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A Surprising Physical Quantity Involved in the Phase Velocity and Energy Levels of the Electron in a Hydrogen Atom

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Abstract

The phase velocity v_{phase} of a material wave is given by the following equation: $v_{\text{phase}} = \lambda\nu$. If this formula is rewritten using the Planck-Einstein relation $E = h\nu$ and the de Broglie formula $p = h/\lambda$, it becomes: $v_{\text{phase}} = E/p$. Next, the values for E and p, obtained from a relation applicable to the electron in a hydrogen atom derived by the author, are substituted into this equation. When that is done, multiple formulas relating to the phase velocity of the electron wave are derived. These formulas contain still-unknown ultra-low energy levels of the hydrogen atom and electron orbital radii r_n^- corresponding to those energy levels. r_n^- also appear in the formula for energy levels of the hydrogen atom derived in this paper. This serves as grounds for the existence of ultra-low energy levels in addition to the already-known energy levels of the hydrogen atom.

Keywords: energy-momentum relationship in a hydrogen atom, material wave, phase velocity, negative energy specific to the electron, ultra-low energy levels in a hydrogen atom

1. Introduction

According to Maxwell's electromagnetism, the following relationship holds between the momentum p and energy E of light.

$$E = cp. \tag{1}$$

Also, Einstein asserted, based on consideration of the photoelectric effect, that light has a particle nature, although it had previously been regarded as a wave.

If a photon as a single particle is assumed to have a frequency ν, Einstein concluded it has the following energy.

$$E = h\nu. \tag{2}$$

Here, h is the Planck constant. Also Formula (2) can be written as follows using the angular frequency ω.

$$E = \hbar\omega, \quad \hbar = \frac{h}{2\pi}. \tag{3}$$

ω is defined as follows.

$$\omega = 2\pi\nu. \tag{4}$$

The following equation can be derived from Formulas (1) and (2).

$$\lambda = \frac{c}{\nu} = \frac{h}{p}. \tag{5}$$

Also, the wavenumber κ is defined as follows.

$$\kappa = \frac{\lambda}{2\pi}. \tag{6}$$

Incidentally, the quantum condition of Bohr, which provided a good explanation of the stability of the hydrogen atom, contains an integer called a quantum number (Bohr, 1913).

In classical physics, integers appear in interference and normal modes. Thus, de Broglie thought that, if light —previously thought to be a wave —has a particle nature, then perhaps the electron—thought to be a particle—has a wave nature. Thus, he applied Formula (5) to matter.

In classical physics, the following relation holds between momentum p and kinetic energy K.

$$K = \frac{1}{2}mv^2 = \frac{p^2}{2m}. \tag{7}$$

Here, if Formulas (3), (5), and (6) are used,

$$\hbar\omega = \frac{p^2}{2m} = \frac{1}{2m}\frac{h^2}{\lambda^2} = \frac{1}{2m}\frac{4\pi^2\hbar^2}{\lambda^2} = \frac{\kappa^2\hbar^2}{2m}. \tag{8}$$

Therefore,

$$\omega = \frac{\hbar k^2}{2m}. \tag{9}$$

The phase velocity v_{phase} and group velocity v_{group} of a material wave are defined as follows (in the following, these may be abbreviated as v_{p}, v_{g}.)

$$v_{\text{phase}} = \frac{\omega}{k}, \quad v_{\text{group}} = \frac{d\omega}{dk}. \tag{10}$$

In light of the above, the phase velocity of the wave is as follows.

$$v_{\text{p}} = \frac{\omega}{k} = \frac{\hbar k}{2m} = \frac{p}{2m}$$

$$= \frac{v}{2}. \tag{11}$$

Also, the group velocity of the wave is as follows.

$$v_{\text{g}} = \frac{d\omega}{dk} = \frac{\hbar 2k}{2m} = \frac{p}{m}$$

$$= v. \tag{12}$$

de Broglie noticed that the velocity of a body is the group velocity of the material wave. He also concluded that the velocity, obtained from the product of the material wave’s wavelength and frequency, is the phase velocity.

Incidentally, the author has already derived a formula for the relativistic energy levels of a hydrogen atom. The only quantum number in this formula is the principal quantum number n. The energy levels derived by the author are quantitatively nowhere near the solutions of the Dirac equation. However, Dirac derived the relativistic wave equation by assuming that Einstein's energy-momentum relationship holds even within the hydrogen atom. On the other hand, the author has already pointed out that Einstein's relation does not hold in the hydrogen atom, where potential energy exists.

Thus far, in deriving the relativistic energy levels of the hydrogen atom, the author has not considered at all the conclusions reached by de Broglie.

Therefore, this paper examines whether new findings are obtained regarding the energy levels of the hydrogen atom when the assertions of de Broglie are taken into account.

In light of these points, Section 5 of this paper derives a formula for the phase velocity of the electron in a hydrogen atom. The formulas needed to achieve that objective are confirmed in the subsequent sections.

2. Formulas for Kinetic Energy and Momentum Derived From the Special Theory of Relativity

According to the special theory of relativity (STR), the following relation holds between the energy and momentum of a body moving in free space (Einstein, 1961).

$$\left(mc^2\right)^2 = \left(m_0c^2\right)^2 + c^2p^2. \tag{13}$$

Here, m_0 is the rest mass of the body, and m is the relativistic mass.

Now, Formula (13) is rewritten as follows.

$$\left(mc^2\right)^2 = m_0^2c^4 + \left(m^2c^4 - m_0^2c^4\right) = \left(m_0c^2\right)^2 + \left(m+m_0\right)\left(mc^2 - m_0c^2\right)c^2. \tag{14}$$

Comparing Formulas (13) and (14), the relativistic momentum p_{re} can be defined as follows.

$$p_{\mathrm{re}}^2 = \left(m+m_0\right)\left(mc^2 - m_0c^2\right). \tag{15}$$

Hence,

$$p_{\mathrm{re}} = c\left(m^2 - m_0^2\right)^{1/2}. \tag{16}$$

The “re” subscript of p_{re} stands for “relativistic.”

