Prime Cordial Labeling for Some Graphs

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Abstract

We present here prime cordial labeling for the graphs obtained by some graph operations on given graphs.

Keywords: Prime cordial labeling, Total graph, Vertex switching

1. Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G),E(G))$. For all standard terminology and notations we follow (Harary F., 1972). We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 If the vertices of the graph are assigned values subject to certain conditions then it is known as *graph labeling*.

For a dynamic survey on graph labeling we refer to (Gallian J., 2009). A detailed study on variety of applications of graph labeling is reported in (Bloom G. S., 1977, p. 562-570).

Definition 1.2 Let G be a graph. A mapping $f: V(G) \rightarrow \{0, 1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the label of the vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^*: E(G) \rightarrow \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f while $e_f(0)$, $e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.3 A binary vertex labeling of a graph G is called a *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is *cordial* if it admits cordial labeling.

The concept of cordial labeling was introduced by (Cahit I.,1987, p.201-207). After this many researchers have investigated graph families or graphs which admit cordial labeling. Some labeling schemes are also introduced with minor variations in cordial theme. Some of them are product cordial labeling, total product cordial labeling and prime cordial labeling. The present work is focused on prime cordial labeling.

Definition 1.4 A prime cordial labeling of a graph G with vertex set $V(G)$ is a bijection $f: V(G) \rightarrow \{1,2,3,\ldots,|V(G)|\}$ defined by

$$f(e = uv) = 1 \quad ; \text{if } \gcd(f(u), f(v)) = 1$$
$$= 0 \quad ; \text{otherwise}$$

and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits prime cordial labeling is called a prime cordial graph.

Definition 1.5 Let G be a graph with two or more vertices then the *total graph* $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.

Definition 1.6 The *composition* of two graphs G_1 and G_2 denoted by $G_1[G_2]$ has vertex set $V(G_1[G_2]) = V(G_1) \times V(G_2)$ and edge set $E(G_1[G_2]) = \{(u_1,v_1)(u_2,v_2) \,/\, u_1u_2 \in E(G_1) \text{ or } [u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)]\}$

Definition 1.7 A vertex switching G_v of a graph G is the graph obtain by taking a vertex v of G, removing all the edges incident to v and adding edges joining v to every other vertex which are not adjacent to v in G.

2. Main Results

Theorem 2.1 $T(P_n)$ is prime cordial graph, $\forall\, n \geq 5$.

Proof : If $v_1, v_2, v_3, \ldots\ldots\ldots v_n$ and $e_1, e_2, e_3, \ldots\ldots\ldots e_n$ be the vertices and edges of P_n then $v_1, v_2, v_3, \ldots\ldots\ldots v_n, e_1, e_2, e_3, \ldots\ldots\ldots e_n$ are vertices of $T(P_n)$.

We define vertex labeling $f: V(T(P_n)) \rightarrow \{1, 2, 3\ldots/V(G)/\}$ as follows. We consider following four cases.

Case 1: $n = 3, 5$

For the graph $T(P_3)$ the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5). Then obviously $e_f(0) = 1$, $e_f(1) = 6$. That is, $|e_f(0) - e_f(1)| = 5$ and in all other possible arrangement of vertex labels $|e_f(0) - e_f(1)| > 5$. Therefore $T(P_3)$ is not a prime cordial graph.

The case when n=5 is to be dealt separately. The graph $T(P_5)$ and its prime cordial labeling is shown in *Fig 1*.

Case 2: n *odd,* $n \geq 7$

$f(v_1) = 2, f(v_2) = 4,$

$f(v_{i+2}) = 2(i + 3), \quad 1 \leq i \leq \lfloor n/2 \rfloor - 2$

$f(v_{\lfloor n/2 \rfloor+1}) = 3, f(v_{\lfloor n/2 \rfloor+2}) = 1, f(v_{\lfloor n/2 \rfloor+3}) = 7,$

$f(v_{\lfloor n/2 \rfloor+2+i}) = 4i + 9, 1 \leq i \leq \lfloor n/2 \rfloor - 2$

$f(e_i) = f(v_{\lfloor n/2 \rfloor}) + 2i, 1 \leq i \leq \lfloor n/2 \rfloor - 1,$

