{3\left(1+3\cos^2\alpha\right)}\left[Rc_{1,2}^2 \pm Ic_{1,2}^2\sqrt{3}\cos\alpha + i\left(Ic_{1,2}^2 \mp Rc_{1,2}^2\sqrt{3}\cos\alpha\right)\right] \\
&= \frac{2\sin\alpha}{3\left(1+3\cos^2\alpha\right)}\left(Rc_{1,2}^2 \pm Ic_{1,2}^2\sqrt{3}\cos\alpha\right) \qquad (17.1)
\end{aligned}$$

Applying now the IRC from (16) to (17.1) we can eliminate $Ic_{1,2}^2$ from (17.1) to obtain for $Rc_{1,2}^2$:

$$Rc_{1,2}^2 : \ \frac{v^2}{z} = \frac{2}{3}\sin\alpha\, Rc_{1,2}^2 \tag{17.2}$$

Again using the IRC from (16) we can eliminate $Rc_{1,2}^2$ from (17.2) and obtain for $Ic_{1,2}^2$:

$$\begin{aligned} Ic_{1,2}^2 \quad &: \quad \frac{v^2}{z} = \frac{2}{3\sqrt{3}}\tan\alpha\,(\pm Ic_{1,2}^2). \\ \frac{v^2}{\pm Ic_{1,2}^2} \quad &= \quad \frac{2\,z\tan\alpha}{3\sqrt{3}} \end{aligned} \tag{17.3}$$

And for c_3^2 from (14) we have $c_3^2 : v^2/z = -\sin\alpha\, c_3^2/3$ which together with (17.2) gives:

$$\begin{aligned} \frac{v^2}{z} \quad &= \quad \frac{2}{3}\sin\alpha\, Rc_{1,2}^2 : \frac{v^2}{Rc_{1,2}^2} = \frac{2}{3}z\sin\alpha\,, \\ \frac{v^2}{z} \quad &= \quad -\frac{\sin\alpha\, c_3^2}{3} : \ \frac{v^2}{c_3^2} = -\frac{z\sin\alpha}{3} \end{aligned} \tag{17.4}$$

The main reason for deriving relations (17.1), (17.2) and (17.3) was to show the validity of the IRC from (16). Once having the IRC one can derive (17.2) and (17.3) and (17.4) directly from (14). One thing to notice from (17.1,2,3,4) and (17.1,2,3) that once one figures out from the particle energy, mass, etc., the particle velocity fractions depend only on the congruent parameter z and congruent angle α.

The general energy expression for the original quadratic limiting velocity solution is deduced from (17.1) as follows

$$z \succeq 1 : E(c_{1,2}) = \frac{\sqrt{3}m\sin\alpha}{(1+3\cos^2\alpha)}\left(Rc_{1,2}^2 + \sqrt{3}\cos\alpha\,(\pm Ic_{1,2}^2)\right) \tag{18}$$

As one sees, in (18), to measure this energy, one has to know both $Rc_{1,2}^2$ and $Ic_{1,2}^2$. However, this energy has the same value as the others that follow from (17.2), (17.3) and (17.4), which unlike (18) will be proportional to just single square of novel particle limiting velocity. Hence, with notations $Rc_{1,2} \equiv \sqrt{Rc_{1,2}^2}$, $Ic_{1,2} = \sqrt{Ic_{1,2}^2}$ and $c_3 = \sqrt{c_3^2}$ from (17.2), (17.3) and (17.4) we obtain the energy expressions as follows:

$$z \succeq 1 : E(Rc_{1,2}) = \sqrt{3}m\sin\alpha\, Rc_{1,2}^2 == m\gamma(Rc_{1,2})Rc_{1,2}^2 = \overline{m}(Rc_{1,2})Rc_{1,2}^2 \tag{18.1,2}$$

$$z \succeq 1 : E(c_3) = \frac{\sqrt{3}}{2}m\sin\alpha\,(-c_3^2) = m\gamma(c_3)(-c_3^2) = \overline{m}(c_3)(-c_3^2) \tag{18.3}$$

$$z \succeq 1 : E(Ic_{1,2}) = m\tan\alpha\,(\pm Ic_{1,2}^2) = m\gamma(Ic_{1,2})\,(\pm Ic_{1,2}^2) = \overline{m}(Ic_{1,2})\,(\pm Ic_{1,2}^2) \tag{18.4}$$