Incidentally, Einstein and Sommerfeld defined the relativistic kinetic energy K_{re} as follows (Sommerfeld, 1923).

$$K_{\mathrm{re}} = mc^2 - m_0c^2. \tag{17}$$

The following relation holds due to Formulas (15) and (17).

$$K_{\mathrm{re}} = \frac{p_{\mathrm{re}}^2}{m+m_0}. \tag{18}$$

The formula for kinetic energy of a body is given by Formula (7) in classical physics, but in the STR, this becomes Formula (18).

Incidentally, the following relationship holds between m_0 and m in the STR.

$$m = m_0\left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{19}$$

Now, if Formula (19) is substituted for m in Formula (16).

$$p_{\mathrm{re}} = c\left[m_0^2\left(1 - \frac{v^2}{c^2}\right)^{-1} - m_0^2\right]^{1/2}. \tag{20}$$

Simplifying this equation,

$$p_{\mathrm{re}} = mv. \tag{21}$$

Momentum is defined in classical physics as m_0v, but in the STR this becomes mv.

Physical quantities in classical physics and the STR are summarized in the following table.

Table 1. Physical quantities described in classical physics and the STR

	Classical Physics	STR
Mass	m_0	$m = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}$
Kinetic Energy	$K = \frac{p^2}{2m_0} = \frac{m_0^2 v^2}{2m_0}$	$K_{\mathrm{re}} = \frac{p_{\mathrm{re}}^2}{m_0 + m} = \frac{m^2 v^2}{m_0 + m}$
	$K = \frac{1}{2} m_0 v^2$	$K_{\mathrm{re}} = mc^2 - m_0 c^2$
Momentum	$p = m_0 v$	$p_{\mathrm{re}} = mv$
		$p_{\mathrm{re}} = c\left(m^2 - m_0^2\right)^{1/2}$

3. Energy Levels of the Electron in a Hydrogen Atom

Section 3 confirms the energy-momentum relationship applicable to the electron in a hydrogen atom.

This relation has already been derived by the author with three types of methods. One of these is presented here (Suto, 2011; Suto, 2018; Suto, 2020a; Suto, 2020b).

The relativistic kinetic energy of an electron in a hydrogen atom is defined as follows by referring to Formulas (17) and (18).

$$K_{\mathrm{re},n} = -E_{\mathrm{re},n} = m_{\mathrm{e}} c^2 - m_n c^2, \quad n = 1, 2, \cdots. \tag{22}$$

$$K_{\mathrm{re},n} = \frac{p_{\mathrm{re},n}^2}{m_{\mathrm{e}} + m_n}. \tag{23}$$

Here, $m_n c^2$ is the relativistic energy of the electron when the principal quantum number is in the state *n*.

Also, $p_{\mathrm{re},n}$ indicates the relativistic momentum of the electron. *n* is the principal quantum number.

This paper defines $E_{\mathrm{re},n}$ as the relativistic energy levels of the hydrogen atom. (The quantum number used here is just the principal quantum number. Therefore, $E_{\mathrm{re},n}$ is not a formula which predicts all the relativistic energy levels of the hydrogen atom.)

However, the term "relativistic" used here does not mean based on the STR. It means that the expression takes into account the fact that the mass of the electron varies due to velocity.

According to the STR, the electron's mass increases when its velocity increases. However, inside the hydrogen atom, the mass of the electron decreases when the velocity of the electron increases. Attention must be paid to the fact that, inside the hydrogen atom, the relativistic mass of the electron m_n is smaller than the rest mass m_{e}.

In this way, two formulas have been obtained for the relativistic kinetic energy of the electron in a hydrogen atom (Formulas (22), and (23)).

The following formula can be derived from Formulas (22) and (23).

$$\frac{p_{\mathrm{re},n}^2}{m_{\mathrm{e}} + m_n} = m_{\mathrm{e}} c^2 - m_n c^2. \tag{24}$$

Rearranging this, the following relationship can be derived.

$$\left(m_n c^2\right)^2 + c^2 p_{\mathrm{re},n}^2 = \left(m_{\mathrm{e}} c^2\right)^2. \tag{25}$$

Formula (25) is the energy-momentum relationship applicable to the electron in a hydrogen atom.

Now, in the past, Dirac derived the following negative solution from Formula (13).

$$E = \pm mc^2 = \pm m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{26}$$

If the same logic is applied to Formula (25), then the following formula can be derived.

$$E_n = \pm m_n c^2 = \pm m_e c^2 \left(1 + \frac{v_n^2}{c^2}\right)^{-1/2}. \tag{27}$$

However, Formula (27) does not incorporate the discontinuity peculiar to the micro world. Therefore, Formula (27) must be rewritten into a relationship where energy is discontinuous.

In order to incorporate discontinuity into Formula (27), the author has previously shown that the following relation can be used (Suto, 2021a). (Appendix)

$$\frac{v_n}{c} = \frac{\alpha}{n}. \tag{28}$$

Here, α is the following fine-structure constant.

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = 7.2973525693 \times 10^{-3}. \tag{29}$$

Using the relation in Formula (28), Formula (27) can be written as follows (Suto, 2014).

$$E_{\mathrm{ab},n}^{\pm} = \pm m_n c^2 = \pm m_e c^2 \left(1 + \frac{\alpha^2}{n^2}\right)^{-1/2}. \tag{30a}$$

$$= \pm m_e c^2 \left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2}. \tag{30b}$$

$E_{\mathrm{ab},n}$ gives the relativistic energy of the electron, but this is also the absolute energy of the electron. The "ab" subscript of $E_{\mathrm{ab},n}$ stands for "absolute."

