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Boundedness of Commutators on Generalized Morrey Spaces

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Abstract

In this paper, we establish the boundedness of strongly singular integrals operators T and commutators T_b on generalized Morrey spaces, where T_b are generated by $BMO(R^n)$ functions b and the strongly singular integrals operators T.

Keywords: Strongly singular integrals, Generalized Morrey spaces, Commutators, BMO functions

Let $\upsilon \in C_c^\infty(R^n)$,supp $\upsilon \subset \{x \in R^n : |x| \le 2\}$.We define strongly singular integrals kernels $K(x) = \frac{e^{i|x|^{-s'}}}{|x|^n}\upsilon(x)$,

where $0 < s < 1, s' = \frac{s}{1-s}$, and the corresponding strongly singular integrals

$$Tf(x) = p.v.\int_{R^n} K(x-y) f(y) dy .$$

Let $b \in BMO(R^n)$. We define the commutators T_b generated by functions b and operators T as the following:

$$T_b f(x) = p.v.\int_{R^n} \big(b(x) - b(y)\big)\ K(x-y) f(y) dy .$$

Wainger S.etc (Wainger, S. 1965.) had studied the boundedness of the operators T on $L^q(R^n)$. Chanillo (Chanillo, S. 1984.) developed weighted $L^q(R^n)$ theory by virtue of a basal lemma which will be mentioned later. Garcia-Cuerva J. etc (Garcia-Cuerva, J. , Harboure, E. , Segovia, C. and Torrea, J. L.) obtained the boundedness of higher-order commutators on weighted $L^q(R^n)$. Morrey (Morrey, C. B. 1938.) proposed the classical Morrey spaces when he studied the properties of local solutions of second order elliptic equations. In Ref. (Mizuhara T.. 1991.) Mizuhara introduced the following generalized Morrey spaces.

Let Φ be positive increasing functions on $(0,\infty)$ satisfying $\Phi(2r) \le D\Phi(r)$ for any $r > 0$, where $D \ge 1$ is a constant independent of r.

Definition 1 (Mizuhara T. 1991.) For $1 \le p < \infty$,we define the generalized Morrey spaces as the following

$$L^{p,\Phi}(R^n) = \left\{ f \in L^p_{loc} : \|f\|_{L^{p,\Phi}} < \infty \right\},$$

Where $\|f\|^p_{L^{p,\Phi}} = \sup_{t \in R^n, r>0} \frac{1}{\Phi(r)} \int_{B(t,r)} |f(x)|^p dx$, $B(t,r)$ is a sphere r in radius whose center is at point t.

We know that when $\Phi(r) = r^\lambda, (0 < \lambda < n)$, $L^{p,\Phi}$ are the classical Morrey spaces.

In this paper we will show the boundedness of strongly singular integrals operators T on generalized Morrey spaces by weighted inequations. Furthermore, we will show the boundedness of the commutators generalized by BMO functions b of operators T by virtue of sharp estimate.

1. Boundedness of strongly singular integrals operators T on generalized Morry spaces

We fix the following notations in Lemma 1 and Theorem 1:

For $\forall x_0 \in R^n, r>0,$ $f(y)=f\chi_{B(x_0,2r)}(y)+\sum_{k=1}^{\infty} f\chi_{B(x_0,2^{k+1}r)\setminus B(x_0,2^k r)}(y)\equiv \sum_{k=0}^{\infty} f_k(y)$.

Lemma 1 For $1\le D(\Phi)<2^n, 1<p<\infty, f\in L^{p,\Phi}$, if T are strongly singular integrals then for $\forall k,(k>0)$ we have:

$$\int_{B(x_0,r)} |T(f_k)(x)|^p dx \le C\left(\frac{D}{2^{\theta n}}\right)^k \|f\|_{L^{p,\Phi}}^p \Phi(r),$$

Where $\theta \in \left(\frac{\ln D}{\ln 2^n},1\right)$.

Proof: By the weighted property of A_p, for any $\alpha\in(0,1)$, $\forall k>0$, $\left(M\chi_{B(x_0,r)}\right)^\alpha \in A_1 \subset A_p, (1<p\le\infty)$, we have

$$\int_{B(x_0,r)} |T(f_k)(x)|^p dx \le C\int_{B(x_0,r)}\left|\int_{B(x_0,2^{k+1}r)}\frac{|f_k(y)|}{|x-y|^n}dy\right|^p dx$$

$$\le C\int_{B(x_0,r)}|M(|f_k|)(x)|^p dx$$

$$\le C\int_{R^n}|M(|f_k|)(x)|^p \left(M\chi_{B(x_0,r)}\right)^\theta dx$$

$$\le C\int_{R^n}|f_k(x)|^p \left(M\chi_{B(x_0,r)}\right)^\theta dx$$

$$\le C\frac{1}{\left(2^{k-1}\right)^{n\theta}}\int_{B(x_0,2^{k+1}r)}|f_k(x)|^p dx$$

$$\le C\frac{1}{\left(2^{k-1}\right)^{n\theta}}\|f\|_{L^{p,\Phi}}^p \Phi(2^{k+1}r)$$

$$\le C\left(\frac{D}{2^{\theta n}}\right)^k \|f\|_{L^{p,\Phi}}^p \Phi(r).$$

Theorem 1 For $1\le D(\Phi)<2^n, 1<p<\infty$, and T are strongly singular integrals then T are bounded on $L^{p,\Phi}(R^n)$.

