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Kepler´s Ellipse Observed from Newton´s Evolute (1687), Horrebow´s Circle (1717), Hamilton´s Pedal Curve (1847), and Two Contrapedal Curves (28.10.2018)

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Abstract

Johannes Kepler discovered the very elegant elliptical path of planets with the Sun in one focus of that ellipse in 1605. Kepler inspired generations of researchers to study properties hidden in those elliptical paths. The visible elliptical paths belong to the Aristotelian World. On the other side there are invisible mathematical objects in the Plato´s Realm that might describe the mechanism behind those elliptical paths. One such curve belonging to the Plato´s Realm discovered Isaac Newton in 1687 - the locus of radii of curvature of that ellipse (the evolute of the ellipse). Are there more curves in the Plato´s Realm that could reveal to us additional information about Kepler´s ellipse? W.R. Hamilton in 1847 discovered the hodograph of the Kepler´s ellipse using the pedal curve with pedal points in both foci (the auxiliary circle of that ellipse). This hodograph depicts the moment of the tangent momentum of orbiting planets. Inspired by the hodograph model we propose newly to use two contrapedal curves of the Kepler´s ellipse with contrapedal points in both the Kepler´s occupied and Ptolemy´s empty foci. Observers travelling along those contrapedal curves might bring new valuable experimental data about the orbital angular velocity of planets and a new version of the Kepler´s area law. Based on these contrapedal curves we have defined the moment of the normal momentum. The first derivation of the moment of the normal momentum reveals the torque of the ellipse. This torque of ellipse should contribute to the precession of the Kepler´s ellipse. In the Library of forgotten works of Old Masters we have re-discovered the Horrebow´s circle (1717) and the Colwell´s anomaly H (1993) that might serve as an intermediate step in the solving of the Kepler´s Equation (KE). Have we found the Arriadne´s Thread leading out of the Labyrinth or are we still lost in the Labyrinth?

Keywords: Kepler´s ellipse, Aristotelian World, Plato´s Realm, hidden mathematical objects, Newton´s evolute, Hamilton´s pedal curve, two contrapedal curves, torque of ellipse, Horrebow´s circle, Colwell´s anomaly, Kepler´s Equation.

1. Introduction

The famous quote of Heraclitus “Nature loves to hide” was described in details by Pierre Hadot in 2008. Hadot in his valuable book gives us many examples how Nature protects Her Secrets. In several situations the enormous research of many generations is strongly needed before the right “recipe” unlocking the true reality can be found.

The Old Masters of the Hellenistic and Alexandrian astronomy used the combinations of perfect circular motions for the description of the planet orbits. Johannes Kepler in 1605 made his great breakthrough when he discovered the elliptical path of Mars with the Sun in one focus of that ellipse. Generations of researchers were inspired by this Kepler´s ellipse and were searching for properties hidden in those elliptical paths. The great step made Isaac Newton in 1687 when he discovered the locus of radii of curvature of that ellipse (the evolute of that ellipse) and applied it for the calculation of the centripetal force. W.R. Hamilton in 1847 discovered a very elegant model of the hodograph using the pedal curve with pedal points located in both foci (the auxiliary circle). However, this classical model of the Kepler´s ellipse could not properly explain the precession of the planets and Albert Einstein in 1915 replaced this classical model with his concept of the elastic spacetime.

A possible chance for further classical development of the Kepler´s ellipse is to penetrate more deeply into the secrets of the Kepler´s ellipse and to reappear with some new hidden properties overlooked by earlier generations of researchers. Our guiding principle is the existence of the Plato´s Realm with invisible mathematical objects that might bring to us some additional information about the visible Kepler´s ellipse in the Aristotelian World. In this contribution we have been working with these mathematical objects from the Plato´s Realm:

1) Ellipse properties discovered by Appolonius of Perga - the Great Geometer and many his scholars.
2) Locus of the radii of curvature (evolute) - Isaac Newton in 1687.
3) Horrebow´s circle and Colwell´s anomaly - Peder Horrebow in 1717 and Peter Colwell in 1993.
4) Pedal curve with the pedal points in both foci - the auxiliary circle - W.R. Hamilton in 1847 and his hodograph.
5) Contrapedal curve with the contrapedal point in the Kepler´s occupied focus (2018).
6) Contrapedal curve with the contrapedal point in the Ptolemy´s empty focus (2018).
7) Torque of the Kepler´s ellipse (2018).