In past papers, the author has written $E_{\mathrm{ab},n}$ as $E_{\mathrm{re},n}$. In this paper, the author would like to clear up this confusion. The following relation holds between $E_{\mathrm{ab},n}$ and $E_{\mathrm{re},n}$.

$$E_{\mathrm{ab},n} = m_e c^2 + E_{\mathrm{re},n}, \quad E_{\mathrm{re},n} < 0. \tag{31}$$

Here, $E_{\mathrm{ab},n} (= m_n c^2)$ is the residual part of the rest mass energy of the electron, and $E_{\mathrm{re},n}$ corresponds to the reduction in rest mass energy of the electron.

The positive solution of Formula (30), i.e., the relativistic energy levels of an ordinary hydrogen atom, can be expressed as follows. (Ordinarily, there is no problem in omitting the + of $E_{\mathrm{re},n}^{+}$. Note the difference between the relativistic energy levels of the hydrogen atom $E_{\mathrm{re},n}$ and the relativistic energy of the electron $E_{\mathrm{ab},n}$.) (Suto, 2019).

$$E_{\mathrm{re},n} = m_n c^2 - m_\mathrm{e} c^2 = m_\mathrm{e} c^2 \left[\left(1 + \frac{\alpha^2}{n^2} \right)^{-1/2} - 1 \right] \tag{32a}$$

$$= m_\mathrm{e} c^2 \left[\left(\frac{n^2}{n^2 + \alpha^2} \right)^{1/2} - 1 \right], \quad n = 1, 2, \cdots. \tag{32b}$$

To simplify the discussion in this paper, the only quantum number addressed is n.

Next, when the part of Formula (32a) in parentheses is expressed as a Taylor expansion,

$$E_{\mathrm{re},n} \approx m_\mathrm{e} c^2 \left[\left(1 - \frac{\alpha^2}{2n^2} + \frac{3\alpha^4}{8n^4} - \frac{5\alpha^6}{16n^6} \right) - 1 \right] \tag{33a}$$

$$\approx -\frac{\alpha^2 m_\mathrm{e} c^2}{2n^2}. \tag{33b}$$

Incidentally, Bohr derived the following formula as the energy levels of the hydrogen atom.

$$E_{\mathrm{BO},n} = -\frac{1}{2} \left(\frac{1}{4\pi\varepsilon_0} \right)^2 \frac{m_\mathrm{e} e^4}{\hbar^2} \cdot \frac{1}{n^2} = -\frac{\alpha^2 m_\mathrm{e} c^2}{2n^2}, \quad n = 1, 2, \cdots. \tag{34}$$

From this, it is evident that Formula (34) derived by Bohr is an approximation of Formula (32).

Incidentally, the fine-structure constant can be written as follows.

$$\alpha = \frac{e^2}{4\pi\varepsilon_0} \frac{2\pi}{h} \frac{1}{c} = \frac{e^2}{4\pi\varepsilon_0} \frac{2\pi}{m_\mathrm{e} c \lambda_\mathrm{C}} \frac{1}{c} = \frac{e^2}{4\pi\varepsilon_0 m_\mathrm{e} c^2} \frac{2\pi}{\lambda_\mathrm{C}} = \frac{2\pi r_\mathrm{e}}{\lambda_\mathrm{C}}. \tag{35}$$

Here, r_e is the following classical electron radius.

$$r_\mathrm{e} = \frac{e^2}{4\pi\varepsilon_0 m_\mathrm{e} c^2}. \tag{36}$$

Also, the following formula was used here (Suto, 2020c).

$$h = m_\mathrm{e} c \lambda_\mathrm{C}. \tag{37}$$

Here, λ_C is the Compton wavelength of the electron.

If α in Formula (35) is substituted into Formula (32a),

$$E_{\mathrm{re},n} = -K_{\mathrm{re},n} = m_\mathrm{e} c^2 \left[\left\{ 1 + \frac{1}{n^2} \left(\frac{2\pi r_\mathrm{e}}{\lambda_\mathrm{C}} \right)^2 \right\}^{-1/2} - 1 \right]. \tag{38}$$

The energy levels (34) of the hydrogen atom derived by Bohr include the Planck constant and fine-structure constant. However, $\hbar$ and α are not included in Formula (38). It is very significant that a formula can be derived for energy levels which does not contain the constants $\hbar$ and α which are important in quantum mechanics.

Incidentally, in Formula (34) for the energy levels of the hydrogen atom derived by Bohr, the energy of an electron at rest infinitely far from the proton was regarded as zero (Figure 1).

The rest mass energy of the electron is not taken into account in Bohr’s theory. Thus, the author derived a formula (32) for the energy levels of the hydrogen atom, taking into account the rest mass energy of the electron (Suto, 2021b). (Figure 2)

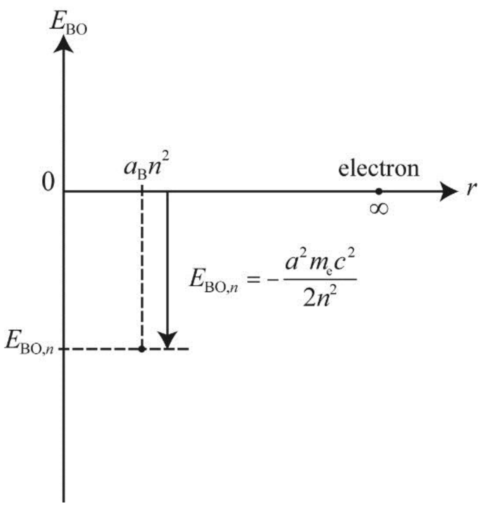

Figure 1

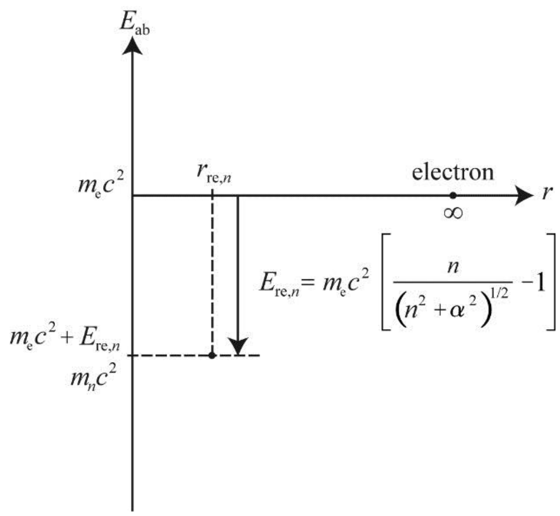

Figure 2

Figure 1. In Bohr's theory, the energy when the electron is at rest at a position infinitely distant from the proton (atomic nucleus) is defined to be zero.

