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Quantization of Damped Systems Using Fractional WKB Approximation

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Abstract

The Hamilton Jacobi theory is used to obtain the fractional Hamilton-Jacobi function for fractional damped systems. The technique of separation of variables is applied here to solve the Hamilton Jacobi partial differential equation for fractional damped systems. The fractional Hamilton-Jacobi function is used to construct the wave function and then to quantize these systems using fractional WKB approximation. The solution of the illustrative example is found to be in exact agreement with the usual classical mechanics for regular Lagrangian when fractional derivatives are replaced with the integer order derivatives and $\gamma \to 0$.

Keywords: Damped Systems, WKB Approximation, Wave Function, Hamilton-Jacobi Function, Hamilton Jacobi Equation

1. Introduction

The Hamilton Jacobi theory provides a bridge between classical and quantum mechanics; it implies that quantum mechanics should reduce to classical mechanics in the limit $\hbar \to 0$. The principal interest in this theory is based on the hope that it might provide some guidance concerning the form of a Schrödinger type quantum theory for constrained fields. The fact that (Arnold, 1989; Goldstein, 1980; Lanczos, 1986) solving the Hamilton Jacobi equation gives a generating function for the family of canonical transformation of the dynamics is the theoretical basis for the powerful technique of exact integration of Hamilton's equations that are often employed with the technique of separation of variables.

The canonical formalism for investigating the first order singular systems has been developed by (Guler, 1992; Rabei et al., 1992, Nawafleh, 1998; Rabei, 1999; Muslih, 2002). The quantization of constrained systems has been studied using the WKB approximation (Rabei et al., 2002, 2005; Hasan et al., 2004). The set of Hamilton Jacobi partial differential equations for these systems has been determined using the canonical method, the Hamilton Jacobi function has been obtained by solving these equations. In addition (Nawafleh et al., 2004; Nawafleh, 2007) calculating the Hamilton Jacobi function enables us to construct the wave function of constrained systems, for which the constraints become conditions on it in the semiclassical limit.

This limit also is known as the WKB approximation and it is named after physicists Wentzel, Kramers and Brillouin who all developed it in 1962.

Recently, the Hamilton Jacobi partial differential equations and WKB approximation have been studied for systems containing fractional derivatives using the canonical method (Rabei et al., 2009, 2010). More recently, a powerful approach, the canonical method, has been developed for dissipative systems (Jarab'ah et al., 2013). In this approach the equations of motion are written as total differential equations and the formulation leads to a set of Hamilton Jacobi partial differential equations which are familiar to regular systems. The purpose of this work is indeed to quantize the damped systems using fractional WKB approximation building on the previous work (Hasan, 2016).

This paper is organized as follows: In section **2**, Hamilton Jacobi formalism and fractional WKB approximation are discussed. In section **3**, one illustrative example is studied in detail. The work closes with some concluding remarks in section **4**.

2. Hamilton-Jacobi Formulation and Fractional WKB Approximation

The Lagrangian formulation for time independent damped systems depending on the fractional derivatives is given by

$$L(q, {}_aD_t^{\alpha}q, {}_tD_b^{\beta}q) = L_{\circ}(q, {}_aD_t^{\alpha}q, {}_tD_b^{\beta}q)e^{\gamma q} \tag{1}$$

$e^{\gamma q}$: time independent damping factor.

The formulation of fractional Euler Lagrange equation is obtained as

$$\frac{\partial L}{\partial q} + {}_tD_b^{\alpha}\frac{\partial L}{\partial\, {}_aD_t^{\alpha}q} + {}_aD_t^{\beta}\frac{\partial L}{\partial\, {}_tD_b^{\beta}q} = 0 \tag{2}$$

Remembering that:

The left Riemann- Liouville fractional derivative is defined as

$${}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha-1} f(\tau)d\tau \tag{3}$$

which is denoted as the LRLFD,

and the right Riemann- Liouville fractional derivative is defined as

$${}_xD_b^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b (\tau-x)^{n-\alpha-1} f(\tau)d\tau \tag{4}$$

which is denoted as the RRLFD,

If α is an integer, these derivatives are defined as follows:

$${}_aD_x^{\alpha} f(x) = \left(\frac{d}{dx}\right)^{\alpha} f(x) \tag{5}$$

$${}_xD_b^{\alpha} f(x) = \left(-\frac{d}{dx}\right)^{\alpha} f(x) \tag{6}$$

$$\alpha = 1, 2, \ldots$$

And the fractional Hamiltonian of damped systems is given by

$$H(q, p_{\alpha}, p_{\beta}) = p_{\alpha}\, {}_aD_t^{\alpha}q + p_{\beta}\, {}_tD_b^{\beta}q - L(q, {}_aD_t^{\alpha}q, {}_tD_b^{\beta}q) \tag{7}$$

