Extended Cesaro Operator from $A^\infty_{\phi}$ to Bloch Space

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Abstract
Let $g$ be a holomorphic function of the unit ball $B$ in several complex variables, and denote by $T_g$ the induced extended Cesaro operator. This paper discussed the boundedness and compactness of $T_g$ acting from $A^\infty_{\phi}$ to Bloch space in the unit ball.

Keywords: Cesaro operator, Unit ball, Bounded, Compact, Bloch space

1. Introduction
Let $B$ be the unit ball of $\mathbb{C}^n$, and $H(B)$ denotes the class of analytic functions in $B$. Let $H^p$ be the standard Hardy space on the unit disc $D$. For $0 < p \leq \infty$, the classical Cesaro operator acting on $f$ is given by the formula

$$C[f](z) = \sum_{j=0}^{\infty} \frac{1}{j+1} \sum_{k=0}^{j} a_k z^j.$$

The study of Cesaro operator has become a major driving force in the development of modern complex analysis. The recent papers are good sources for information on much of the developments in the theory of Cesaro operators up to the middle of last decade. In the recent years, boundedness and compactness of extended Cesaro operator between several spaces of holomorphic functions have been studied by many mathematicians. It is well known that the operator $C$ is bounded on the usual Hardy spaces $H^p$ and Bergman space, as well as the Dirichlet space. Basic results facts on Hardy spaces can be found in Durn(1970). For $0 < p < \infty$, Siskakis (1987) studied the spectrum of $C$, as a by-product he obtained that $C$ is bounded on $H^p(D)$. For $p = 1$, the boundedness of $C$ was given also by Siskakis (1990) by a particularly elegant method, independent of spectrum theory, a different proof of the result can be found in Giang and Morrisz(1995). After that, for $0 < p < 1$, Miao(1992) proved $C$ is also bounded. For $p = \infty$, the boundedness of $C$ was given by Danikas and Sisakis(1993).

A little calculation shows $C[f](z) = \frac{1}{z} \int_{0}^{\infty} f(t)(\ln \frac{1}{1-t}) dt$. From this point of view, if $g \in H(B)$, it is natural to consider the extended Cesaro operator defined by

$$T_g(f)(z) = \int_{0}^{\infty} f(t)R_g(tz) \frac{dt}{t},$$

where $R_g(z) = \sum_{j=0}^{\infty} z_j \partial_{z_j}^{2} g(z)$.

It is easy to see that $T_g$ take $H(B)$ into itself. In general, there is no easy way to determine when an extended Cesaro operator is bounded or compact.

The boundedness and compactness of this operator on weighted Bergman, mixed norm, Bloch, and Dirichlet spaces in the unit ball have been studied by Xiao and Hu. In this paper, we continue this line of research.

Now we introduce some spaces first. We define Bloch space $B$ as the space of holomorphic functions $f \in H(B)$ such that $\|f\|_{B} = \|f(0)\| + \sup_{z \in \mathbb{D}} (1-|z|^2)|Rf(z)|$.

Let $\phi$ denote a strictly decreasing continuous function $\phi:[0,1] \rightarrow \mathbb{R}$, with $\phi(0) = 1$. For $z \in B$, $\phi(z)$ will denote $\phi(|z|)$. The Banach space of all analytic functions $f \in H(B)$ which satisfy $\|f\|_{\phi} := \sup_{z \in \mathbb{D}} |\phi(z)||f(z)||$ will be denoted by $A^\infty_{\phi}$. For example, $\phi(r) = (1-r^2)^\alpha$ with $\alpha > 0$. When $\phi = 1$, it becomes the classical bounded function space.
2. Some Lemmas

In the following, we will use the symbol $C$ or $M$ to denote a finite positive number which does not depend on variable $z$ and may depend on some norms and parameters $f$, not necessarily the same at each occurrence.

By Montel theorem and the definition of compact operator, the following lemma follows.

2.1 Assume that $g \in H(B)$. Then $T_g : A^\infty \rightarrow \text{Bloch}$ is compact if and only if $T_g$ is bounded and for any bounded sequence $f_k (k \in N)$ in $A_g^\infty$ which converges to zero uniformly on compact subsets of $B$ as $k \rightarrow \infty$, $\| T_g f_k \|_{\text{Bloch}} \rightarrow 0$ as $k \rightarrow \infty$.