$f(e_{\lfloor n/2 \rfloor}) = 6, f(e_{\lfloor n/2 \rfloor+1}) = 9, f(e_{\lfloor n/2 \rfloor+2}) = 5,$

$f(e_{\lfloor n/2 \rfloor+i+1}) = 4i+7, 1 \leq i \leq \lfloor n/2 \rfloor - 2$

In this case we have $e_f(0) = e_f(1) + 1 = 2(n-1)$

Case 3: $n = 2, 4, 6$

For the graph $T(P_2)$ the possible pairs of labels of adjacent vertices are (1,2), (1,3), (2,3). Then obviously $e_f(0) = 0$, $e_f(1) = 3$. Therefore $T(P_2)$ is not a prime cordial graph.

For the graph $T(P_4)$ the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (2,3), (2,4), (2,5), (2,6), (2,7), (3,4), (3,5), (3,6), (3,7), (4,5), (4,6), (4,7), (5,6), (5,7), (6,7). Then obviously $e_f(0) = 4$, $e_f(1) = 7$. That is, $|e_f(0) - e_f(1)| = 3$ and in all other possible arrangement of vertex labels $|e_f(0) - e_f(1)| > 3$. Thus $T(P_4)$ is not a prime cordial graph.

The case when n=6 is to be dealt separately. The graph $T(P_6)$ and its prime cordial labeling is shown in *Fig 2*.

Case 4 n even, $n \geq 8$

$f(v_1) = 2, f(v_2) = 4,$

$f(v_{i+2}) = 2(i + 3), \quad 1 \leq i \leq n/2 - 3$

$f(v_{n/2}) = 6, f(v_{n/2+1}) = 9, f(v_{n/2+2}) = 5,$

$f(v_{n/2+2+i}) = 4i + 7, 1 \leq i \leq n/2 - 2$

$f(e_i) = f(v_{n/2-1}) + 2i, 1 \leq i \leq n/2 - 1,$

$f(e_{n/2}) = 3, f(e_{n/2+1}) = 1, f(e_{n/2+2}) = 7,$

$f(e_{n/2+2+i}) = 4i + 9, 1 \leq i \leq n/2 - 3$

In this case we have $e_f(0) = e_f(1) + 1 = 2(n-1)$

That is, $T(P_n)$ is a prime cordial graph, $\forall\, n \geq 5$.

Illustration 2.2 Consider the graph $T(P_7)$. The labeling is as shown in *Fig 3*.

Theorem 2.3 $T(C_n)$ is prime cordial graph, $\forall\, n \geq 5$.

Proof : If $v_1, v_2, v_3, \ldots\ldots\ldots v_n$ and $e_1, e_2, e_3, \ldots\ldots\ldots e_n$ be the vertices and edges of C_n then $v_1, v_2, v_3, \ldots\ldots\ldots v_n, e_1, e_2, e_3, \ldots\ldots\ldots e_n$ are vertices of $T(C_n)$.

We define vertex labeling $f: V(T(C_n)) \rightarrow \{1, 2, 3, \ldots\ldots/V(G)/\}$ as follows. We consider following four cases.

Case 1: $n = 4$

For the graph $T(C_4)$ the possible pair of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (3,4), (3,5), (3,6), (3,7), (3,8), (4,5), (4,6), (4,7), (4,8), (5,6), (5,7), (5,8), (6,7), (6,8), (7,8) . Then obviously $e_f(0) = 6$, $e_f(1) = 10$. That is, $|e_f(0) - e_f(1)| = 4$ and all other possible arrangement of vertex labels will yield $|e_f(0) - e_f(1)| > 4$. Thus $T(C_4)$ is not a prime cordial graph.