The conjugate momenta can be written as

$$p_{\alpha} = \frac{\partial L}{\partial\, {}_aD_t^{\alpha}q} \tag{8}$$

$$p_{\beta} = \frac{\partial L}{\partial\, {}_tD_b^{\beta}q} \tag{9}$$

$$D_t^1 = \frac{d}{dt} \tag{10}$$

Following to the canonical method (Rabei et al., 1992) the Hamilton Jacobi partial differential equation reads as

$$H' = p_{\circ} + H \tag{11}$$

Remembering that

$$p_{\circ} = \frac{\partial S}{\partial t} \tag{12}$$

Where **S** is the Hamilton Jacobi function and takes this form

$$S = S({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, E_1, E_2, t) \tag{13}$$

so that equations (11, 12) can be written in compact form as

$$H' = \frac{\partial S}{\partial t} + H = 0 \tag{14}$$

The solution of the above Hamilton Jacobi partial differential equation can be constructed as

$$S = S({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, E_1, E_2, t) = f(t) + W_1(E_1, {}_aD_t^{\alpha-1}q) + W_2(E_2, {}_tD_b^{\beta-1}q) + A \tag{15}$$

Where E_1 and E_2 are the constants of integration and **A** is some other constant. Thus, the equations of motion can be obtained using the canonical transformations as follows

$$\eta_1 = \frac{\partial S}{\partial E_1} = {}_aD_t^{\alpha-1}Q \tag{16}$$

$$\eta_2 = \frac{\partial S}{\partial E_2} = {}_tD_b^{\beta-1}Q \tag{17}$$

$$p_\alpha = \frac{\partial S}{\partial\, {}_aD_t^{\alpha-1}q} = \frac{\partial W_1}{\partial\, {}_aD_t^{\alpha-1}q} \tag{18}$$

$$p_\beta = \frac{\partial S}{\partial\, {}_tD_b^{\beta-1}q} = \frac{\partial W_2}{\partial\, {}_tD_b^{\beta-1}q} \tag{19}$$

Where η_1 and η_2 are constants and can be determined from the initial conditions

The semiclassical expansion (WKB approximation) of Hamilton Jacobi function of constrained systems has been investigated by (Rabei et al., 2002). Following this reference the wave function can be constructed as

$$\psi(q,t) = \prod_{i=1}^{N} \psi_{0i}(q_i) \exp\left[\frac{i}{\hbar} S(q,t)\right] \tag{20}$$

where $\psi_{0i}(q_i)$ is the amplitude of the wave function, which is defined as

$$\psi_{0i}(q_i) = \frac{1}{\sqrt{p_i}} \tag{21}$$

The wave function $\psi(q,t)$ satisfies the condition

$$\hat{H}'\psi = 0 \tag{22}$$

In the semiclassical limits $\hbar \to 0$.

Thus, the wave function for damped systems in the fractional form can be written as

$$\psi({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, t) = \frac{1}{\sqrt{p_\alpha p_\beta}} \exp\left[\frac{i}{\hbar} S({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, E_1, E_2, t)\right] \tag{23}$$

And the momenta operators

$$\hat{p}_\alpha = \frac{\hbar}{i} \frac{\partial}{\partial\, {}_aD_t^{\alpha-1}q} \tag{24}$$

$$\hat{p}_\beta = \frac{\hbar}{i} \frac{\partial}{\partial\, {}_tD_b^{\beta-1}q} \tag{25}$$

$$\hat{p}_0 = \frac{\hbar}{i} \frac{\partial}{\partial t} \tag{26}$$

Note that,

$$D_t^0 = 1$$

It is important to notice that if α and β are equal unity, the results are found to be inexact agreement with the results that obtained by conventional methods.

3. Example

Let us discuss the motion of a pendulum of mass **m** and length l with angular displacement **θ** from the vertical (Fowles, 1993).

The Lagrangian which describes this example is given by:

$$L = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl(1 - \cos\theta)$$

In the presence of damping the Lagrangian becomes

$$L = [\frac{1}{2} ml^2 \dot{\theta}^2 - mgl(1 - \cos\theta)]e^{\gamma\theta} \tag{27}$$

For small θ, we have approximately $\cos\theta = 1 - \frac{\theta^2}{2}$

Thus, this Lagrangian reads

$$L = [\frac{1}{2} ml^2 \dot{\theta}^2 - \frac{1}{2} mgl\theta^2]e^{\gamma\theta} \tag{28}$$

The Lagrangian in fractional form can be written as

$$L = [\frac{1}{2} ml^2 ({}_0D_t^\alpha \theta)^2 - \frac{1}{2} mgl\theta^2]e^{\gamma\theta} \tag{29}$$

The Hamiltonian of this system reads

$$H = \frac{p_\theta^2}{2ml^2} e^{-\gamma\theta} + e^{\gamma\theta} \frac{mgl}{2} \theta^2 \tag{30}$$