Figure 2. According to the STR, the energy of an electron at rest at a position where $r=\infty$ is $m_e c^2$. $E_{re,n}$ is given by the difference between $m_e c^2$ and $m_n c^2$. That is, $m_n c^2 - E_{re,n} = m_e c^2$.

Here, the physical quantities of an electron in a hydrogen atom derived from classical physics and the STR are as indicated in the following table.

Table 2. Physical quantities of an electron in a hydrogen atom based on classical physics and the STR

	Classical Physics	This Paper
Electron Mass	m_e	$m_n = m_e\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}$
Kinetic Energy	$K = \frac{p^2}{2m_e}$	$K_{re,n} = \frac{p_{re,n}^2}{m_e + m_n} = \frac{m_n^2 v_n^2}{m_e + m_n}$
	$K = \frac{1}{2} m_e v^2$	$K_{re,n} = m_e c^2 - m_n c^2$
		$K_{re,n} = m_e c^2\left[1-\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}\right]$
Momentum	$p = m_e v$	$p_{re,n} = m_n v_n$
		$p_{re,n} = c\left(m_e^2 - m_n^2\right)^{1/2}$
		$p_{re,n} = m_e c\left(\frac{\alpha^2}{n^2+\alpha^2}\right)^{1/2}$

4. Orbital Radius of an Electron in a Hydrogen Atom

The total mechanical energy of the hydrogen atom is given by the following formula.

$$E_{\mathrm{re},n} = K_{\mathrm{re},n} + V\left(r_n\right) = -K_{\mathrm{re},n}. \tag{39}$$

Also, if the formula for potential energy is used, then $E_{\mathrm{re},n}$ can be written as follows (Suto, 2018b).

$$E_{\mathrm{re},n} = \frac{1}{2}V\left(r_n\right) = -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_n} = -\frac{1}{2}m_{\mathrm{e}}c^2\frac{r_{\mathrm{e}}}{r_n} = -m_{\mathrm{e}}c^2\left(\frac{r_{\mathrm{e}}/2}{r_n}\right). \tag{40}$$

From Formula (40), $m_n c^2$ is:

$$m_n c^2 = E_{\mathrm{ab},n} = m_{\mathrm{e}}c^2 + E_{\mathrm{re},n} = m_{\mathrm{e}}c^2 - m_{\mathrm{e}}c^2\left(\frac{r_{\mathrm{e}}/2}{r_n}\right) = m_{\mathrm{e}}c^2\left(\frac{r_n - r_{\mathrm{e}}/2}{r_n}\right). \tag{41}$$

Here, the following energy is obtained if $r_{\mathrm{e}}/2$ and $r_{\mathrm{e}}/4$ are substituted for r_n in Formula (41).

$$\text{When } r = \frac{r_{\mathrm{e}}}{2},\ E_{\mathrm{ab}} = 0, \text{ and when } r = \frac{r_{\mathrm{e}}}{4},\ E_{\mathrm{ab}} = -m_{\mathrm{e}}c^2. \tag{42}$$

Incidentally, the following equation holds due to Formulas (30b) and (41).

$$\frac{n^2}{n^2+\alpha^2} = \left(\frac{r_n - r_{\mathrm{e}}/2}{r_n}\right)^2. \tag{43}$$

From this, the following quadratic equation is obtained.

$$r_n^2 - \left(\frac{n^2+\alpha^2}{\alpha^2}\right)r_{\mathrm{e}}r_n + \left(\frac{n^2+\alpha^2}{\alpha^2}\right)\frac{r_{\mathrm{e}}^2}{4} = 0. \tag{44}$$

If this equation is solved for r_n ,

$$r_n^{\pm} = \frac{r_{\mathrm{e}}}{2}\left(1+\frac{n^2}{\alpha^2}\right)\left[1 \pm \left(1+\frac{\alpha^2}{n^2}\right)^{-1/2}\right]. \tag{45}$$

Next, if the electron orbital radii corresponding to the energy levels in Formula (30) are taken to be, respectively, r_n^+ and r_n^-,

$$r_n^+ = \frac{r_{\mathrm{e}}}{2}\frac{\left(n^2+\alpha^2\right)^{1/2}}{\left(n^2+\alpha^2\right)^{1/2} - n}. \tag{46}$$

$$r_n^- = \frac{r_{\mathrm{e}}}{2}\frac{\left(n^2+\alpha^2\right)^{1/2}}{\left(n^2+\alpha^2\right)^{1/2} + n}. \tag{47}$$

Also, Formulas (46) and (47) can be written as follows (Suto, 2017a).

$$r_n^+ = \frac{r_{\mathrm{e}}}{2}\left[1+\frac{n}{\left(n^2+\alpha^2\right)^{1/2} - n}\right]. \tag{48}$$

$$r_n^- = \frac{r_{\mathrm{e}}}{2}\left[1-\frac{n}{\left(n^2+\alpha^2\right)^{1/2} + n}\right]. \tag{49}$$

In Formula (49), the electron approaches toward $r_{\mathrm{e}}/4$ as *n* increases. Therefore,

$$\frac{r_e}{4} < r_n^- \leq \frac{r_e}{2}. \tag{50}$$

$$\frac{r_e}{2} \leq r_n^+. \tag{51}$$

In this paper, r_n^+ is called the orbital radius, as is customary. However, a picture of the motion of the electron cannot be drawn, even if that motion is discussed at the level of classical quantum theory. The electron in a hydrogen atom is not in orbital motion around the atomic nucleus. The domain of the ordinary hydrogen atom that we all know starts from $r = r_e / 2\ (E_{ab} = 0)$ (The equality sign holds in Formulas (50) and (51) when $n = 0$).