Proof: $\forall x_0\in R^n, r>0,$ by the boundedness of T on $L^p(p>1)$ (Wainger, S. 1965.), we get

$$\int_{B(x_0,r)}|T(f_0)(x)|^p dx \le C\int_{B(x_0,2r)}|f(x)|^p dx \le C\|f\|_{L^{p,\Phi}}^p \Phi(r).$$

And by Lemma 1 we have

$$\left[\int_{B(x_0,r)}|T(f)(x)|^p dx\right]^{\frac{1}{p}} \le \sum_{k=0}^{\infty}\left[\int_{B(x_0,r)}|T(f_k)(x)|^p dx\right]^{\frac{1}{p}}$$

$$\le C\left[1+\sum_{k=1}^{\infty}\left(\frac{D}{2^{n\theta}}\right)^{\frac{k}{p}}\right]\|f\|_{L^{p,\Phi}}\Phi^{\frac{1}{p}}(r)$$

$$\le C\|f\|_{L^{p,\Phi}}\Phi^{\frac{1}{p}}(r).$$

We used $\theta\in\left(\frac{\ln D}{\ln 2^n},1\right)$ in the last inequation.

2. Boundedness of the commutators on generalized Morrey spaces

Lemma 2 (Mizuhara T. 1991.) For $1\le D(\Phi)<2^n, 1\le q<p<\infty$, then $M_q f(x)=\left\{\left(M|f|^q\right)(x)\right\}^{\frac{1}{q}}$ are bounded on $L^{p,\Phi}(R^n)$, where M are H-L maximum operators.

Lemma 3 (Ding, Y. 1997.) For $1 \le D(\Phi) < 2^n, 1 < p < \infty$, then for $\forall f \in L^{p,\Phi}(R^n)$ we have

$$\|Mf\|_{L^{p,\Phi}} \le C\|f^{\#}\|_{L^{p,\Phi}},$$

where $f^{\#}(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y) - f_Q| dy$.

Lemma 4 If T are strongly singular integrals, $1 < r, s < \infty$, $b \in BMO(R^n)$, then for $a.e. x \in R^n$, $\forall f \in L^{p,\Phi}(R^n)$, $(1 < p < \infty)$ there exists a constant C independent of b, f so that the following inequations hold

$$(T_b f)^{\#}(x) \le C\|b\|_* \left\{ \left(M|Tf|^r\right)^{\frac{1}{r}}(x) + \left(M|f|^t\right)^{\frac{1}{t}}(x) \right\}.$$

Proof: We only need to show for any $x_0 \in R^n$ the following inequations hold

$$(T_b f)^{\#}(x_0) \le C\|b\|_* \left\{ \left(M|Tf|^r\right)^{\frac{1}{r}}(x_0) + \left(M|f|^t\right)^{\frac{1}{t}}(x_0) \right\}.$$

We choose a cube Q with side length δ, $s.t. x_0 \in Q$. Let $4\delta_0 = \delta_0^{\frac{1}{1+s'}}$ and we know δ_0 is a constant less than 1.

Case 1: $\delta < \delta_0$.

Suppose cube $\tilde{Q}$ with side length $\delta^{\frac{1}{1+s'}}$ has the same center as Q. We have

$$f = f\chi_{4Q} + f\chi_{\tilde{Q}\backslash 4Q} + f\chi_{R^n\backslash\tilde{Q}} \equiv f_1 + f_2 + f_3.$$

$$T_b f(x) = \int_{R^n} (b_Q - b(y))k(x-y)f(y)dy - (b_Q - b(x))\int_{R^n} k(x-y)f(y)dy.$$