Where the fractional canonical momentum is

$$p_\theta = \frac{\partial S}{\partial\, {}_0D_t^{\alpha-1}\theta} \tag{31}$$

Using equations (30 and 31), equation (14) becomes

$$H' = p_\circ + H = \frac{\partial S}{\partial t} + \frac{(\partial S / \partial\, {}_0D_t^{\alpha-1}\theta)^2}{2ml^2} e^{-\gamma\theta} + e^{\gamma\theta} \frac{mgl}{2} \theta^2 = 0 \tag{32}$$

From equation (15), the Hamilton Jacobi function takes the following form

$$S = S({}_0D_t^{\alpha-1}\theta, E_\theta, t) = f(t) + W_\theta(E_\theta, {}_0D_t^{\alpha-1}\theta) + A \tag{33}$$

Where, $f(t) = -E_\theta t$

Inserting equation (33) into equation (32), we obtain

$$-E_\theta + \frac{(\partial W_\theta / \partial\, {}_0D_t^{\alpha-1}\theta)^2}{2ml^2} e^{-\gamma\theta} + e^{\gamma\theta} \frac{mgl}{2} \theta^2 = 0 \tag{34}$$

Using separation of variables, we get

$$\frac{(\partial W_\theta / \partial\, {}_0D_t^{\alpha-1}\theta)^2}{2ml^2} e^{-\gamma\theta} = E_\theta - \frac{mgl}{2} \theta^2 e^{\gamma\theta} \tag{35}$$

Where E_θ is constant.

Solving equation (35), we obtain

$$W_{\theta} = \int \sqrt{2ml^2 (e^{\gamma\theta} E_{\theta} - \frac{mgl}{2} \theta^2 e^{2\gamma\theta})} d\, {}_0D_t^{\alpha-1}\theta \tag{36}$$

Then, the Hamilton Jacobi function is

$$S = -E_{\theta} t + \int \sqrt{2ml^2 (e^{\gamma\theta} E_{\theta} - \frac{mgl}{2} \theta^2 e^{2\gamma\theta})} d\, {}_0D_t^{\alpha-1}\theta + A \tag{37}$$

Making use of equation (16), the equation of motion is

$$\eta_{\theta} = \frac{\partial S}{\partial E_{\theta}} = -t + \int \frac{ml^2 e^{\gamma\theta}}{\sqrt{2ml^2 (e^{\gamma\theta} E_{\theta} - \frac{mgl}{2} \theta^2 e^{2\gamma\theta})}} d\, {}_0D_t^{\alpha-1}\theta \tag{38}$$

From equation (31) we get

$$p_y = \frac{\partial S}{\partial\, {}_0D_t^{\alpha-1} y} = \frac{\partial W_y}{\partial\, {}_0D_t^{\alpha-1} y} = \sqrt{2ml^2 (e^{\gamma\theta} E_{\theta} - \frac{mgl}{2} \theta^2 e^{2\gamma\theta})} \tag{39}$$

We are now in a position to quantize our system. The wave function of this example is given by:

$$\psi({}_0D_t^{\alpha-1}\theta, t) = \frac{1}{\sqrt{p_{\theta}}} \exp\left[\frac{i}{\hbar} S({}_0D_t^{\alpha-1}\theta, E_{\theta}, t)\right] \tag{40}$$

$$\psi = [2ml^2 (e^{\gamma\theta} E_{\theta} - \frac{mgl}{2} \theta^2 e^{2\gamma\theta})]^{\frac{-1}{4}} \exp\left\{\frac{i}{\hbar} [-E_{\theta} t + \int \sqrt{2ml^2 (e^{\gamma\theta} E_{\theta} - \frac{mgl}{2} \theta^2 e^{2\gamma\theta})} d\, {}_0D_t^{\alpha-1}\theta + A]\right\} \tag{41}$$

The Schrödinger equation takes the form

$$\hat{H}'\psi = \left\{\frac{\hbar}{i} \frac{\partial}{\partial t} - e^{-\gamma\theta} \frac{\hbar^2}{2ml^2} \frac{\partial^2}{\partial ({}_0D_t^{\alpha-1}\theta)^2} + \frac{mgl}{2} \theta^2 e^{\gamma\theta}\right\}\psi = 0 \tag{42}$$

After some calculations, it is easy to show that in the semiclassical limit $\hbar \to 0$, $\hat{H}\psi = E\psi$

4. Conclusion

The damped systems are investigated using the Hamilton Jacobi quantization scheme. The fractional Hamilton-Jacobi function **S** is determined using the method of separation of variables in the same manner as for regular systems. The equations of motion were derived from this function. Further, this function enables us to formulate the wave function; this meant that the quantization using the fractional WKB approximation had been completed. The solution of the illustrative example is found to be in exact agreement with the usual classical mechanics for regular Lagrangian when α, β are equal unity only. Also in the semiclassical limit $\hbar \to 0$, the quantum results are found to be in exact agreement with the classical results.

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