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Existence of Nonoscillatory Solutions for a Class of N-order Neutral Differential Systems

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Abstract

Using fixed point theorem through the method of structural compression mapping on set of bounded continuous function, this paper qualitative studies the existence of nonoscillatory solutions for a class of n-order neutral differential systems, and obtains some sufficient conditions of nonoscillation existence of solution for this class systems.

Keywords: Neutral, Differential system, Nonoscillatory solution, Compression mapping, Fixed point theorem

1. Introduction

Consider the existence of nonoscillatory solutions for n-order neutral differential systems

$$[x(t) - \sum_{i=1}^{l} p_i(t)x(t-\tau_i(t))]^{(n)} + (-1)^{n+1}\sum_{j=1}^{r} Q_j(t) f_j(t, x(\sigma_j(t))) = 0 \tag{1.1}$$

where n is a positive integer and the followings are always satisfied:

(A_1) $p_i(t), Q_j(t) \in C\left[[t_0,\infty), R^+\right], \tau_i(t), \sigma_j(t) \in C\left[[t_0,\infty), R\right]$, $\lim_{t\to\infty}(t-\tau_i(t)) = +\infty$,

$\lim_{t\to\infty}\sigma_j(t) = +\infty \quad (i = 1,2\cdots l; j = 1,2\cdots r)$.

(A_2) $f_j(t,y) \in C\left[[t_0,\infty)\times R, R\right]$,for enough big t,when $y \neq 0$, $yf_j(t,y) > 0$, $(j = 1,2\cdots r)$ holds.

(A_3) Function $f_j(t,y)$ satisfies local *Lipschitz* condtion for y ,i.e. exists constant $L > 0$ and $\delta > 0$,which agrees that $\left|f_j(t,y_1) - f_j(t,y_2)\right| \le L\left|y_1 - y_2\right|$ when $-\delta < y_1, y_2 < \delta$, $(j = 1,2\cdots r)$.

The equation (1.1) solution is oscillatory, if it has arbitrary big zero; Otherwise, it is nonoscillatory.

On the century 90's, it appeared many papers (Yan, Jurang. 1990.)(Wang, Zhibin. 1995.)(Wang, Lianwen. 1995.)(Li, Guanghua, Yu, Yuanhong, & Lin, ShiZhong. 1997)(Wang, Guangpei. 2000.)(Li, Hongfei, & Wang, Zhibin. 2000.) about oscillatory solutions for higher order neutral differential equation. These articles have given some solution oscillatory conditions. However, there have few researches for nonoscillatory solution. The most earliest research is discussed by Zhang Binggen in paper (Zhang , Binggen. 1996.) for the existence of nonoscillatory solution

$$\left[x(t) - h(t)x(\tau(t))\right]^{(n)} + \sum_{i=1}^{m} p_i(t)x(g_i(t)) = 0 \quad , \tag{1.2}$$

Afterwards, Shen Jianhu, Yu Jianshe studies the more generaler equation than (1.2) in paper (Shen, Jianhua, & Yu, Jianshe. 1996.):

$$\left[x(t)-p(t)x(\tau(t)\right]^{(n)}+f(t,x(t-\sigma_1(t)),\cdots x(t-\sigma_m(t)))=0 \tag{1.3}$$

This paper established the sufficient condition for bounded positive solution existence of equation (1.3)when neutral item $p(t)\equiv 1$ and $P(t)\equiv p\neq 1$. In recent years, papers (Tang, Xianhua, & Yu, Jianshe. 2000.)(Liu, Kaiyu. 2000.) have considered the special situation of equation (1.3)

$$\left[x(t)-p(t)x(t-\tau)\right]^{(n)}+Q(t)x(t-\sigma)=0 \tag{1.4}$$

which established the sufficient condition of solution existence of equation (1.4) when neutral item $p(t)$ has initial change among 1.

This paper uses *Banach* compression reflection principle, discusses nonoscillation existence of solution for the generaler equation than equation (1.1)of papers (Zhang, Binggen. 1996.)(Shen, Jianhua, & Yu, Jianshe. 1996.)(Tang, Xianhua, & Yu, Jianshe. 2000.)(Liu, Kaiyu. 2000), and obtains the sufficient conditions of nonoscillation existence of solution for this class systems, whose conclusions are different from papers (Zhang, Binggen. 1996.)(Shen, Jianhua, & Yu, Jianshe. 1996.)(Tang, Xianhua, & Yu, Jianshe. 2000.)(Liu, Kaiyu. 2000).