There are several new concepts in this contribution - two travelers moving along those contrapedal curves and observing planets on their elliptical paths, the introduction of the moment of the normal momentum as an additional quantity for the calculation of the planet precession, the torque of the Kepler´s ellipse. Horrebow´s circle reveals to us the Colwell´s anomaly H as the intermediate step for the solution of the Kepler´s Equation (KE) when we have given the mean anomaly M and should find the eccentric anomaly E.

The experimental analysis of properties of these two contrapedal curves and the Horrebow´s circle should evaluate if we have found the Arriadne´s Thread leading out of the Labyrinth or we are still lost in the Labyrinth.

(We are aware of the famous quote of Richard Feynman from the year 1962: “There´s certain irrationality to any work in gravitation, so it is hard to explain why you do any of it.”)

2. Some trigonometric properties of the ellipse

The discovery of ellipse, parabola, and hyperbola is attributed to Menaechmus. Apollonius of Perga - the Great Geometer - was the top Ancient Greek mathematician specialized on the conic sections. Pappus of Alexandria and Anthemius of Tralles were the last of great Ancient Greek mathematicians that contributed to this topic. After one thousand years this “geometric treasure” passed into the hands of Johannes Kepler, Isaac Newton, Peder Horrebow, W.R. Hamilton, Richard Feynman and many others.

Figure 1 and Figure 2 shows some trigonometrical properties of the ellipse that might be used for the description of motion of planets around the Sun.

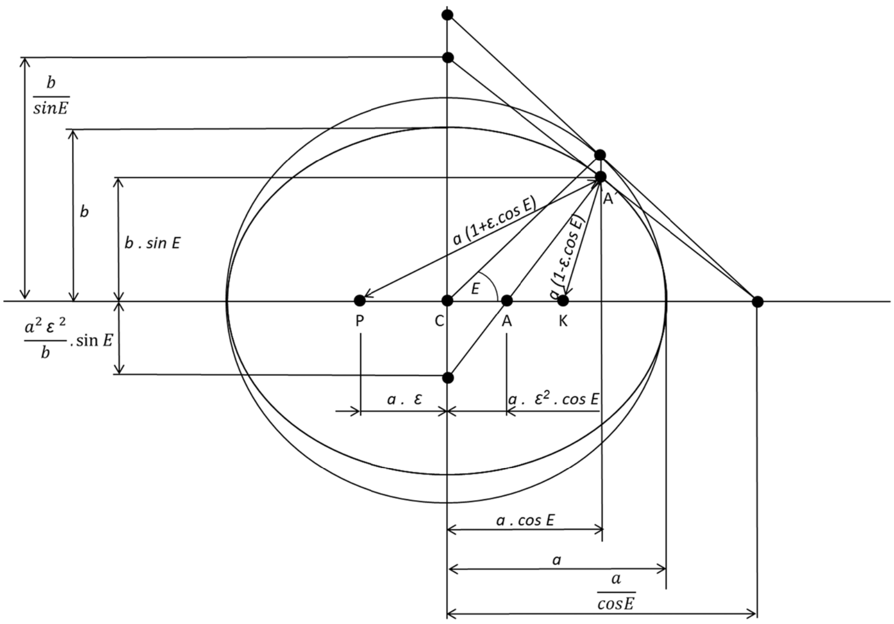

Figure 1. Some trigonometric relations derived for the eccentric anomaly E - we have corrected mistakes in Figure 1 in Stávek (2018)