Case 2: n even, $n \geq 6$

$f(v_1) = 2, f(v_2) = 8,$

$f(v_{i+2}) = 4i + 10, \quad 1 \leq i \leq n/2 - 3$

$f(v_{n/2}) = 12, f(v_{n/2+1}) = 3, f(v_{n/2+2}) = 9, f(v_{n/2+3}) = 7,$

$f(v_{n/2+2+i}) = 4i + 9, 1 \leq i \leq n/2 - 3$

$f(e_1) = 4, f(e_2) = 10,$

$f(e_{i+2}) = 4(i + 3), 1 \leq i \leq n/2 - 3,$

$f(e_{n/2}) = 6, f(e_{n/2+1}) = 1, f(e_{n/2+2}) = 5,$

$f(e_{n/2+1+i}) = 4i + 7, 1 \leq i \leq n/2 - 2$

In view of the labeling pattern defined above we have

$e_f(0) = e_f(1) = 2n.$

Case 3: $n = 3$

For the graph $T(C_3)$ the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6). Then obviously $e_f(0) = 4$, $e_f(1) = 8$. That is, $|e_f(0) - e_f(1)| = 4$ and all other possible arrangement of vertex labels will yield $|e_f(0) - e_f(1)| > 4$. Thus $T(C_3)$ is not a prime cordial graph.

Case 4: n odd, $n \geq 5$

$f(v_1) = 2,$

$f(v_{1+i}) = 4(i + 1), \quad 1 \leq i \leq \lfloor n/2 \rfloor - 1$

$f(v_{\lfloor n/2 \rfloor+1}) = 6, f(v_{\lfloor n/2 \rfloor+2}) = 9, f(v_{\lfloor n/2 \rfloor+3}) = 5,$

$f(v_{\lfloor n/2 \rfloor+3+i}) = 4i + 7, 1 \leq i \leq n - \lfloor n/2 \rfloor - 3$

$f(e_1) = 4,$

$f(e_{1+i}) = 4i + 6, 1 \leq i \leq \lfloor n/2 \rfloor - 1,$

$f(e_{\lfloor n/2 \rfloor+1}) = 3, f(e_{\lfloor n/2 \rfloor+2}) = 1, f(e_{\lfloor n/2 \rfloor+3}) = 7,$

$f(e_{\lfloor n/2 \rfloor+3+i}) = 4i+9, 1 \leq i \leq n - \lfloor n/2 \rfloor - 3$

In view of the labeling pattern defined above we have

$e_f(0) = e_f(1) = 2n.$

Thus f is a prime cordial labeling of $T(C_n)$.

Illustration 2.4 Consider the graph $T(C_6)$. The labeling is as shown in *Fig 4*.

Theorem 2.5 $P_2 [P_m]$ is prime cordial graph $\forall$ m ≥ 5.

Proof : Let $u_1, u_2, u_3, \ldots\ldots u_m$ be the vertices of P_m and v_1, v_2 be the vertices of P_2. We define vertex labeling $f: V(P_2 [P_m]) \rightarrow \{1, 2, 3, \ldots\ldots |V(G)|\}$ as follows. We consider following four cases.

Case 1: m = 2, 4

For the graph $P_2 [P_2]$ the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (2,3), (2,4), (3,4). Then obviously $e_f(0) = 1$, $e_f(1) = 5$. That is, $|e_f(0) - e_f(1)| = 4$ and in all other possible arrangement of vertex labels we have $|e_f(0) - e_f(1)| > 4$. Therefore $P_2 [P_2]$ is not a prime cordial graph.

For the graph $P_2 [P_4]$ the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8) (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (3,4), (3,5), (3,6), (3,7), (3,8), (4,5), (4,6), (4,7), (4,8), (5,6), (5,7), (5,8), (6,7), (6,8), (7,8) . Then obviously $e_f(0) = 7$, $e_f(1) = 9$. i.e. $|e_f(0) - e_f(1)| = 2$ and in all other possible arrangement of vertex labels we have $|e_f(0) - e_f(1)| > 2$. Thus $P_2 [P_4]$ is not a prime cordial graph.

Case 2: m even, $m \geq 6$

$f(u_1, v_1) = 2, f(u_2, v_1) = 8,$

$f(u_{2+i}, v_1) = 4i + 10, \quad 1 \leq i \leq m/2 - 3$

$f(u_{m/2}, v_1) = 12,$

$f(u_{m/2+i}, v_1) = 4i - 3, \quad , \quad 1 \leq i \leq m/2$

$f(u_1, v_2) = 4, f(u_2, v_2) = 10,$

$f(u_{2+i}, v_2) = 4i + 12, \quad 1 \leq i \leq m/2 - 3$

$f(u_{m/2}, v_2) = 6, f(u_{m/2+1}, v_2) = 3,$

$f(u_{m/2+1+i}, v_2) = 4i + 3, \quad 1 \leq i \leq m/2 - 1$

Using above pattern we have

$e_f(0) = e_f(1) = \frac{5n-4}{2}$

<u>*Case 3:* $m = 3$</u>

For the graph $P_2[P_3]$ the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6). Then obviously $e_f(0) = 4$, $e_f(1) = 7$. That is, $|e_f(0) - e_f(1)| = 3$ and in all other possible arrangement of vertex labels we have $|e_f(0) - e_f(1)| > 3$. Thus $P_2[P_3]$ is not a prime cordial graph.