Comparing the energy expressions for ordinary particles, $z \preceq 1$, (18.1,2,3) with the energy expressions for novel particles (18) and (18.1,2,3,4) we notice that they cannot be all matched. The ones that can be matched belong to the squares of limiting velocities as indicated:

$$c_1^2(z \preceq 1) : Rc_1^2(z \succeq 1);\ c_2^2(z \preceq 1) : c_3^2(z \succeq 1);\ c_3^2(z \preceq 1) : Rc_2^2(z \succeq 1) \tag{19}$$

The energy of novel (some as DM) particle in (18) contains a combination of two limiting velocity squares and will be discussed last. The energy in (18.4) is due entirely to the imaginary limiting velocity square, however with the same numerical energy value as the others. However, one can easily see from relation (18), that in practical calculations with real quantities the congruent angle α will not depart very far from the value of $\pi/2$ keeping cos$\alpha \leqslant \sin\alpha$. At this point, also here we indicate that energy relations from (18.1,2,3,4), since they depend on respective limiting velocity squared, real or imaginary, also tells us how much energy is carried away by the particle involved with specific limiting velocity, Rc_1, $Rc_2, Ic_1, Ic_{,2}$ and c_3. These energy transfers will be discussed in more detail in the Numerology Section.

To continue, the inertial masses with inertial γ factors follow with fixed $z \succeq 1$:

$$\gamma(Rc_{1,2}) = \sqrt{3}\sin\alpha\;,\;\overline{m}(Rc_{1,2}) = \gamma(Rc_{1,2})m \tag{20.1,2}$$

$$\gamma(c_3) = \frac{\sqrt{3}}{2}\sin\alpha,\;\overline{m}(c_3) = \gamma(c_3)m \tag{20.3}$$

$$\gamma(Ic_{1,2}) = \tan\alpha\;,\overline{m}((Ic_{1,2}) = \gamma(Ic_{1,2})m \tag{20.4}$$

And as was done for the ordinary particles, here also for the novel particles, $z \succeq 1$, the kinetic energy of the particle, say, associated with the limiting velocities $Rc_{1,2}$ for instance, are:

$$T(Rc_{1,2}) = mRc^2_{1,2}(\gamma(Rc_{1,2}) - 1) \tag{20.5}$$

For novel particles, the proper velocity vectors and the time-like proper velocities are respectively, as follows:

$$\begin{aligned}\overrightarrow{\eta}(Rc_{1,2}) &= \gamma(Rc_{1,2})\overrightarrow{v},\;\overrightarrow{\eta}(c_3) = \gamma(c_3)\overrightarrow{v} &(21.1)\\ \eta^0(Rc_{1,2}) &= \gamma(Rc_{1,2})Rc_{1,2},\;\eta^0(c_3) = \gamma(c_3)c_3 &(21.2)\end{aligned}$$

Despite the fact that the novel particle energy (18.3) with imaginary limiting velocity c_3 is positive, never the less, the meaning of the time-like $\eta^0(c_3)$ proper velocity needs to be studied further. Likewise that applies to any other quantity that depends on imaginary c_3. From (21.1,2), as discussed in the Introduction by utilizing 4×4 Mikowski metric, for novel particles, $z \succeq 1$, we formulate the proper velocity four vector forms together with their scalar products:

$$\begin{aligned}\eta^\mu(Rc_{1,2}) &= \gamma(Rc_{1,2})\,(Rc_{1,2},\overrightarrow{v})\,,\\ \eta^2(Rc_{1,2}) &= \eta_\mu(Rc_{1,2})\eta^\mu(Rc_{1,2}) = \gamma^2(Rc_{1,2})\left(Rc^2_{1,2} - \overrightarrow{v}^2\right)\\ &= Rc^2_{1,2}\sin^2\alpha\;(3 - 2z\sin\alpha)\end{aligned} \tag{21.3}$$

$$\begin{aligned}\eta^\mu(c_3) &= \gamma(c_3)(c_3,\overrightarrow{v}),\\ \eta^2(c_3) &= \eta_\mu(c_3)\eta^\mu(c_3) = \gamma^2(c_3)(c_3^2 - \overrightarrow{v}^2) = c_3^2\frac{(3 + z\sin\alpha)}{4}\end{aligned} \tag{21.4}$$