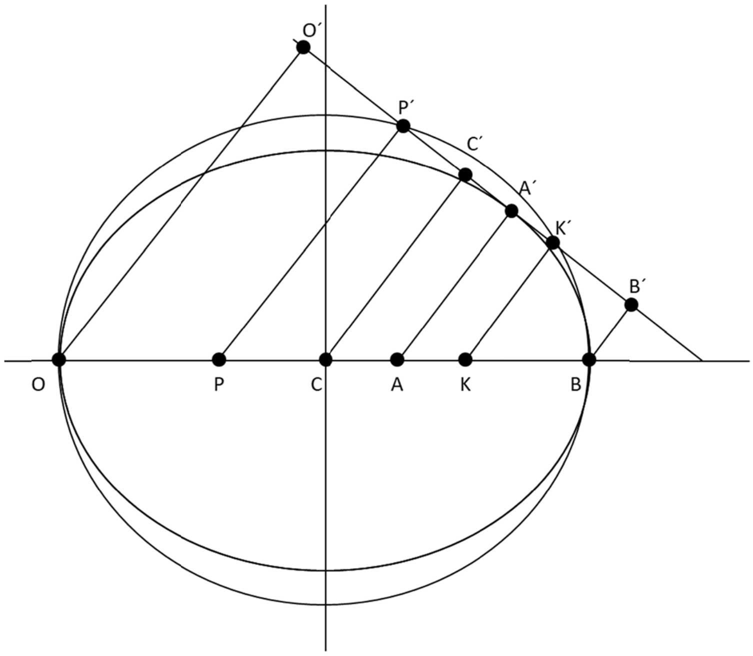

Figure 2. Distances from Osiander´s point, Ptolemy´s empty focus, Copernicus´ center of the auxiliary circle, Archimedean point, Kepler´s occupied focus, and Brahe´s point to the tangent

Table I summarizes some relations derived for the eccentric anomaly because of the complex behavior of the ellipse.

Table I. Some trigonometrical properties of the ellipse

Some trigonometrical properties of the ellipse
a … semi-major axis
b … semi-minor axis
E … eccentric anomaly
O … Osiander´s point
P … Ptolemy´s empty focus
C … Copernicus´ center of the auxiliary circle
A … Archimedean point
K … Kepler´s occupied focus
B … Brahe´s point
A´ … planet
$PA' = a\left(1+\varepsilon\cos E\right)$
$KA' = a\left(1-\varepsilon\cos E\right)$
$\frac{PA}{KA} = \frac{PA'}{KA'} = \frac{\left(1+\varepsilon\cos E\right)}{\left(1-\varepsilon\cos E\right)}$
$CA' = a\sqrt{1-\varepsilon^2\sin^2 E}$
$OO' = b\frac{1+\cos E}{\sqrt{1-\varepsilon^2\cos^2 E}}$

$$PP' = b\frac{\sqrt{1+\varepsilon\cos E}}{\sqrt{1-\varepsilon\cos E}}$$

$$CC' = b\frac{1}{\sqrt{1-\varepsilon^2\cos^2 E}}$$

$$AA' = b\sqrt{1-\varepsilon^2\cos^2 E}$$

$$KK' = b\frac{\sqrt{1-\varepsilon\cos E}}{\sqrt{1+\varepsilon\cos E}}$$

$$BB' = b\frac{1-\cos E}{\sqrt{1-\varepsilon^2\cos^2 E}}$$

3. Kepler´s Ellipse Observed from the Newton´s Evolute

Newton discovered several important properties hidden in the Kepler´s ellipse in his Principia in 1687. For the centripetal force F, he derived formula:

$$F = m\frac{v_T^2}{\rho} \tag{1}$$

where m is the mass of the planet, v_T is the tangent velocity of the planet and ρ is the radius of curvature of that ellipse. The locus of radii of curvature is termed as the evolute. This equation opened completely new possibilities in the understanding of the Kepler´s ellipse.

In the standard procedure both quantities v_T and ρ are found by the derivation method invented independently by Newton and Leibniz.

We will present here the trigonometric approach to these two quantities (v_T in the next chapter). The radius of curvature of the ellipse ρ can be derived in the trigonometric way shown in Figure 3. Figure 3 describes an interplay between the normal to the tangent and the line connecting the Sun and orbiting planet.