Proof. Assume that $T_g$ is compact and suppose $f_k (k \in N)$ is a sequence in $A_g^\infty$ with $\sup_{n \in N} \| f \|_g < \infty$ and $f_k \rightarrow 0$ uniformly on compact subsets of $B$. By the compactness of $T_g$, we have that $T_g f_k$ has a subsequence $T_g f_{km}$ which converges in $A^\infty$, say, to $h$. We have that for any compact set $K \subset B$, there is a positive constant $C_k$ independent of $f$ such that $\| T_g f_k (z) - h(z) \|_g \leq C_k \| T_g f_k - h \|_{\text{Bloch}}$ for all $z \in K$. This implies that $T_g f_k (z) - h(z) \rightarrow 0$ uniformly on compact sets of $B$. Since $K$ is a compact subset of $B$, by the hypothesis and the definition of $T_g$, $T_g f_k (z)$ converges to zero uniformly on $K$. It follows from the arbitrary of $K$ that the limit function $h$ is equal to 0.

Since it's true for arbitrary subsequence of $f_k$, we see that $T_g f_k$ converges to 0 in $\text{Bloch}$.

Conversely, if $\{ f_k \} \in K_r = B^{\infty}_{\phi}(0, r)$, where $B^{\infty}_{\phi}(0, r)$ is a ball in $A_g^\infty$, then by Montel's Lemma, $f_k$ is a normal family, therefore there is a subsequence $f_{km}$ which converges uniformly to $f \in H(B)$ on compact subsets of $B$.

Denote $B_r = B(0, 1 - \frac{1}{k}) \subset C^\infty$, and since $\{ f_k \} \in K_r = B^{\infty}_{\phi}(0, r)$ then

$\phi(z) \mid f(z) = \lim_{m \rightarrow \infty} \phi(z) \mid f_{km}(z) \mid < r$. Hence the sequence $\{ f_{km} - f \}$ is such that

$\| f_{km} - f \|_g \leq 2r < \infty$ and converges to 0 on compact subsets of $B$, by the hypothesis of this lemma, we have that $T_g f_{km} \rightarrow T_g f$ in $\text{Bloch}$. Thus the set $T_g (K_r)$ is relatively compact, so $T_g$ is compact, finishing the proof.

2.2 Let $g \in H(B)$, then $RT_g f(z) = f(z) Rg(z)$ for any $f \in H(B)$ and $z \in B$.

Proof: Suppose the holomorphic function $f Rg$ has the Taylor expansion

$(f Rg)(z) = \sum_{\alpha \in \mathbb{N}} a_{\alpha} z^\alpha$.

Then we have

$RT_g f(z) = R \left[ \int_0^1 f(tz) Rg(tz) dt \right] = \int_0^1 \sum_{\alpha \in \mathbb{N}} a_{\alpha} (tz)\alpha dt$

$= R \left[ \sum_{\alpha \in \mathbb{N}} a_{\alpha} z^\alpha \right] = \sum_{\alpha \in \mathbb{N}} a_{\alpha} z^\alpha = (f Rg)(z)$.

3. Main Theorem

3.1 Suppose $g \in H(B)$, then $T_g : A^\infty \rightarrow \text{Bloch}$ is bounded if and only if

$\sup_{z \in B} (1 - | z |^2) \mid Rg(z) \mid < +\infty$.

Proof: We proof the sufficiency frist. Since $T_g f(0) = 0$ and

$\sup_{z \in B} (1 - | z |^2) \mid RT_g f(z) \mid \leq \sup_{z \in B} (1 - | z |^2) \mid f(z) \mid \| Rg(z) \| \leq \| f \|_g \sup_{z \in B} (1 - | z |^2) \mid Rg(z) \mid$

Therefore, $\sup_{z \in B} (1 - | z |^2) \mid Rg(z) \mid < +\infty$ implies that $T_g$ is bounded.
Now we turn to the necessity. Setting the test function, for any \( w \in B \), let \( f_w(z) = \frac{1}{\phi(w)} \frac{1-|z|}{1-z,w} \). Then it is easy to see that \( \sup_{w,d} \|f_w\|_\phi = M < \infty \).