Linear space-like, time-like and four momenta of novel (some as DM) particles are

$$\begin{aligned}\overrightarrow{p}(Rc_{1,2}) &= m\gamma(Rc_{1,2})\overrightarrow{v} = m\overrightarrow{\eta}(Rc_{1,2}),\\ p^0(Rc_{1,2}) &= m\gamma(Rc_{1,2})Rc_{1,2} = m\eta^0(Rc_{1,2}), p^\mu(Rc_{1,2}) = m\eta^\mu(Rc_{1,2}),\\ p^2(Rc_{1,2}) &= p_\mu\,(Rc_{1,2})\,p^\mu\,(Rc_{1,2}) = p^{02}(Rc_{1,2}) - \overrightarrow{p}^2(Rc_{1,2})\\ &= m^2R^2c_{1,2}\sin^2\alpha\,(3 - 2z\sin\alpha)\end{aligned} \tag{21.5}$$

$$\begin{aligned}\overrightarrow{p}(c_3) &= m\gamma(c_3)\overrightarrow{v} = m\overrightarrow{\eta}(c_3),\;p^0(c_3) = m\gamma(c_3)c_3 = m\eta^0(c_3),\\ p^\mu(c_3) &= m\eta^\mu(c_3),\;p^2(c_3) = p_\mu(c_3)p^\mu(c_3) = p^{02}(c_3) - \overrightarrow{p}^2(c_3)\\ &= m^2c_3^2\frac{(3 + z\sin\alpha)}{4}\end{aligned} \tag{21.6}$$

4. Numerology

Here, we wish to demonstrate on specific examples the validity of space-time connections, derived either analytically or phenomenologically. As far as the ordinary particles,$z \preceq 1$, analytical derivations are straightforward, while for the novel (some as DM) particles, $z \succeq 1$, the phenomenological methods are naturally accepted. However, for both the ordinary and novel particles the space-time connections to hold globally, is assured by having many physical quantities exhibiting the smoothness at the congruent parameter dividing point at $z = 1$ between the ordinary and novel particles. On specific level, this will be demonstrated with variety of physical quantities, such as, the limiting velocity solutions, particle energies, scaling actors, proper velocities, particle momenta, etc. Of particular interest in this endeavor, for both ordinary and novel particles, will be the ratios of particle velocities to respective limiting velocities; and those which with maxima of unity define the limiting velocities from respective maximum particle velocities.

The comparable evaluations of the same physical quantities for ordinary particles, $z \preceq 1$, and novel (some as DM) particles, $z \succeq 1$, at the congruent parameter dividing point of $z = 1$, are presented in pairs of tables, (Table 1.1; Table 1.2) and (Table 2.1; Table 2.2) where the first one contains ordinary particles and second the novel (some as DM particles).

Table 1.1, Ordinary particles , $z \preceq 1 \longrightarrow 1$: $\dfrac{c_1^2}{v^2} = \dfrac{3}{2}, \dfrac{c_2^2}{v^2} = -3, \dfrac{c_1^2}{v^2} = \dfrac{3}{2}; E(c_1) = \sqrt{3}mc_1^2 = \dfrac{3\sqrt{3}}{2}mv^2; E(c_2) = \dfrac{\sqrt{3}}{2}m(-c_2^2) = \dfrac{3\sqrt{3}}{2}mv^2; E(c_3) = \sqrt{3}mc_3^2 = \dfrac{3\sqrt{3}}{2}mv^2; \gamma(c_1) = \sqrt{3}, \gamma(c_2) = \dfrac{\sqrt{3}}{2}, \gamma(c_3) = \sqrt{3}$

Table 1.2, Novel particles, $z \succeq 1 \longrightarrow 1$:$\dfrac{Rc_1^2}{v^2} = \dfrac{3}{2}, \dfrac{c_3^2}{v^2} = -3, \dfrac{Rc_2^2}{v^2} = \dfrac{3}{2};$

$E(Rc_1) = \sqrt{3}mRc_1^2 = \dfrac{3\sqrt{3}}{2}mv^2, ; E(c_3) = \dfrac{\sqrt{3}}{2}m(-c_3^2) = \dfrac{3\sqrt{3}}{2}mv^2; E(Rc_2) = \sqrt{3}mRc_2^2 = \dfrac{3\sqrt{3}}{2}mv^2; \gamma(Rc_1) = \sqrt{3}, \gamma(c_3) = \dfrac{\sqrt{3}}{2}, \gamma(Rc_2) = \sqrt{3}.$