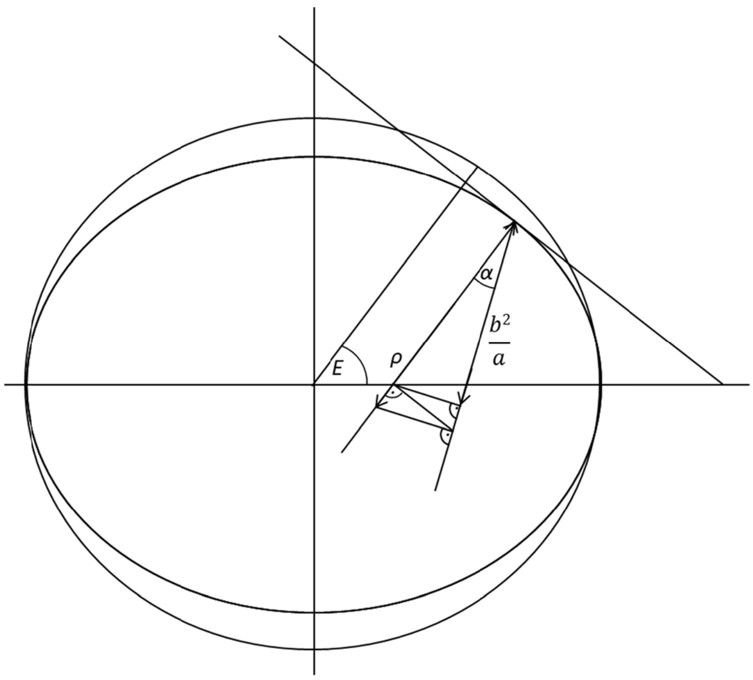

Figure 3. Trigonometric approach to reveal the expression for the radius of curvature ρ

We have used the deep knowledge of ellipse properties of Issac Todhunter (1881) and Anthony Rynne (2006) and extracted from Figure 3 this "remarkable" expression for the radius of curvature:

$$\rho = \frac{b^2}{a} \times \frac{1}{(\cos\alpha)^3} = \frac{a^2}{b} \times \left(1 - \varepsilon^2 \cos^2 E\right)^{3/2} \tag{2}$$

(Several hours after this trigonometric discovery on 17 Oct 2018 we have found in the Arxiv the paper of W.Y. Hsiang and E. Straume (2014) who derived the first part of this "remarkable" formula using the derivation technique on 28 Aug 2014).

(The quantities expressed in the trigonometric language are simpler and Nature can talk with us in this trigonometric language that could be depicted in simple Figures without words).

4. Kepler´s Ellipse Observed from the Hamilton´s Pedal Curve with Pedal Points in both Foci

The pedal curve of the Kepler´s ellipse is the locus of the feet of the perpendiculars from both foci to the tangent of that ellipse. In this case the pedal curve is the famous auxiliary circle of the ellipse.

W.R. Hamilton in 1847 discovered this concept that is known as the hodograph. This approach was several times forgotten and its beauty was several times rediscovered by many researchers. E.g., Richard Feynman in his "Lost lecture" made this concept very well known for our generation.

W.R. Hamilton elegantly interpreted the quantity of the tangent planet velocity as the perpendicular from the Ptolemy´s empty focus to the tangent. Figure 4 depicts both perpendiculars from the Ptolemy´s empty focus and the Kepler´s occupied focus. The vector of PP´ is rotated anticlockwise by the angle $\pi/2$.