Then \( M \| T_g \|_\phi \geq \| T_g f_w \|_{\text{Bloch}} = \sup_{z \in B} (1-|z|^2) \cdot |RT_g f_w(z)| \)

\[
\geq (1-|w|^2) |f_w(w)| \cdot |Rg(w)| \geq \frac{1}{2} \left( \frac{1-|w|^2}{\phi(w)} \right) |Rg(w)|
\]

Since \( w \in B \) is arbitrary, we get the necessity.

Remark: note that by take the test function \( f = 1 \), we can get \( g \in \text{Bloch} \).

3.2 Suppose \( g \in H(B) \), and \( \phi(z) = 1-|z|^2 \), then \( T_{g,} : A_\phi^\infty \rightarrow \text{Bloch} \) is bounded if and only if \( \sup_{z \in B} |Rg(z)| < \infty \), that is \( Rg \) belongs to the class of bounded holomorphic functions.

Proof: It is obvious from the 3.1.

3.3 Suppose \( g \in H(B) \), then \( T_{g,} : A_\phi^\infty \rightarrow \text{Bloch} \) is compact if and only if

\[
\lim_{|z| \to \infty} \frac{1-|z|^2}{\phi(z)} |Rg(z)| = 0.
\]

Proof: We consider the sufficiency first. Assume the condition holds, then for any given \( \varepsilon > 0 \), there exists a \( \delta(0 < \delta < 1) \), such that \( \frac{1-|z|^2}{\phi(z)} |Rg(z)| < \varepsilon \) when \( |z| > \delta \). Let \( K = \{ z \in B : |z| \leq \delta \} \), and for any sequence \( f_k \) with \( \|f_k\| \leq C \) and \( f_k \) converges to 0 uniformly on compact subsets of \( B \). Notice that \( T_g f(0) = 0 \), then

\[
\| T_g f_k \|_{\text{Bloch}} = \sup_{z \in B} (1-|z|^2) |f_k(z)| |Rg(z)| \leq \sup_{z \in B} (1-|z|^2) |f_k(z)| |Rg(z)| + \sup_{z \in B} (1-|z|^2) |f_k(z)| |Rg(z)|
\]

\[
\leq \|g\|_{\text{Bloch}} \sup_{z \in K} |f_k(z)| + \frac{1-|z|^2}{\phi(z)} |Rg(z)| \|f_k\|_\phi.
\]

With the uniform convergence of \( f_k \) we get \( \| T_g f_k \| \to 0 \) as \( k \to \infty \). Owing to Lemma 1, \( T_g \) is compact.

Now we turn to the necessity. For the necessity, we choose the test functions as follows. For any sequence \( \{z_j\} \) with \( |z_j| \to 1 \) as \( j \to \infty \). We set \( h_j(z) = \frac{1}{\phi(z_j)} \frac{1-|z|}{1-z,z_j} \). It is easy to check that \( \sup_{z \in B} \|h_j\|_\phi < \infty \) and \( h_j \) uniformly converges to 0 in any compact subset of \( B \). That is to say \( h_j \) satisfy the condition of lemma 2.1, then we have

\[
\| T_g h_j \|_{\text{Bloch}} \to 0 \quad \text{as} \quad j \to \infty.
\]

Therefore, \( 0 \leq \| T_g h_j \|_{\text{Bloch}} = \sup_{z \in B} (1-|z|^2) |RT_g h_j(z)| = \sup_{z \in B} (1-|z|^2) |h_j(z)| \cdot |Rg(z)| \)

\[
\geq (1-|z_j|^2) |h_j(z_j)| \cdot |Rg(z_j)| \geq \frac{1}{2} \left( \frac{1-|z_j|^2}{\phi(z_j)} \right) |Rg(z_j)|.
\]

The conclusion follows by the arbitrary of the sequence \( \{z_j\} \).

3.4 Suppose \( g \in H(B) \), and \( \phi(z) = 1-|z|^2 \), then \( T_{g,} : A_\phi^\infty \rightarrow \text{Bloch} \) is compact if and only if \( g \) is a constant.

Proof: By 3.3 we can obtain that \( \lim_{|z| \to \infty} |Rg(z)| = 0 \), by the maximum module of principle we have \( Rg = 0 \), and by the formula that \( f(z) - f(0) = \oint_0^{\frac{Rg(z)}{t}} dt \), we must have \( g \) is a constant.

References