<u>*Case 4:* m *odd,* $m \geq 5$</u>

$f(u_i, v_1) = 4(1+i), \quad 1 \leq i \leq \lfloor n/2 \rfloor - 1$

$f(u_{\lfloor n/2 \rfloor}, v_1) = 2,$

$f(u_{\lfloor n/2 \rfloor+1}, v_1) = 6, \quad f(u_{\lfloor n/2 \rfloor+2}, v_1) = 9, \quad f(u_{\lfloor n/2 \rfloor+3}, v_1) = 5,$

$f(u_{\lfloor n/2 \rfloor+2+i}, v_1) = 4i + 7, \quad 1 \leq i \leq \lfloor n/2 \rfloor - 2$

$f(u_1, v_2) = 4,$

$f(u_{1+i}, v_2) = 4i + 6, \quad 1 \leq i \leq \lfloor n/2 \rfloor - 1$

$f(u_{\lfloor n/2 \rfloor+1}, v_2) = 3, \quad f(u_{\lfloor n/2 \rfloor+2}, v_2) = 1, \quad f(u_{\lfloor n/2 \rfloor+3}, v_2) = 7,$

$f(u_{\lfloor n/2 \rfloor+2+i}, v_2) = 4i + 9, \quad 1 \leq i \leq \lfloor n/2 \rfloor - 2$

Using above pattern we have

$e_f(0) = e_f(1) + 1 = 2n + \lfloor n/2 \rfloor - 1.$

Thus in case 2 and case 4 the graph $P_2[P_m]$ satisfies the condition $|e_f(0) - e_f(1)| \leq 1$.

That is, $P_2[P_m]$ is a prime cordial graph $\forall m \geq 5$.

Illustration 2.6 Consider the graph $P_2[P_5]$. The prime cordial labeling is as shown in *Fig 5*.

Theorem 2.7 Two cycles joined by a path P_m is a prime cordial graph.

Proof: Let G be the graph obtained by joining two cycles C_n and C'_n by a path P_m. Let $v_1, v_2, v_3, \ldots\ldots v_n$, $v'_1, v'_2, v'_3 \ldots v'_n$ be the vertices of C_n, C'_n respectively. Here $u_1, u_2, u_3, \ldots$ are the vertices of P_m. We define vertex labeling $f: V(G) \rightarrow \{1, 2, 3, \ldots |V(G)|\}$ as follows. We consider following four cases.

<u>*Case 1:* m *odd,* $m \geq 5$</u>

$f(u_1) = f(v_1) = 2, f(v_2) = 4,$

$f(v_{i+2}) = 2(i+3), \quad 1 \leq i \leq n-2$

$f(u_{i+1}) = f(v_n) + 2i, \quad 1 \leq i \leq \lfloor m/2 \rfloor - 2$

$f(u_{\lfloor m/2 \rfloor}) = 6, \quad f(u_{\lfloor m/2 \rfloor+1}) = 3, \quad f(u_{\lfloor m/2 \rfloor+2}) = 1$

$f(u_{\lfloor m/2 \rfloor+2+i}) = 2i + 3, \quad 1 \leq i \leq \lfloor m/2 \rfloor - 1$