Table 2.1, Ordinary particles, $z \preceq 1 \longrightarrow 1 : \eta^\mu(c_1) = \sqrt{3}(c_1, \overrightarrow{v}); \eta^\mu(c_2) = \dfrac{\sqrt{3}}{2}(c_2, \overrightarrow{v}); \eta^\mu(c_3) = \sqrt{3}(c_3, \overrightarrow{v});$

$p^\mu(c_1) = m\sqrt{3}(c_1, \overrightarrow{v}); p^\mu(c_2) = m\dfrac{\sqrt{3}}{2}(c_2, \overrightarrow{v}); p^\mu(c_3) = m\sqrt{3}(c_3, \overrightarrow{v});$

Table 2.2, Novel particles, $z \succeq 1 \longrightarrow 1 : \eta^\mu(Rc_1) = \sqrt{3}(Rc_1, \overrightarrow{v}); \eta^\mu(c_3) = \dfrac{\sqrt{3}}{2}(c_3, \overrightarrow{v}); \eta^\mu(Rc_2) = \sqrt{3}(Rc_2, \overrightarrow{v});$

$p^\mu(Rc_1) = m\sqrt{3}(Rc_1, \overrightarrow{v}); p^\mu(c_3) = m\dfrac{\sqrt{3}}{2}(c_3, \overrightarrow{v}); p^\mu(Rc_2) = m\sqrt{3}(Rc_2, \overrightarrow{v}).$

Direct comparisons within these tables, demonstrates the smoothness of all physical quantities between the ordinary and novel (some as DM) particles. This indicates that they operate in the same Universe. In (Šoln, J., 2021), it was shown explicitly the consistency for ordinary particles bicubic equation limiting velocities descriptions with the Special Theory of Relativity. The smoothness demonstrations above, strongly suggests that also the novel particles bicubic equation limiting velocities descriptions are consistent with the Special Theory of Relativity,however phenomenologically, rather than analytically.

Next on the agenda is to see what are, if any, possible limits of applicability for both ordinary, $z \preceq 1$, and novel, $z \succeq 1$, particles, parametrized in terms of the congruent parameter z and congruent angle α. Of particular interest here is to compare the fractions of particle velocities with respect to their limiting velocities for ordinary particles, $z \preceq 1$, versus novel particles, $z \succeq 1$. From parametrizing values of $z \preceq 1$ for ordinary particles and $z \succeq 1$ for novel particles, we should be able to deduce which are usable segments of z for ordinary and novel particles. For the ordinary particles, $z \preceq 1$, to calculate the velocity fractions, $v^2/c_i^2, i = 1, 2, 3$, we use relations from (7), while for the novel particles, $z \succeq 1$, the velocity fractions are calculated from $v^2/Rc_i^2, i = 1, 2$ and v^2/c_3^2 from relations (17.2,3,4). Since they all involve $z \sin\alpha$ we combine relations (12.1) and (12.2) to express $z \sin\alpha$ just in terms of the congruent parameter z :

$$z \sin\alpha = z \sin\left(2\tan^{-1}(\tan(\frac{1}{2}\sin^{-1}\frac{1}{z}))^{\frac{1}{3}}\right) \tag{22.1}$$

In Tables 3.1, 3.2 and 3.3, for ordinary particles, $z \preceq 1$, wee list with decreasing z, respectively, $\frac{v^2}{c_1^2}$, $\frac{v^2}{c_2^2}$ and $\frac{v^2}{c_3^2}$. While in Tables 4.1,2 and 4.3, for novel (some as DM) particles, $z \succeq 1$, we list with increasing z, respectively, $\frac{v^2}{Rc_{1,2}^2}$ and $\frac{v^2}{c_3^2}$.

z	$\frac{v^2}{c_1^2}$	$\frac{v}{c_1}$
1	$\frac{2}{3}$	0.816
0.9	0.48	0.69
0.6	0.267	0.516
0.4	0.169	0.411
0.2	0.08	0.283
0.05	0.019	0.139
10^{-10}	0.385×10^{-4}	0.62×10^{-5}

:

Table 3.1, $z \leq 1$

z	$\frac{v^2}{c_2^2}$	$\frac{v}{c_2}$
1	$-\frac{1}{3}$	$i0.58$
0.9	-0.3	$i0.58$
0.6	-0.21	$i458$
0.4	-0.14	$i0.38$
0.2	-0.074	$i0.272$
0.05	-0.074	$i0.272$
10^{-10}	-0.385×10^{-4}	$i0.62 \times 10^{-5}$

Table 3.2, $z \leq 1$

z	$\frac{v^2}{c_3^2}$	$\frac{v}{c_3}$
1	$\frac{2}{3}$	0.82
0.9	0.823	0.91
0.6	0.94	0.97
0.4	0.975	0.987
0.2	0.994	0.997
0.05	0996	0.9998
10^{-10}	1	1

Table 3.3, $z \leq 1$

z	α	$\frac{v^2}{Rc_{1,2}^2}$	$\frac{v}{Rc_{1,2}}$
1	$\frac{\pi}{2}$	$\frac{2}{3}$	0.816
1.1	$\frac{\pi}{2.21}$	0.7254	0.852
1.2	$\frac{\pi}{2.3}$	0.783	0.885
1.3	$\frac{\pi}{2.38}$	0.84	0.916
1.4	$\frac{\pi}{2.44}$	0.9	$'0.95$
1.5	$\frac{\pi}{2.5}$	0.95	0.96
1.59	$\frac{\pi}{2.55}$	1	1

Table 4.1,2,$z \succeq 1$

z	α	$\frac{v^2}{c_3^2}$	$\frac{v}{c_3}$
1	$\frac{\pi}{2}$	$-\frac{1}{3}$	$i0.58$
1.1	$\frac{\pi}{2.21}$	-0.363	$i0.6$
1.2	$\frac{\pi}{2.3}$	-0.4	$i0.63$
1.3	$\frac{\pi}{2.38}$	-0.42	$i0.65$
1.4	$\frac{\pi}{2.44}$	-0.45	$i0.67$
1.5	$\frac{\pi}{2.5}$	-0.475	$i0.69$
1.59	$\frac{\pi}{2.55}$	-0.5	$i0.707$

Table 4.3,$z \succeq 1$

The simple analysis shows that for either ordinary or novel (some as DM) particle, there is just one value of velocity $v_{\max}$ which yields $v^2_{\max}/c^2_i = 1$, specifically for i=3 of the ordinary particle, which can be denoted as $v_{\max} = c$; And similarly $v_{\max}/Rc^2_i$ =1, for i=1,2 of the novel particle whose $v_{\max}$ can be denoted as $v_{\max} = c^{\bullet}$. To be specific, from Table 3.3, we read that at $z = 0, c_3$ =$v \max = c$, while from Table 4.1,2 we read that at at $z = 1.59$, $Rc_{1,2} = v \max = c^{\bullet}$. Hence, the limiting velocity for the normal particles we denote simply with $c_3 = c$, the velocity of light, as was already argued in (Šoln, J., (2014, 2021) consistent with (Adams T. at al. (2012), Stecker F. W. (2015)). The limiting velocity of the novel (some as DM) particle, we denote simply with $Rc_{1,2} = c^{\bullet}$ as it may be different from c, although being equal to c would be rather desirable. The novel particle fraction $\frac{v^2}{Ic^2_{1,2}}$ was not discussed separately as it can be calculated, with $IR\,C$connection (16): $Ic^2_{1,2} = \pm\sqrt{3}\cos\alpha\, Rc^2_{1,2}$, from $\frac{v^2}{Rc^2_{1,2}}$. Namely, from $IR\,C$connection,$\frac{v^2}{Ic^2_{1,2}} = \sqrt{3}\ \frac{2}{3}z\sin\alpha\, Rc^2_{1,2}$ which for $z = 1.59$ and $\alpha = \frac{\pi}{2.55}$, gives $\frac{v \max^2}{Ic^2_{1,2}} = \sqrt{3}\frac{v \max^2}{Rc^2_{1,2}} = \sqrt{3} \# 1$. This is not surprising as particles following imaginary square limiting velocity can have also only the positive energy.