Let $C_Q = \frac{1}{|Q|}\int_Q \int_{R^n} (b_Q - b(y))k(z-y)f_3(y)dydz$. It is obvious that C_Q is a constant. So

$$\frac{1}{|Q|}\int_Q |T_b(f)(x) - C_Q| dx$$

$$\le \frac{1}{|Q|}\int_Q |(b_Q - b(x))Tf(x)| dx + \frac{1}{|Q|}\int_Q \left|\int_{R^n} (b_Q - b(y))k(x-y)f_1(y)dy\right| dx$$

$$+ \frac{1}{|Q|}\int_Q \left|\int_{R^n} (b_Q - b(y))k(x-y)f_2(y)dy\right| dx$$

$$+ \frac{1}{|Q|}\int_Q \left(\frac{1}{|Q|}\int_Q \left|\int_{R^n} (b_Q - b(y))\big(k(x-y) - k(z-y)\big)f_3(y)dy\right| dz \right) dx$$

$$= A_1 + A_2 + A_3 + A_4.$$

As for A_1, we have

$$A_1 \le C\left(\frac{1}{|Q|}\int_Q |b_Q - b(x)|^{r'} dx\right)^{\frac{1}{r'}} \left(\frac{1}{|Q|}\int_Q |Tf(x)|^r dx\right)^{\frac{1}{r}}$$

$$\le \|b\|_* \left(M(|Tf|)^r\right)^{\frac{1}{r}}(x_0).$$

As for A_2, for $1 < q < t, u > 1$ so that $qu = t$, by *Hölder* inequation we have

$$A_2 \le C\left\{\frac{1}{|Q|}\int_Q |T((b_Q - b)f_1)(x)|^q dx\right\}^{\frac{1}{q}}$$

$$\leq C\left\{\frac{1}{|Q|}\int_{4Q}\left|(b_Q - b)f(x)\right|^q dx\right\}^{\frac{1}{q}}$$

$$\leq C\left\{\frac{1}{|Q|}\int_{4Q}\left|(b_Q - b(x)\right|^{qu'} dx\right\}^{\frac{1}{qu'}}\left\{\frac{1}{|Q|}\int_{4Q}\left|f(x)\right|^{qu} dx\right\}^{\frac{1}{qu}}$$

$$\leq C\|b\|_*\left(M|f|^t\right)^{\frac{1}{t}}(x_0).$$

As for A_4 ,by noting $x \in Q, z \in Q, y \notin \widetilde{Q}$, and mean value theorem, we have

$$|k(x-y)-k(z-y)| \leq \frac{C\delta}{|y-z|^{n+s'+1}}.$$

$$A_4 \leq C\frac{1}{|Q|}\int_Q\left(\delta\int_{|y-x_0|>\frac{1}{2}\delta^{\frac{1}{1+s'}}}|b_Q - b(y)||y-z|^{-(n+s'+1)}|f(y)|dy\right)dz$$

$$\leq C\frac{1}{|Q|}\int_Q\left(\sum_{k=0}^{\infty}2^{-k}\left(2^k\delta\right)^{\frac{-n}{1+s'}}\int_{|y-z|<2\left(2^k\delta\right)^{\frac{1}{1+s'}}}|b_Q - b(y)||f(y)|dy\right)dz$$

$$\leq C\frac{1}{|Q|}\int_Q dz\sum_{k=0}^{\infty}2^{-k}\left\{\left(2^k\delta\right)^{\frac{-n}{1+s'}}\int_{|y-z|<2\left(2^k\delta\right)^{\frac{1}{1+s'}}}|b_Q - b(y)|^{t'}dy\right\}^{\frac{1}{t'}}\left\{\left(2^k\delta\right)^{\frac{-n}{1+s'}}\int_{|y-z|<2\left(2^k\delta\right)^{\frac{1}{1+s'}}}|f(y)|^t dy\right\}^{\frac{1}{t}}$$

$$\leq C\|b\|_*\left(M|f|^t\right)^{\frac{1}{t}}(x_0).$$

As for A_3 ,we use the similar ways in chanillo (Chanillo, S. 1984.). Let $l > 1, m > 1$ satisfying $2+s' < l', lm = t$,then

$$\int_{R^n}(b_Q - b(y))k(x-y)f_2(y)dy$$

$$= \int_{R^n}\frac{e^{i|x-y|^{-s'}}v(x-y)}{|x-y|^{\frac{n(2+s')}{l'}}}\left(\frac{1}{|x-y|^{n(1-\frac{2+s'}{l'})}} - \frac{1}{|x_0-y|^{n(1-\frac{2+s'}{l'})}}\right)(b_Q - b(y))f_2(y)dy$$

$$+ \int_{R^n}\frac{e^{i|x-y|^{-s'}}v(x-y)}{|x-y|^{\frac{n(2+s')}{l'}}}\frac{(b_Q - b(y))f_2(y)}{|x_0-y|^{n(1-\frac{2+s'}{l'})}}dy$$

$$= B_1 + B_2.$$

As for B_1 ,by the mean value theorem, we get

$$B_1 \leq C\sum_{k=0}^{\infty}2^{-k}\left(2^k\delta\right)^{-n}\int_{|y-x_0|<2^{k+1}\delta}|b_Q - b(y)||f(y)|dy$$

$$\leq C\|b\|_*\left(M|f|^t\right)^{\frac{1}{t}}(x_0).$$

As for the estimate of B_2 , we need the following lemma.