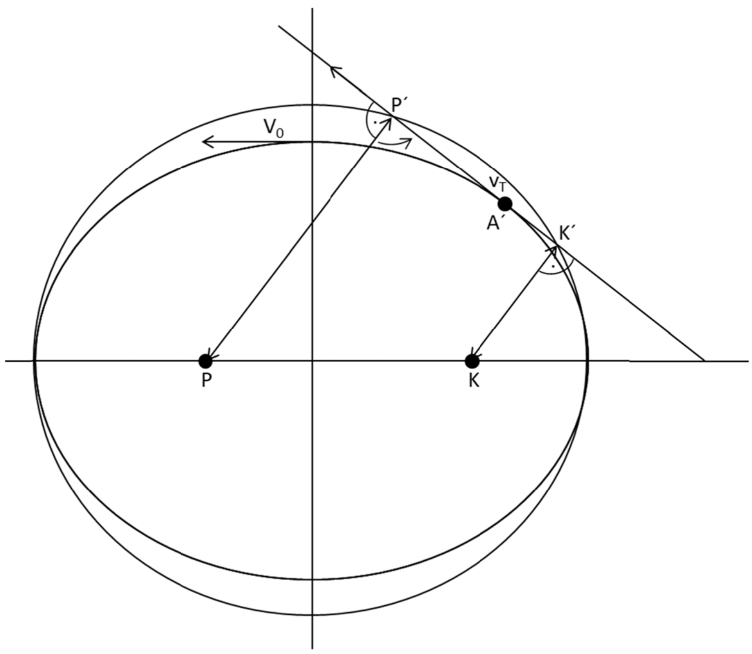

Figure 4. Trigonometric approach to reveal the expression for the tangent velocity v_T

We get from Figure 4 the tangent velocity v_T as:

$$v_T = v_0 \frac{\sqrt{1 + \varepsilon \cos E}}{\sqrt{1 - \varepsilon \cos E}} \tag{3}$$

where v_0 is the tangent orbital velocity at the end of the minor axes.

Now, we can test the validity of the Newton´s formula expressed in the trigonometric language and compare these trigonometric formulae with formulae obtained in other mathematical languages. A very good inspiration can be found in the valuable book of Arjun Tan (2008).

The famous Newton´s formula can be trigonometrically expressed as:

$$F = \frac{GMm}{R^2} = m\,\frac{v_T^2}{\rho}\frac{1}{\cos\alpha} =$$
$$= m\,\frac{v_0^2\left(\frac{1+\varepsilon\cos E}{1-\varepsilon\cos E}\right)}{\frac{a^2}{b}\left(1-\varepsilon^2\cos^2 E\right)^{3/2}}\,\frac{a\sqrt{1-\varepsilon^2\cos^2 E}}{b} = \frac{m\,v_0^2 a}{R^2}$$

At the end we have obtained the standard gravitational parameter μ:

$$\mu = GM = a\,v_0^2$$

where a is the semi-major axis and v_0 is the orbital velocity at the end of minor axis.

5. Kepler´s Ellipse Observed from Two Contrapedal Curves with Contrapedal Points in both Foci

The contrapedal curve of the Kepler´s ellipse is the locus of the feet of the perpendiculars from both foci to the normal of that ellipse. In this case we will get two contrapedal curves. (Until now we could not find in the literature more data about these two curves. We will leave this topic to Readers of this Journal more experienced in the mathematics).

In this case we will employ the analogy with Hamilton´s hodograph - can we extract a valuable information about the Kepler´s ellipse and to find an expression for the normal velocity of the planet v_N?

Figure 5 shows a schema of two contrapedal points J and S. We propose to use the distance PJ between the Ptolemy´s empty focus P and the contrapedal point J as the measure for the normal velocity v_N:

$$v_N = v_0\varepsilon\,\sin E\,\frac{\sqrt{1+\varepsilon\cos E}}{\sqrt{1-\varepsilon\cos E}} \tag{6}$$

where v_0 is the tangent orbital velocity at the end of the minor axes. This is a new hodograph for the normal velocities of planets orbiting in the Kepler´s ellipses.