The next compares the orbital radii of an electron in a hydrogen atom r_n^+ and the orbital radii of an electron with a negative mass r_n^-.

The following ratio is obtained from Formulas (46) and (47).

$$\frac{r_n^-}{r_n^+} = \frac{\left(n^2 + \alpha^2\right)^{1/2} - n}{\left(n^2 + \alpha^2\right)^{1/2} + n}. \tag{52}$$

Here, if we set $n = 1$,

$$\frac{r_1^-}{r_1^+} = \frac{\left(1 + \alpha^2\right)^{1/2} - 1}{\left(1 + \alpha^2\right)^{1/2} + 1} = 1.3312484168 \times 10^{-5} \approx \frac{1}{75120}. \tag{53}$$

The author pointed out that an electron with negative mass forming dark hydrogen atom exists near the atomic nucleus (proton) (Suto, 2017b; Suto, 2021c).

5. Phase Velocity of the Electron Wave in a Hydrogen Atom

The important formulas derived in Sections 2 to 4 were all derived in the past by the author. In Section 5, the phase velocity of the electron wave in the hydrogen atom is derived from the standpoint of relativity theory based on the discussion in the previous sections.

First, the electron's phase velocity $v_{p,n}$ is given by the following formula.

$$v_{p,n} = \lambda_n \nu_n. \tag{54}$$

Here, $v_{p,n}$ is the phase velocity of the electron wave when the principal quantum number is in the n state. Also, λ_n and ν_n are the wavelength and frequency of the electron wave.

Formula (54) can be written as follows using the relationship of Formulas (2) and (5) (velocity and frequency are easily confused, so caution is necessary).

$$v_{p,n} = \lambda_n \nu_n = \frac{h}{p_n}\frac{K_n}{h} = \frac{K_n}{p_n}. \tag{55}$$

Due to the above, the formula for the relativistic kinetic energy of the electron corresponding to Formula (1) is as follows.

$$K_{re,n} = -E_{re,n} = v_{p,n} p_{re,n}. \tag{56}$$

$$K_{\mathrm{re},n} = -E_{\mathrm{re},n} = m_n v_{\mathrm{g},n} v_{\mathrm{p},n}. \tag{57}$$

The energy of a photon is found as the product of the photon's momentum and the speed of light. The kinetic energy of an electron, in contrast, is determined by the product of the electron's momentum and its phase velocity.

In Formula (57), the velocity of the electron as a wave and its velocity as a particle are both involved in the relativistic kinetic energy of the electron. A single formula incorporates the particle/wave duality.

Incidentally, there are multiple formulas for the kinetic energy and momentum of the electron, as is also evident from Table 2. Here, the phase velocity of the electron wave is derived with three methods by appropriately combining those formulas.

First,

$$v_{\mathrm{p},n} = \frac{K_{\mathrm{re},n}}{p_{\mathrm{re},n}} = \frac{m_n^2 v_{\mathrm{g},n}^2}{m_\mathrm{e} + m_n} \frac{1}{m_n v_{\mathrm{g},n}} = \frac{m_n}{m_\mathrm{e} + m_n} v_{\mathrm{g},n}. \tag{58}$$

The following relation is used here.

$$m_n = m_\mathrm{e} \left(1 + \frac{\alpha^2}{n^2}\right)^{-1/2}. \tag{59}$$

When that is done, Formula (58) can be written as follows.

$$v_{\mathrm{p},n} = \frac{m_n}{m_n \left[\left(1 + \frac{\alpha^2}{n^2}\right)^{1/2} + 1\right]} v_{\mathrm{g},n} = \frac{\left(1 + \frac{\alpha^2}{n^2}\right)^{1/2} - 1}{\left[\left(1 + \frac{\alpha^2}{n^2}\right)^{1/2} + 1\right]\left[\left(1 + \frac{\alpha^2}{n^2}\right)^{1/2} - 1\right]} v_{\mathrm{g},n} = \frac{n^2}{\alpha^2}\left[\left(1 + \frac{\alpha^2}{n^2}\right)^{1/2} - 1\right] v_{\mathrm{g},n}. \tag{60}$$

Next, the following equation obtained from Formula (28) is used.

$$v_{\mathrm{g},n} = \frac{\alpha c}{n}. \tag{61}$$

Then,

$$v_{\mathrm{p},n} = \frac{nc}{\alpha}\left[\left(1 + \frac{\alpha^2}{n^2}\right)^{1/2} - 1\right]. \tag{62}$$

Formula (62) can also be written as follows.

$$v_{\mathrm{p},n} = \frac{c}{\alpha}\left[\left(n^2 + \alpha^2\right)^{1/2} - n\right]. \tag{63}$$

In the second method, the phase velocity is defined as follows.

$$v_{\mathrm{p},n} = \frac{c^2 \left(m_\mathrm{e} - m_n\right)}{c\left(m_\mathrm{e}^2 - m_n^2\right)^{1/2}}. \tag{64}$$

Rearranging this equation,

$$v_{\mathrm{p},n} = \frac{c\left(m_\mathrm{e} - m_n\right)^{1/2}\left(m_\mathrm{e} - m_n\right)^{1/2}}{\left(m_\mathrm{e} - m_n\right)^{1/2}\left(m_\mathrm{e} + m_n\right)^{1/2}}$$

$$= c\left(\frac{m_\mathrm{e} - m_n}{m_\mathrm{e} + m_n}\right)^{1/2}. \tag{65}$$