$f(v'_1) = f(u_m), \quad f(v'_{i+1}) = f(v'_1) + 2i, \quad 1 \leq i \leq n-1$

In view of the above defined labeling pattern we have

$e_f(0) = e_f(1) = n + \lfloor m/2 \rfloor.$

<u>*Case 2:* $m = 3$</u>

$f(u_1) = f(v_1) = 6, f(v_2) = 2, f(v_3) = 4,$

$f(v_{i+3}) = 2(i+3), \quad 1 \leq i \leq n-3$

$f(u_2) = 3, f(v'_1) = f(u_3) = 1, \quad f(v'_2) = 5$

$f(v'_{2+i}) = 2i + 5, \ 1 \leq i \leq n-2$

In view of the above defined labeling pattern we have

$e_f\ (0) = e_f\ (1) = n + 1$

Case 3: m even, m ≥ 4

$f(u_1) = f(v_1) = 2, f(v_2) = 4,$

$f(v_{i+2}) = 2(i + 3), \quad 1 \le i \le n - 2$

$f(u_{i+1}) = f(v_n) + 2i, \quad 1 \le i \le m/2 - 2$

$f(u_{m/2}) = 6, \quad f(u_{m/2+1}) = 3, \quad f(u_{m/2+2}) = 1$

$f(u_{m/2+2+i}) = 2i + 3, \quad 1 \le i \le m/2 - 2$

$f(v'_1) = f(u_m), \quad f(v'_{i+1}) = f(v'_1) + 2i, \quad 1 \le i \le n - 1$

In view of the above defined labeling pattern we have

$e_f\ (0) = e_f\ (1) + 1 = n + m/2$

Case 4: m = 2

$f(u_1) = f(v_1) = 2$

$f(v_{1+i}) = 2(i + 1), \quad 1 \le i \le n - 1$

$f(v'_1) = f(u_2) = 1$

$f(v'_{1+i}) = 2i+1, 1 \le i \le n - 1$

In view of the above defined labeling pattern we have

$e_f\ (0) + 1 = e_f\ (1) = n + 1.$

Thus in all cases graph G satisfies the condition $/ e_f\ (0) - e_f\ (1) / \le 1$.

That is G is a prime cordial graph.

Illustration 2.8 Consider the graph joining to copies of C_5 by the path P_7. The prime cordial labeling is as shown in *Fig 6.*

Theorem 2.9 The graph obtained by switching of an arbitrary vertex in cycle C_n admits prime cordial labeling except $n = 5$.

Proof : Let $v_1, v_2, ...v_n$ be the successive vertices of C_n and G_v denotes the graph obtained by switching of a vertex v. Without loss of generality let the switched vertex be v_1 and we initiate the labeling from the switched vertex v_1.

To define $f: V(Gv_1) \rightarrow \{1, 2, 3 ... /V(G)/\}$ we consider following four cases.

Case 1: n = 4

The case when $n=4$ is to be dealt separately. The graph Gv_1 and its prime cordial labeling is shown in *Fig* 7.

Case 2: n even, n ≥ 6

$f(v_1) = 2, f(v_2) = 1, f(v_3) = 4$

$f(v_{3+i}) = 2(i+3), \quad 1 \le i \le n/2 - 3$

$f(v_{n/2+1}) = 6, f(v_{n/2+2}) = 3$

$f(v_{n/2+2+i}) = 2i + 3, 1 \le i \le n/2 - 2$

Using above pattern we have

$e_f\ (0) = e_f\ (1) + 1 = n - 2$

Case 3: n = 5

For the graph Gv_1 the possible pairs of labels of adjacent vertices are (1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5). Then obviously $e_f(0) = 1$, $e_f(1) = 4$. That is, $|e_f(0) - e_f(1)| = 3$ and in all other possible arrangement of vertex labels we have $|e_f(0) - e_f(1)| > 3$. Thus, Gv_1 is not a prime cordial graph.

Case 4: n odd, n ≥ 7

$f(v_1) = 2, f(v_2) = 1, f(v_3) = 4$

$f(v_{3+i}) = 2(i+3), \quad 1 \le i \le \lfloor n/2 \rfloor - 3$

$f(v_{\lfloor n/2 \rfloor+1}) = 6, f(v_{\lfloor n/2 \rfloor+2}) = 3$

$f(v_{\lfloor n/2 \rfloor + 2 + i}) = 2i + 3,\ 1 \le i \le \lfloor n/2 \rfloor - 1$

Using above pattern we have

$e_f\ (0) + 1 = e_f\ (1) = n - 2$

Thus in cases 1, 2 and 4 f satisfies the condition for prime cordial labeling. That is, Gv_1 is a prime cordial graph.