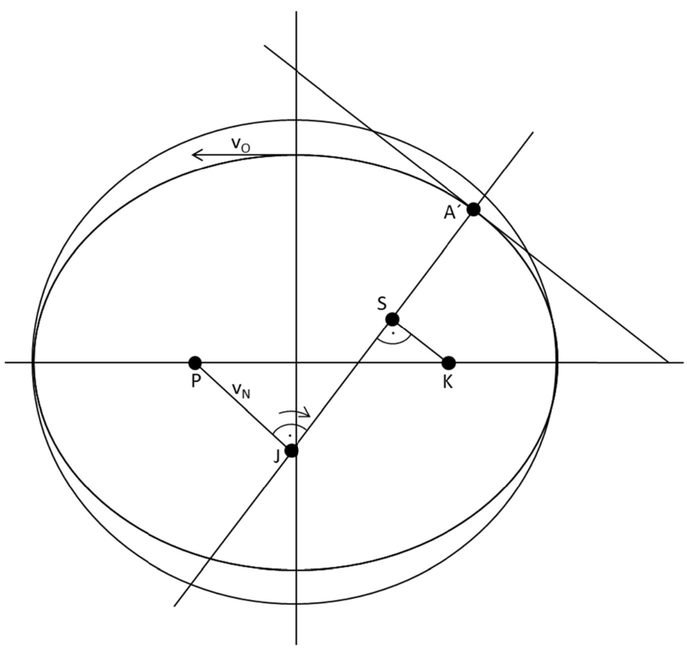

Figure 5. Trigonometric approach to reveal the expression for the normal velocity v_N

There is another interesting property of those contrapedal points J and S: if an observer moving along those contrapedal curves observes the planet orbiting in the Kepler´s ellipse he might get the experimental data given in Table II. These two observes J and S will get different values for the orbital angular velocity of that planet and the modified Kepler´s area law between that observer and the planet.

Table II Observers J and S collecting experimental data along the contrapedal curves

Contrapedal curve with the contrapedal point in the Ptolemy´s empty focus - constant orbital angular velocity
$\omega_E = \frac{v_T}{PP^{/}} = \frac{v_0 \frac{\sqrt{1+\varepsilon\cos E}}{\sqrt{1-\varepsilon\cos E}}}{b\frac{\sqrt{1+\varepsilon\cos E}}{\sqrt{1-\varepsilon\cos E}}} = \frac{v_0}{b}$
Contrapedal curve with the contrapedal point in the Ptolemy´s empty focus - modified Kepler´s area law
$\dot{A}_E = v_T PP^{/} = v_0 \frac{\sqrt{1+\varepsilon\cos E}}{\sqrt{1-\varepsilon\cos E}} b \frac{\sqrt{1+\varepsilon\cos E}}{\sqrt{1-\varepsilon\cos E}} = v_0 b \frac{1+\varepsilon\cos E}{1-\varepsilon\cos E}$
Contrapedal curve with the contrapedal point in the Kepler´s occupied focus - orbital angular velocity
$\omega_E = \frac{v_T}{KK^{/}} = \frac{v_0 \frac{\sqrt{1+\varepsilon\cos E}}{\sqrt{1-\varepsilon\cos E}}}{b\frac{\sqrt{1-\varepsilon\cos E}}{\sqrt{1+\varepsilon\cos E}}} = \frac{v_0}{b}\frac{1+\varepsilon\cos E}{1-\varepsilon\cos E}$
Contrapedal curve with the contrapedal point in the Kepler´s occupied focus - constant Kepler´s area law
$\dot{A}_E = v_T KK^{/} = v_0 \frac{\sqrt{1+\varepsilon\cos E}}{\sqrt{1-\varepsilon\cos E}} b \frac{\sqrt{1-\varepsilon\cos E}}{\sqrt{1+\varepsilon\cos E}} = v_0 b$

6. Moment of the tangent momentum and moment of the normal momentum of the Kepler´s Ellipse

Based on the formulae in Table II we can evaluate the moment of the tangent momentum L_T and to introduce a new physical quantity - the moment of the normal momentum L_N.

The moment of momentum L is defined as the product of the linear momentum with the length of the moment arm, a line dropped perpendicularly from the origin onto the path of the particle. It is this definition: L = (length of moment arm) x (linear momentum).

The moment of the tangent momentum L_T is given as:

$$L_T = mv_T KK^{/} = mv_0 \frac{\sqrt{1+\varepsilon\cos E}}{\sqrt{1-\varepsilon\cos E}} b \frac{\sqrt{1-\varepsilon\cos E}}{\sqrt{1+\varepsilon\cos E}} = mv_0 b \qquad (7)$$

where m is the mass of the planet, v_T the tangent velocity of the planet and KK´ is the length of the moment arm (the distance between the Kepler´s occupied focus and the tangent). The moment of the tangent momentum L_T is constant during the complete path of the Kepler´s ellipse. Therefore, there is no contribution to the torque from this moment of the tangent momentum. This is very well-known experimental fact documented in the existing literature.