Furthermore, these Tables are useful as they also show in detail how the limiting velocities of ordinary and novel particles change with particle velocity as z changes. In fact, consistent with Table 3.3 and 4.1,2, we deduce directly from (3.3) and (17.2), respectively, for ordinary and novel particles:

$$
\begin{aligned}
z &\le 1, \frac{c^2}{c_3^2} = 1 = \frac{z}{3\sin(\frac{1}{3}\sin^{-1} z)} : z = 0; \\
z &\succeq 1, \frac{c^{*2}}{Rc^2_{1,2}} = 1 = \frac{2}{3}z\sin\alpha = 1, z = \frac{4 - 3\sin^2\alpha}{\sin^3\alpha}, \alpha = \frac{\pi}{2.55}, z = 1.59 \qquad (22.2)
\end{aligned}
$$

Here, it should be noticed the importance of the congruent parameter z. Namely as relations (7.1,2,3) and (17.1,2,3,4) indicate that the ordinary and novel particle velocity fractions depend, respectively, basically on congruent parameters z, and z with α, consistent with particle energies and masses. In fact, at various $z's$ we can calculate limiting velocities now with $v_{\max} = c$ for ordinary particles and with $v_{\max} = c^{\bullet}$ for novel particles. In fact, although the ordinary particle which with $v_{\max} = c$ defined the limiting velocity $c_3 = v_{\max} = c$ may, when tied to c_1, cause $c_1$to achieve very large value. Here we present similar examples for both ordinary and novel particles:

$$z = 10^{-10} : \frac{v_{\max}}{c_3} = \frac{c}{c_3} = 1, \frac{v_{\max}}{c_1} = \frac{c}{c_1} = 0.62 \times 10^{-5}; c_1 = 1.6 \times 10^5 c. \qquad (23.1)$$

$$z = 10^{-10} : \frac{v_{\max}}{c_2} = \frac{c}{c_2} = i0.62 \times 10^{-5}; c_2 = i1.5 \times 0^5 c. \qquad (23.2)$$

$$z = 1.59, \alpha = \frac{\pi}{2.55} : \frac{v_{\max}}{Rc_1} = \frac{c^{\bullet}}{Rc_1} = 1, \frac{v_{\max}}{c_3} = \frac{c^{\bullet}}{c_3} = i0.77, c_3 = -i1.4c^*. \qquad (23.3)$$

$$z = 1.2, \alpha = \frac{\pi}{2.3} : \frac{v_{\max}}{Rc_1} = \frac{c^{\bullet}}{Rc_1} = 0.885, Rc_1 = 1.3c^{\bullet} \qquad (23.4)$$

In (23.1) and (23.2), consistent with ordinary particle limiting velocity solutions, (6.1) (6.2) and (6.3) we now take the particle with its $v_{\max} = c = c_3$ at $z = 10^{-10}$ to give the solutions also at $z = 10^{-10}$ for $c_1$and $c_2$which are rather different from c_3. Likewise, in (23.3), consistent with novel particle limiting velocity solutions in (17.4) we take particle with its $v_{\max} = c^{\bullet} = Rc_1$ at $z = 1.59, \alpha = \frac{\pi}{2.55}$ to give at the same z and α the value for imaginary c_3 limiting velocity. In (23.4) we notice that now Rc_1 changes to larger value with smaller $z = 1.2$ and larger $\alpha = \frac{\pi}{2.3}$.

The energies, for the ordinary, $z \le 1$, and the novel, $z \succeq 1$, particles can be calculated in terms limited velocities, respectively, from, $z \le, 1$, (8.1, 2,3) and,$z \succeq 1$, (18.1,2,3,4). Here, we wish to take advantage of, $z \le 1$, Tables: 3.1,3.2 and 3.3, plus,$z \succeq 1$,Tables: 4.1,2,3 and rewrite the nergies from $z \le, 1$, (8.1, 2,3) and,$z \succeq 1$, (18.1,2,3,4) as:

$$z \le ,1 : E(c_i, z) = \frac{3\sqrt{3}}{2z}\frac{v^2}{c_i^2} mc_i^2 = \gamma(c_i, z)\, mc_i^2,\ i = 1, 2, 3 \qquad (24.1)$$

$$
\begin{aligned}
z &\succeq 1 : E(Rc_i, z) = \frac{3\sqrt{3}}{2z}\frac{v^2}{Rc_i^2} mRc_i^2 = \frac{3\sqrt{3}}{2z}\frac{2}{3}z\sin\alpha\, mRc_i^2 \\
&= \sqrt{3}\sin\alpha\, mRc_i^2 = \gamma(Rc_i, z)\, mRc_i^2,\ , i = 1, 2, \qquad (24.2)
\end{aligned}
$$

$$z \succeq 1 : E(c_3, z) = \frac{3\sqrt{3}}{2z}\frac{v^2}{c_3^2} mc_3^2 = \frac{3\sqrt{3}}{2z}(-\frac{z\sin\alpha}{3}) mc_3^2. = \gamma(c_3, z) m(-c_3^2) \qquad (24.3)$$