Lemma 5 (Chanillo, S.) For $0 < s < 1, s' = \dfrac{s}{1-s}$, $t > 1$, $\dfrac{s'+2}{t} < 1$ $\widetilde{K}_{s',t} \equiv \dfrac{e^{i|x|^{-s'}}}{|x|^{\frac{n(s'+2)}{t}}}$,we have $\forall f \in L^{t'}(R^n)$,

$$\left\|\widetilde{K}_{s',t} * f\right\|_{L^t(R^n)} \leq C\|f\|_{L^{t'}(R^n)}.$$

By Lemma 5 and using the similar ways in chanillo (Chanillo, S. 1984.), we get

$$\frac{1}{|Q|}\int_Q |B_2(x)|dx \le C|Q|^{-\frac{1}{l'}}\left(\int_{R^n}\frac{|b_Q-b(y)|^l|f_2(y)|^l}{|x_0-y|^{nl(1-\frac{2+s'}{l'})}}dy\right)^{\frac{1}{l}}.$$

And $\frac{1}{|Q|}\int_Q |B_2(x)|dx$

$$\le C|Q|^{-\frac{1}{l'}}\left\{\sum_{k=0}^{k_0}\left(2^k\delta\right)^{(l-1)(1+s')n}\left(2^k\delta\right)^{-n}\int_{|y-x_0|<2^{k+1}\delta}|b_Q-b(y)|^l|f(y)|^l dy\right\}^{\frac{1}{l}}$$

Where k_0 satisfies $2^{k_0}\delta<\delta^{\frac{1}{1+s'}}<2^{k_0+1}\delta$.

$$\le C|Q|^{-\frac{1}{l'}}\sum_{k=0}^{k_0}\left(2^k\delta\right)^{\frac{(l-1)(1+s')n}{l}}\left\{\left(2^k\delta\right)^{-n}\int_{|y-x_0|<2^{k+1}\delta}|b_Q-b(y)|^{lm'}dy\right\}^{\frac{1}{lm'}}$$

$$\times\left\{\left(2^k\delta\right)^{-n}\int_{|y-x_0|<2^{k+1}\delta}|f(y)|^{lm}dy\right\}^{\frac{1}{lm}}$$

$$\le C\|b\|_*\left(M|f|^t\right)^{\frac{1}{t}}(x_0).$$

Case 2: $\delta\ge\delta_0$.

Let $\gamma=5\delta_0^{-1}, f=f\chi_{\gamma Q}+f\chi_{R^n\setminus\gamma Q}$.

$$\frac{1}{|Q|}\int_Q|T_b(f)(x)|dx\le\frac{1}{|Q|}\int_Q|(b_Q-b(x))Tf(x)|dx$$

$$+\quad\frac{1}{|Q|}\int_Q\left|\int_{\gamma Q}(b_Q-b(y))k(x-y)f(y)dy\right|dx$$

$$+\quad\frac{1}{|Q|}\int_Q\left|\int_{R^n\setminus\gamma Q}(b_Q-b(y))k(x-y)f(y)dy\right|dx$$

$$=D_1+D_2+D_3.$$

And the discussion of D_1 is similar with A_1, D_2 is similar with A_2, so we omit the proof here..

We note that when $x\in Q, y\notin\gamma Q, |x-y|>2$, $k(x-y)$ is equal to 0.

So we get the lemma.

Theorem 2: For $1\le D(\Phi)<2^n, 1<p<\infty$, if T are strongly singular integrals,then T_b are bounded on $L^{p,\Phi}(R^n)$.

Proof: By Lemma 2,3,4 and Theorem 1, we get

$$\|T_bf\|_{L^{p,\Phi}}\le\|M(T_bf)\|_{L^{p,\Phi}}\le C\left\|(T_bf)^{\#}\right\|_{L^{p,\Phi}}\le C\|b\|_*\left\|\left(M\left(|Tf|^r\right)\right)^{\frac{1}{r}}+\left(M\left(|f|^s\right)\right)^{\frac{1}{s}}\right\|_{L^{p,\Phi}},$$

where $\left\|\left(M\left(|Tf|^r\right)\right)^{\frac{1}{r}}\right\|_{L^{p,\Phi}}\le\|Tf\|_{L^{p,\Phi}}\le\|f\|_{L^{p,\Phi}}$, and we can choose $r<p$ by Lemma 4. So

$$\left\| \left(M\left(|f|^s \right) \right)^{\frac{1}{s}} \right\|_{L^{p,\Phi}} \leq \|f\|_{L^{p,\Phi}} .$$

We get the theorem.

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