The moment of the normal momentum L_N is given as:

$$\begin{aligned} L_N &= mv_N SK^{/} = \\ &= mv_0 \varepsilon \sin E \frac{\sqrt{1+\varepsilon\cos E}}{\sqrt{1-\varepsilon\cos E}} a\, \varepsilon \sin E \frac{\sqrt{1-\varepsilon\cos E}}{\sqrt{1+\varepsilon\cos E}} = \\ &= mv_0 a\, \varepsilon^2 \sin^2 E \end{aligned} \qquad (8)$$

where m is the mass of the planet, v_N the normal velocity of the planet and SK´ is the length of the moment arm (the distance between the Kepler´s occupied focus and the contrapedal curve). The moment of the normal momentum is not constant during the complete path of the Kepler´s ellipse. Therefore, we expect a contribution to the torque of the Kepler´s ellipse. (We did not study in details the properties of the curve $a\varepsilon^2\sin^2E$.)

7. Torque of the Kepler´s Ellipse (Moment of Force)

Torque is defined mathematically as the rate of the change of the moment of the momentum. As long as the moment of the tangent momentum is constant then there is no net torque applied. However, what about the moment of the normal momentum?

The derivation of the formula for the torque caused by the moment of the normal momentum would be as:

$$\tau = \frac{d\,L_N}{dt} = \frac{d\,L_N}{dE}\frac{d\,E}{dt} = \tag{9}$$
$$= m\,v_0 a\,\varepsilon^2 \sin(2E)\,\frac{v_0}{b}\frac{(1+\varepsilon\cos E)}{(1-\varepsilon\cos E)} = \frac{m\,v_0^2}{b}a\,\varepsilon^2\sin(2E)\,\frac{(1+\varepsilon\cos E)}{(1-\varepsilon\cos E)}$$

where m is the mass of the planet, v_0 is the tangent velocity at the end of the minor axis, b is the semi-minor axis, a is the semi-major axis, ε is the eccentricity of the ellipse and E is the eccentric anomaly.

(We did not study in details the properties of the curve of the torque). We expect that this newly discovered quantity τ - the torque of the Kepler´s ellipse - might contribute to the ellipse precession. We want to pass this concept into the hands of experienced Readers of this Journal. Have we found the Arriadne´s Thread leading out of the Labyrinth or are we still lost in the Labyrinth?

Isaac Newton in his Propositions 43-45 of Book I in his Principia derived a formula for the force causing the precession of planets. The astrophysicist Subrahmayan Chandrasekhar in 1995 in his comments to Principia remarked that this Theorem remained largely unknown and undeveloped for over three centuries.

8. Horrebow´s circle (1717) and Colwell´s anomaly (1993)

Kepler´s Equation (KE) has been in the focus in the modern science for four centuries. This topic passed through hands of numerous great researchers. Peter Colwell (1993) surveyed different mathematical techniques and styles to solve the KE - for a given mean anomaly M to find the eccentric anomaly E.

One of those original styles of Old Masters presented N.T. Jorgenson (1974) who re-discovered the method of Peder Horrebow (1717). We have found that the Peder Horrebow´s method (1717) could be very stimulating technique for our hypothesis. Figure 6 shows the Horrebow´s idea with the second auxiliary circle constructed around the Ptolemy´s empty focus. Peter Colwell added into the original Horrebow´s schema one additional line connecting the Kepler´s occupied focus K with the point F on the second auxiliary circle. We will call the angle FKC as the Colwell´s anomaly H. (It has to be noted that Peter Colwell missed the opportunity to use this angle in his analysis).