In the Tables, regardless weather, the ordinary or novel (some as DM) particle limiting velocity is real or imaginary, the energy for the particle associated with it is always real. How to calculate the mass and deduct its velocity from an energy in (24) with the help from Tables Table: 3.1, 2,3 and Table: 4.1,2,3 is mostly by trial and error until we get particle mass and velocity that are sensible and acceptable.

As an γ scaling factor example,we take the energy of the muon neutrino $E(\nu) = 17GeV$ from the muon neutrino, υ, luminal velocity measurements with the OPERA detector (Adams, T. et al., 2012). With the help of Table: 3.3, by the trial and error, we wish to find the most plausible values of neutrino velocity and the mass. Let us try in Table: 3.3 with $v^2/c_3^2 = 0.823$ with $z = 0.9$. Now with (24.1) we derive $17GeV = 2.4m(\nu)c_3^2$, yielding, after dividing by the inertial scaling factor γ, $m(\nu)c_3^2 = 0.56GeV$,which is way too big a mass energy. So we go down the scale and stop at $v_{\max}^2/c_3^2 = v_{\max}^2/c^2 = 1$ at $z = 10^{-10}$ and with (24.1) write $17GeV = 2.6 \times 10^{10}m(\nu)c_3^2$ yielding, after dividing by the inertial scaling factor γ , $m(\nu)c^2 = 0.65eV$, which is acceptable mass energy.

For a simple comparison, we now turn to a novel, say, Dark Matter particle, $z \succeq 1$. DM particle has energy of $E(DM) = 1e^{\bullet}V$, where $e^{\bullet}V$ is the energy unit of the novel DM particle. The energy units of novel DM particle versus ordinary particle rescale the same way as the corresponding limiting velocities: $e^{\bullet}V/eV = c^{\bullet 2}/c^2$. And now, assuming that it has velocity $v(DM)\max = c^{\bullet}$ and as such from Table 4.1 it has $z = 1.59$ and $\alpha = \pi/2.55$. From (24.2), we have then $E(DM) = 1e^{\bullet}V = \sqrt{3}\sin(\pi/2.55)m(DM)Rc_1^2$, yielding, after dividing by the inertial scaling factor γ, the mass energy of $m(DM)Rc_1^2 = m(DM)c^{\bullet 2} = 0.61e^{\bullet}V$. This mass energy of $m(DM)c^{\bullet 2} = 0.61e^{\bullet}V$ being so close to $E(DM) = 1e^{\bullet}V$, of DM particle energy, makes this novel DM particle rather lethargic and a plausible example of a gravitational DM particle. As we see the novel particles are good candidates to describe the Dark Matter particles.

Namely, many new novel particles already appear as the Dark Matter particles in the energy range from keV to GeV and above. A typical example is a novel DM sterile neutrino complementing ordinary neutrino (Jaramilo, J., 2022). Experimentalists are also busy pursuing novel DM particles. For example, EDELWEISS is searching for just light novel DM particles (Lattaud, L., 2022). Also in pursuit of the Dark Photon, one is looking at $\gamma\gamma \to e^+e^-$ process (Xu, I.et al., 2022). It is fair to say that novel as DM particles are taken more and more seriously every day as there are quite a few meetings dedicated to them in search of new ideas (Battaglieri, M. et al., 2017).

While the roles of real limiting velocities are clear, the roles of imaginary limiting velocities with real energies should be possible to discuss additionally with the help of limiting velocity algebras from (5.1.2,3,4). Furthermore, regardless of weather, an ordinary or novel limiting velocity is real or imaginary, the energy for the particle associated with it is always real and positive. Systematically that can be easily verified with the help of Tables: 3.1,2,3 and 4.1,2,3.

5. Conclusion

We believe that these rather large collections of ordinary and novel particle limiting velocities together with a Dark Matter particle limiting velocity, real or imaginary, most of them with large absolute values some possibly exceeding c,the velocity of light, together with the energies of particles associated with these limiting velocities, should to some extent answer in part the question posed by Dawson and Percival (K. Dawson and W. Percival, 2021) as to why the Universe has such large distances.