Rearranging further,

$$v_{\mathrm{p},n}=c\left[m_{\mathrm{e}}-\frac{m_{\mathrm{e}}}{\left(1+\alpha^2/n^2\right)^{1/2}}\right]^{1/2}\left[m_{\mathrm{e}}+\frac{m_{\mathrm{e}}}{\left(1+\alpha^2/n^2\right)^{1/2}}\right]^{-1/2}$$

$$=c\left[\frac{\left(n^2+\alpha^2\right)^{1/2}-n}{\left(n^2+\alpha^2\right)^{1/2}}\right]^{1/2}\left[\frac{\left(n^2+\alpha^2\right)^{1/2}+n}{\left(n^2+\alpha^2\right)^{1/2}}\right]^{-1/2}$$

$$=c\left[\frac{\left(n^2+\alpha^2\right)^{1/2}-n}{\left(n^2+\alpha^2\right)^{1/2}+n}\right]^{1/2}. \quad (66)$$

Thus,

$$v_{\mathrm{p},n}=c\frac{\left(n^2+\alpha^2\right)^{1/2}-n}{\left[\left(n^2+\alpha^2\right)^{1/2}+n\right]^{1/2}\left[\left(n^2+\alpha^2\right)^{1/2}-n\right]^{1/2}}$$

$$=\frac{nc}{\alpha}\left[\left(1+\frac{\alpha^2}{n^2}\right)^{1/2}-1\right]. \quad (67)$$

In the third, phase velocity is defined as follows.

$$v_{\mathrm{p},n}=\frac{\left(m_{\mathrm{e}}-m_n\right)c^2}{m_n v_n}. \quad (68)$$

Rearranging, the following is obtained.

$$v_{\mathrm{p},n}=\frac{m_{\mathrm{e}}c^2\left[1-\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}\right]}{m_{\mathrm{e}}c\left(\frac{\alpha^2}{n^2+\alpha^2}\right)^{1/2}}=\frac{c\left[\left(n^2+\alpha^2\right)^{1/2}-n\right]\left(n^2+\alpha^2\right)^{1/2}}{\left(n^2+\alpha^2\right)^{1/2}\cdot\alpha}$$

$$=\frac{nc}{\alpha}\left[\left(1+\frac{\alpha^2}{n^2}\right)^{1/2}-1\right]. \quad (69)$$

Naturally, the phase velocities derived with the three methods all agree.
Also, the following formula can be derived from Formulas (52) and (66).

$$v_{\mathrm{p},n}=c\left(\frac{r_n^-}{r_n^+}\right)^{1/2}. \quad (70)$$

Taking the ratio of $v_{\mathrm{p},n}$ and $v_{\mathrm{g},n}$ from Formula (60), the result is as follows.

$$\frac{v_{\mathrm{p},n}}{v_{\mathrm{g},n}}=\frac{n^2}{\alpha^2}\left[\left(1+\frac{\alpha^2}{n^2}\right)^{1/2}-1\right]. \quad (71)$$

Developing the Taylor expansion of Formula (71),

$$\frac{v_{\mathrm{p},n}}{v_{\mathrm{g},n}}=\frac{n^2}{\alpha^2}\left[\left(1+\frac{\alpha^2}{n^2}\right)^{1/2}-1\right]=\frac{n^2}{\alpha^2}\left[\left(1+\frac{\alpha^2}{2n^2}-\cdots\right)-1\right]\approx\frac{1}{2}. \quad (72)$$

Thus,

$$v_{\mathrm{p},n} \approx \frac{v_{\mathrm{g},n}}{2}. \tag{73}$$

Also, in classical physics the mass of the electron is constant, so if Formulas (11) and (12) are taken into account,

$$E_{\mathrm{re},n} = -m_n v_{\mathrm{g},n} v_{\mathrm{p},n} \approx -\frac{1}{2} m_n v_{\mathrm{g},n}^2 \approx -\frac{1}{2} m_{\mathrm{e}} v_{\mathrm{g},n}^2. \tag{74}$$

Thus,

$$E_{\mathrm{re},n} \approx E_{\mathrm{BO},n}. \tag{75}$$

Incidentally, Formula (62) can be written as follows by using Formula (61).

$$v_{\mathrm{p},n} = \frac{c^2}{v_{\mathrm{g},n}} \left[\left(1 + \frac{\alpha^2}{n^2} \right)^{1/2} - 1 \right]. \tag{76}$$

Hence,

$$m_n v_{\mathrm{g},n} v_{\mathrm{p},n} = m_n c^2 \left[\left(1 + \frac{\alpha^2}{n^2} \right)^{1/2} - 1 \right]. \tag{77}$$

$$E_{\mathrm{re},n} = -m_n v_{\mathrm{g},n} v_{\mathrm{p},n} = m_n c^2 - m_{\mathrm{e}} c^2. \tag{78}$$

Next, let us consider the kinetic energy of the electron.

First, from Formulas (56) and (65),

$$K_{\mathrm{re},n} = -E_{\mathrm{re},n} = v_{\mathrm{p},n} p_{\mathrm{re},n} = c p_{\mathrm{re},n} \left(\frac{m_{\mathrm{e}} - m_n}{m_{\mathrm{e}} + m_n} \right)^{1/2}. \tag{79}$$

Next, from Formulas (56) and (66),

$$K_{\mathrm{re},n} = -E_{\mathrm{re},n} = c p_{\mathrm{re},n} \left[\frac{\left(n^2 + \alpha^2 \right)^{1/2} - n}{\left(n^2 + \alpha^2 \right)^{1/2} + n} \right]^{1/2}. \tag{80}$$

Also, from Formulas (56) and (70),

$$K_{\mathrm{re},n} = -E_{\mathrm{re},n} = c p_{\mathrm{re},n} \left(\frac{r_n^-}{r_n^+} \right)^{1/2}. \tag{81}$$

6. Discussion

This paper considers the meaning of the following equation obtained from Formulas (65) and (70).

$$\frac{r_n^-}{r_n^+} = \frac{m_{\mathrm{e}} c^2 - m_n c^2}{m_{\mathrm{e}} c^2 + m_n c^2}. \tag{82}$$

The numerator and denominator on the right side of Formula (82) express some kind of energy levels, and it is convenient if r_n^+ corresponds with the energy levels of $m_{\mathrm{e}} c^2 + m_n c^2$ (the same is also true for $m_{\mathrm{e}} c^2 - m_n c^2$).