Illustration 2.10 Consider the graph obtained by switching the vertex in C_7. The prime cordial labeling is as shown in *Fig 8.*

3. Concluding Remarks

It is always interesting to investigate whether any graph or graph families admit a particular type of graph labeling? Here we investigate five results corresponding to prime cordial labeling. Analogous work can be carried out for other graph families and in the context of different graph labeling problems.

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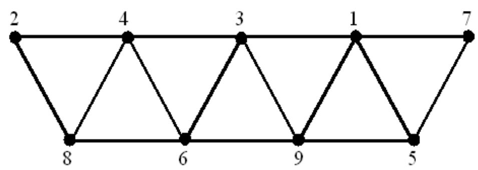

Figure 1. $T(P_5)$ and its prime cordial labeling

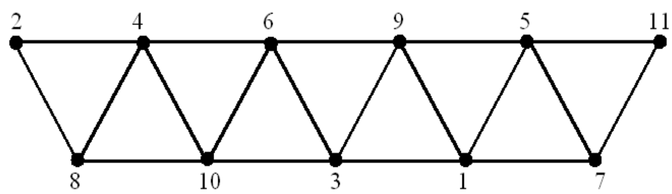

Figure 2. $T(P_6)$ and its prime cordial labeling

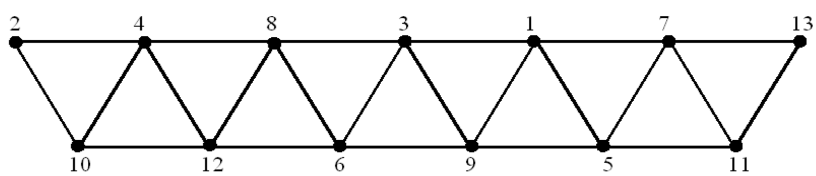

Figure 3. $T(P_7)$ and its prime cordial labeling

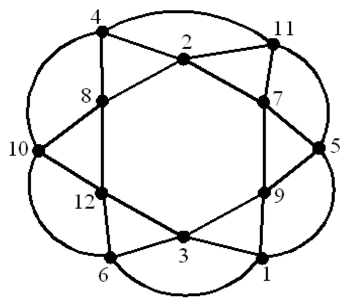

Figure 4. $T(C_6)$ and its prime cordial labeling

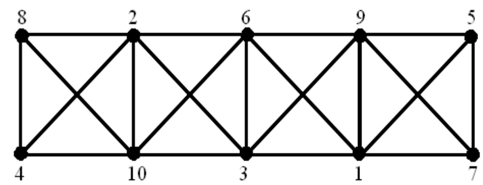

Figure 5. $P_2 [P_5]$ and its prime cordial labeling

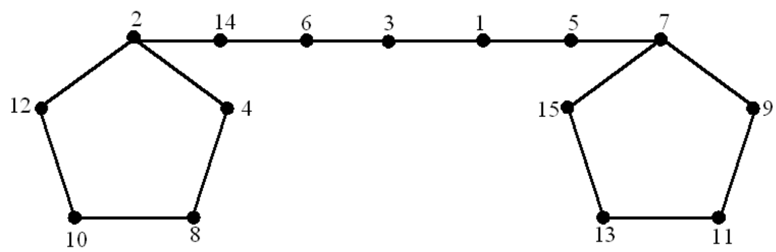

Figure 6. Two cycles C_5 join by P_7 and its prime cordial labeling

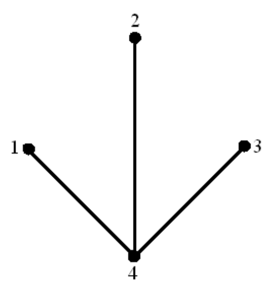

Figure 7. Vertex switching in C_4 and its prime cordial labeling

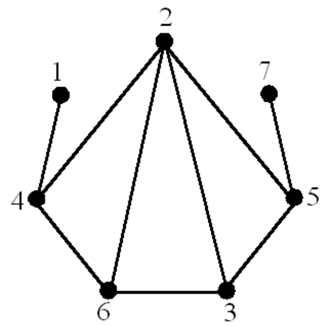

Figure 8. Vertex switching in C_7 and its prime cordial labeling