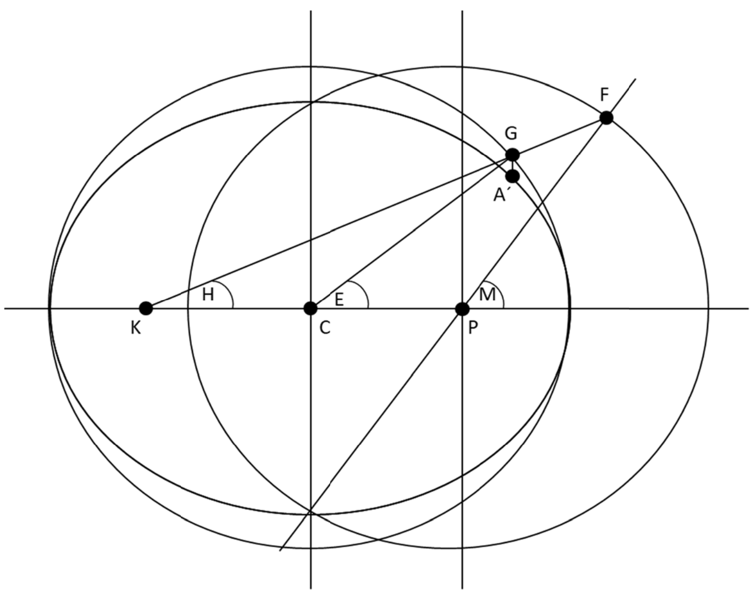

Figure 6. Horrebow´s circle with the mean anomaly M, the eccentric anomaly E, and Colwell´s anomaly H

In Horrebow´s method we have Kepler´s ellipse with the Sun in the left focus K (Kepler´s occupied focus) (the way the Old Masters depicted the Kepler´s ellipse with the aphelion on the right side)) together with the auxiliary circle with the center C (Copernicus center) and the radius equal to the semimajor axis of the orbit. Horrebow inserted the second auxiliary circle with the center in the Ptolemy´s empty focus P and the radius equal to the semimajor axis. In Figure 6 we can see the three anomalies: M - the mean anomaly, E - the eccentric anomaly and H - the Colwell´s anomaly. The Colwell´s anomaly H represent an intermediate step in the determination of the eccentric anomaly E. In Figure 6 we have the Colwell´s anomaly H equaled as:

$$\tan H = \frac{\sin M}{\cos M + 2\varepsilon} \tag{10}$$

Once we know the Colwell´s anomaly H then with the help of the analytical geometry we will get the precise value of the eccentric anomaly E without any iteration process. From the known equation of the line KF and the auxiliary circle with the center C we will get the intersection point G (cosE, sinE). From the known eccentric anomaly E, we will get easily the desired true anomaly Θ.

Our hypothesis is based on an idea that the contrapedal curve with the contrapedal point in the Ptolemy´s empty focus condensed into the Ptolemy´s empty focus and we have got the mean anomaly M depicted by the second auxiliary circle with the center in the Ptolemy´s empty focus. This hypothesis has to be mathematically proved by the Readers of this Journal.

8. "Antikythera Mechanism" in the Solar System

We propose to use the very-well known Antikythera Mechanism as an analogy for the visible Kepler´s ellipse - a part of our Aristotelian World - connected deeply with invisible curves from the Plato´s Realm - Newton´s evolute (1687), Horrebow´s circle (1717) and Colwell´s anomaly (1993), Hamilton´s pedal curve (1847), two contrapedal curves (2018), there are two more curves describing the moment of the normal momentum and the torque of the Kepler´s ellipse (2018).

Are there some more hidden curves in the Plato´s Realm connected to the Kepler´s ellipse?

7. Conclusions

1) We have presented some quantitative properties of the Kepler´s ellipse in Table I and Figures 1 and 2.
2) We have discovered a new trigonometric formula for the radius of curvature in the Newton´s evolute of the Kepler´s ellipse.
3) In the pedal curve with the pedal points in both foci (the auxiliary circle) we have depicted the Hamilton´s hodograph.
4) We have observed the Kepler´s ellipse from two contrapedal curves with contrapedal points in both foci and presented some relationships in Table II.
5) We have derived formulae for the moment of the tangent momentum and the moment of the normal momentum.
6) We have derived the formula for the torque of the Kepler´s ellipse.
7) In the Horrebow´ circle we have presented a hypothesis about the condensation of the contrapedal curve with the contrapedal point in the Ptolemy´s empty focus into the Ptolemy´s empty focus.
8) Are there some more hidden curves in the Plato´s Realm connected to the Kepler´s ellipse?

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