Incidentally, despite the fact that energy is described with an absolute scale in Formula (30), there is a negative solution.

This problem can be solved by considering the situation in the following way. The electron has a latent negative energy of $-m_{e}c^{2}$ (Suto, 2020b).

In the state where all of the photon energy of the electron has been discharged, the electron energy is $-m_{e}c^{2}$ not zero.

That is, the mass specific to the electron is $-m_{e}$. However, this mass is the value obtained by assuming that the electron approaches from the center of the proton ($r=0$) to a distance of $r_{e}/4$.

In the state $E_{ab}=0$, the photon energy of the electron $m_{e}c^{2}$ and the specific negative energy $-m_{e}c^{2}$ cancel out. That is,

$$E_{ab}=-m_{e}c^{2}+m_{e}c^{2}=0. \tag{83}$$

Considered in this way, it is possible for an electron in state $E_{ab}=0$ to emit a photon and transition to negative energy levels.

Here, the true photon energy $E_{tab,n}^{\pm}$ of the electron is defined as follows.

$$E_{tab,n}^{+}=m_{e}c^{2}+E_{ab,n}=\left(m_{e}+m_{n}\right)c^{2}. \tag{84}$$

$$E_{tab,n}^{-}=\left(m_{e}-m_{n}\right)c^{2}. \tag{85}$$

The “tab” subscript of this energy indicates the true, absolute photon energy. The descriptor “tab” is applied because absolute energy $E_{ab,n}$ has already been defined.

$E_{tab,n}^{+}$ indicates the true photon energy of electrons at energy levels $E_{ab,n}^{+}$ ($m_{n}c^{2}$). The true photon energy is the energy obtained by adding, in addition to the photon energy thought to be ordinarily possessed by the electron, the photon energy $m_{e}c^{2}$ canceled out by the negative energy specific to the electron. Also, $E_{tab,n}^{-}$ is the true photon energy of an electron whose energy levels are at $E_{ab,n}^{-}$ ($-m_{n}c^{2}$).

In other words, the relation between $E_{tab,n}$ and $E_{ab,n}$ is as follows.

$$-m_{e}c^{2}+E_{tab,n}=-m_{e}c^{2}+m_{e}c^{2}+E_{ab,n}=E_{ab,n}. \tag{86}$$

$E_{tab,n}$ is the photon’s true energy of the electron. In contrast, $E_{ab,n}$ is the sum of the photon’s true energy of the electron and the negative energy specific to the electron.

Summarizing the above, formulas for the energy levels $E_{re,n}^{+}$, $E_{ab,n}^{+}$, and $E_{tab,n}^{+}$ of a hydrogen atom are as follows.

$$E_{re,n}^{+}=m_{n}c^{2}-m_{e}c^{2}=m_{e}c^{2}\left[\left(\frac{n^{2}}{n^{2}+\alpha^{2}}\right)^{1/2}-1\right]. \tag{87}$$

$$E_{ab,n}^{+}=m_{n}c^{2}=m_{e}c^{2}\left(\frac{n^{2}}{n^{2}+\alpha^{2}}\right)^{1/2}. \tag{88}$$

$$E_{\text{tab},n}^{+} = (m_{\text{e}} + m_n)c^2 = m_{\text{e}}c^2\left[1 + \left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}\right]. \tag{89}$$

Also, regarding ultra-low energy levels,

$$E_{\text{re},n}^{-} = -2m_{\text{e}}c^2 - E_{\text{re},n} = -m_{\text{e}}c^2 - m_n c^2 = -m_{\text{e}}c^2\left[1 + \left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}\right]. \tag{90}$$

$$E_{\text{ab},n}^{-} = -m_n c^2 = -m_{\text{e}}c^2\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}. \tag{91}$$

$$E_{\text{tab},n}^{-} = (m_{\text{e}} - m_n)c^2 = m_{\text{e}}c^2\left[1 - \left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}\right]. \tag{92}$$

Furthermore, the relation between positive and negative energy levels is as follows.

$$E_{\text{re},n}^{+} + E_{\text{re},n}^{-} = -2m_{\text{e}}c^2. \tag{93}$$

$$E_{\text{ab},n}^{+} + E_{\text{ab},n}^{-} = 0. \tag{94}$$

$$E_{\text{tab},n}^{+} + E_{\text{tab},n}^{-} = 2m_{\text{e}}c^2. \tag{95}$$

The following illustrates the relationship of the three types of energy levels (Figure 3).

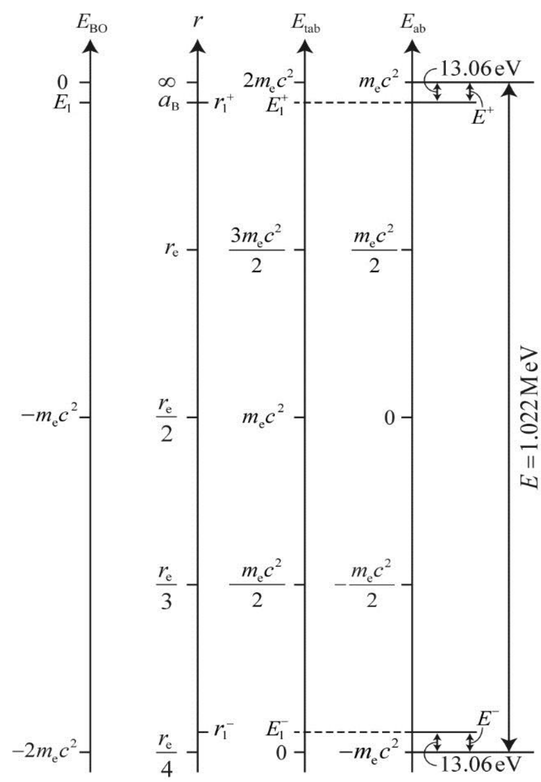

Figure 3. Relationship of three types of energy

7. Conclusion

A. Previously, the phase velocity of a material wave has been defined with Formulas (10) and (54). However, this paper derived the following formula as the wave phase velocity of the electron in a hydrogen atom.