Still, it would be definitively appropriate to pursue in the direction of Dark Matter exploration with the theory of novel as DM particles developed here.

In conclusion, we dare to say: ”The Universe has no beginning and no end, it transforms continuously into itself.”

Acknowledgement
We are grateful to Carlos Fuentes for his help with scientific program settings on the computer.

References

Adams, T. *et al.*. (2012). Measurement of the neutrino velocity with OPERA detector in the CNGS beam. *JHEP*, 093, ARXIV: 1212, 1276.

Barut, A. D. (1964). Electrodynamics and Classical Theory of Fields and Particles. *Dover*, 47.

Battaglieri, M. *et al.*. (2017). *New Ideas in Dark Matter. U. S. Cosmic Vision* (2017). arXiv: 1707. 04591 (hep-th).

Dawson, K., & Percival, W. (2021, May). A new map of the Universe. *Scientific American*, 34.

Griffiths, D. (1987). *Introduction to Elementary Particles.* John Wiley &Sons, 81.

Jaramillo, J. (2022). *Reviving keV sterile neutrino Dark Matter.* arXiv: 2207. 11269 (hep-ph).

Lattaud, L. (2022). *Sub GeV-Dark Matter searches with EDELWEISS: New results and prospects.*arXiv: 2211.04176 (astro-ph).

Rade, L., & Westergren, B. (1999). *Mathematical Handbook*. Springer, 65. https://doi.org/10.1007/978-3-662-03556-6

Sard, R. D. (1970). *Relativistic Mechanics.* W. A. Benjamin, 136.

Stecker, F. W. (2015). Limiting superluminal electron and neutrino velocity using 2010 Crab Nebula Flare and the Ice Cube PeV neutrino event. *Astroparticle Physics, 56,* 16-18, ARXIV: 1306, 6095. https://doi.org/10.1016/j.astropartphys.214.02.007

Šoln, J. (2014). Theoretical particle limiting velocity from the bicubic equation: Neutrino example. *Physics Essays, 27*(3), 448, ARXIV: 1403. 2683. https://doi.org/10.4006/0836-1398-27.3.448

Šoln, J. (2015). Particle limiting velocities from the bicubic equation derived from the Einstein's kinematics: PeV electron case. *Applied Physics Research, 7*(4), 37. https://doi.org/10.5539/apr.v7n4p37 Šoln, J. (2016). Limiting velocities of primary, obscure and normal particles: Self-annihilating obscure particle as an example of dark matter particle. *Applied Physics Research, 8*(5), 1. https://doi.org/10.5539/apr.v8n5p1

Šoln, J. (2017). Connecting dark matter particles with the primary, obscure and normal particles through implicit causality. *Applied Physics Research, 9*(3), 1. https://doi.org/10.5539/apr.v9n3p1

Šoln, J. (2018). Positive and negative particle masses in the bicubic equation limiting particle velocity formalism. *Applied Physics Research, 10*(1), 14. https://doi.org/10.5539/apr.v10n1p14

Šoln, J. (2018). Similarities and differences between positive and negative particle masses in the bicubic equation limiting particle velocity formalism: positive or negative muon neutrino mass?. *Applied Physics Research , 10*(5), 40. https://doi.org/10.5539/apr.v10n540

Šoln, J. (2019). Formation of particle real energy in the bicubic equation limiting particle velocity formalism with possible applications to light dark matter. *Applied Physics Research, 11*(2), 22. https://doi.org/10.5539/apr.v11n2p92

Šoln, J. (2020). Complex limiting velocity expressions as likely characteristics of dark matter particles. *Applied Physics Research, 12*(4), 107. https://doi.org/10.5539/apr.v12n4p107

Šoln, J. (2021). Real energy dark matter particles with mostly complex limiting velocities in either quadratic or linear forms. *European Journal of Applied Physics, 3*(1), 71. https://doi.org/10.24018.3.1.46

Šoln, J. (2021). Ordinary versus novel particles with the example of their spontaneous transition. *Applied Physics Research, 13*(3), 15. https://doi.org/10.5539/apr.v13n3p15

Xu, I. *et al.*. (2022). Search for Dark Photon in $\gamma\gamma \to e^+e^-$ at RHIC. arXiv:2211.0211.02132 (hep-ex).