$$v_{\mathrm{p},n} = \frac{nc}{\alpha}\left[\left(1+\frac{\alpha^2}{n^2}\right)^{1/2} - 1\right]. \tag{96}$$

A formula like the following was also derived as the formula for electron phase velocity.

$$v_{\mathrm{p},n} = c\left(\frac{m_\mathrm{e} - m_n}{m_\mathrm{e} + m_n}\right)^{1/2}. \tag{97}$$

$$v_{\mathrm{p},n} = c\left[\frac{\left(n^2+\alpha^2\right)^{1/2} - n}{\left(n^2+\alpha^2\right)^{1/2} + n}\right]^{1/2}. \tag{98}$$

$$v_{\mathrm{p},n} = c\left(\frac{r_n^-}{r_n^+}\right)^{1/2}. \tag{99}$$

Also, Formula (97) can be expressed as follows.

$$v_{\mathrm{p},n} = c\left(\frac{E_{\mathrm{tab},n}^-}{E_{\mathrm{tab},n}^+}\right)^{1/2}. \tag{100}$$

B. In this paper, the following formulas were derived as formulas for the energy levels of a hydrogen atom.

$$E_{\mathrm{re},n} = -m_n v_{\mathrm{g},n} v_{\mathrm{p},n} = m_n c^2 - m_\mathrm{e} c^2. \tag{101}$$

$$E_{\mathrm{re},n} = -v_{\mathrm{p},n}\, p_{\mathrm{re},n} = -c p_{\mathrm{re},n}\left(\frac{r_n^-}{r_n^+}\right)^{1/2}. \tag{102}$$

Next, the following relation holds due to Formula (101).

$$m_n c^2 + m_n v_{\mathrm{g},n} v_{\mathrm{p},n} = m_\mathrm{e} c^2. \tag{103}$$

Therefore,

$$m_n = \frac{m_\mathrm{e} c^2}{c^2 + v_{\mathrm{g},n} v_{\mathrm{p},n}} = \frac{m_\mathrm{e}}{1+\frac{v_{\mathrm{g},n} v_{\mathrm{p},n}}{c^2}} \tag{104}$$

Comparing here with Formula (59), it is evident that the following relation holds.

$$1+\frac{v_{\mathrm{g},n} v_{\mathrm{p},n}}{c^2} = \left(1+\frac{\alpha^2}{n^2}\right)^{1/2}. \tag{105}$$

In Formulas (100) and (102), $E_{\mathrm{rab},n}^-$ and r_n^- are incorporated in the formulas for the electron phase velocity, and the energy levels of the hydrogen atom. The existence of these states has been predicted mathematically, but has not yet been experimentally verified. However, due to the discussion in this paper, it has been confirmed that ultra-low energy levels of the hydrogen atom definitely exist.

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Appendix

Rewriting Formula (25) into a relation for momentum yields the following.

$$\left(m_n c\right)^2 + p_{\mathrm{re},n}^2 = \left(m_{\mathrm{e}} c\right)^2. \tag{A1}$$

Also, $p_{\mathrm{re},n}$ can be written as follows.

$$p_{\mathrm{re},n} = m_n v_n = m_{\mathrm{e}} \left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2} \frac{\alpha c}{n} = m_{\mathrm{e}} c \left(\frac{\alpha^2}{n^2+\alpha^2}\right)^{1/2}. \tag{A2}$$

Therefore, Formula (A1) can be written:

$$\left[m_{\mathrm{e}} c \left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}\right]^2 + \left[m_{\mathrm{e}} c \left(\frac{\alpha^2}{n^2+\alpha^2}\right)^{1/2}\right]^2 = \left(m_{\mathrm{e}} c\right)^2. \tag{A3}$$

Here, taking the ratio of the first term and the momentum of the second term on the left side of Formula (A1),

$$\frac{p_{\mathrm{re},n}}{m_n c} = \frac{v_n}{c}. \tag{A4}$$

Similarly, taking the ratio of the first term and the momentum of the second term on the left side of Formula (A3),

$$m_{\mathrm{e}} c \left(\frac{\alpha^2}{n^2+\alpha^2}\right)^{1/2} \cdot \frac{1}{m_{\mathrm{e}} c} \left(\frac{n^2}{n^2+\alpha^2}\right)^{-1/2} = \frac{\alpha}{n}. \tag{A5}$$

From Formulas (A4) and (A5),

$$\frac{v_n}{c} = \frac{\alpha}{n}. \tag{A6}$$

$$\frac{p_{\mathrm{re},n}}{m_n c} = \frac{m_n v_n}{m_n c} = \frac{\alpha}{n} = \frac{1}{n} \frac{2\pi r_{\mathrm{e}}}{\lambda_{\mathrm{C}}}. \tag{A7}$$

The author has previously presented Formula (A6) as a new quantum condition to replace Bohr's quantum condition. However, the reason why Formula (A6) holds is because Formula (A7) holds. Therefore, Formula (A7) is actually a quantum condition to replace the quantum condition of Bohr. Also, α is given by Formula (35), and thus the Planck constant, regarded as an essential constant for quantum mechanics, is actually a constant unnecessary for the quantum condition. Here, $m_n c$ is the momentum corresponding to the photon energy of the electron in a hydrogen atom.

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