2. Conclusions and Proof

Lemma 2.1: (*Banach* compression mapping principle) Supposes mapping $A:\Omega\to\Omega$ is a compression mapping in the complete distance space $(\Omega,\|\bullet\|)$, then A must exist only a fixed point $x\in\Omega$ satisfying $Ax=x$.

Theorem 2.1: Suppose (A_1),(A_2), (A_3) hold, and exist positive constant c and nonnegative number k which satisfying

$$\limsup_{t\to\infty}\left[\sum_{i=1}^{l}p_i(t)e^{k\tau_i(t)}+\frac{Le^{kt}}{(n-1)!}\sum_{j=1}^{r}\int_t^\infty (s-t)^{n-1}Q_j(s)e^{-k\sigma_j(s)}ds\right] \tag{2.1}$$
$$<1$$

,then the equation exists eventually positive solution $x(t)$ and $0<x(t)\le ce^{-kt}$.

Proof: From theorem, there are $T>t_0$ and $0<\alpha<1$ satisfying the following equation when $t\ge T$

$$t-\tau_i(t)\ge t_0,\sigma_j(t)\ge t_0, \qquad (i=1,2\cdots l;j=1,2\cdots r)$$

$$\sum_{i=1}^{l}p_i(t)e^{k\tau_i(t)}+\frac{Le^{kt}}{(n-1)!}\sum_{j=1}^{r}\int_t^\infty (s-t)^{n-1}Q_j(s)e^{-k\sigma_j(s)}ds<\alpha\ . \tag{2.2}$$

Denote set $C_B=\{x(t):[t_0,\infty)\to R\}$ is a bounded continuous function set in $[t_0,\infty)\to R$,and define $\|x(t)\|=\sup_{t\ge t_0}|x(t)|$ in C_B ,then C_B is a *Banach* space.

Denote $\Omega=\{x(t)\in C_B:0\le x(t)\le 1,t>t_0\}$.

the following proof is that the set Ω is a bounded closed convex subset in C_B.

It is obvious that Ω is boundary.

For arbitrary $x,y\in\Omega$,since $0\le ax(t)+(1-a)y(t)\le 1, a\in[0,1]$, Ω is a convex set.

If $x_n\in\Omega$ and $\lim_{n\to\infty}\|x_n-x_0\|=0$, when $n\to\infty$,we have

$$|x_0|\le|x_0-x_n|+|x_n|\le\|x_0-x_n\|+1\to 1\ ,\ 0\le x_n\le\|x_n-x_0\|+x_0\to x_0$$

Hence, $x_0\in\Omega$. From the above, Ω is a bounded closed convex subset in C_B.

Define mapping in Ω

$$(Ax)(t)=\begin{cases}\sum_{i=1}^{l}p_i(t)e^{k\tau_i(t)}x(t-\tau_i(t))+\dfrac{e^{kt}}{c(n-1)!}\sum_{j=1}^{r}\int_t^{\infty}(s-t)^{n-1}Q_j(s)\\ \qquad f_j(s,ce^{-k\sigma_j(s)}x(\sigma_j(s)))ds,t\ge T;\\ 1-\dfrac{t}{T}+\dfrac{t}{T}(Ax)(T),t_0\le t\le T,\end{cases}\tag{2.3}$$

where c,k,δ satisfy inequality $0<ce^{-kt}<\delta$.

The following will prove that mapping A is self and compressive .

(1)For arbitrary $x(t)\in\Omega$ and $t>T$, from (2.2)and (2.3), we have

$$\begin{aligned}(Ax)(t)&=\sum_{i=1}^{l}p_i(t)e^{k\tau_i(t)}x(t-\tau_i(t))\\&+\frac{e^{kt}}{c(n-1)!}\sum_{j=1}^{r}\int_t^{\infty}(s-t)^{n-1}Q_j(s)f_j(s,ce^{-k\sigma_j(s)}x(\sigma_j(s)))ds\\&\le\sum_{i=1}^{l}p_i(t)e^{k\tau_i(t)}+\frac{Le^{kt}}{(n-1)!}\sum_{j=1}^{r}\int_t^{\infty}(s-t)^{n-1}Q_j(s)e^{-k\sigma_j(s)}ds\\&<\alpha\\&<1.\end{aligned}\tag{2.4}$$

When $t_0\le t\le T$,we have

$$(Ax)(t)=1-\frac{t}{T}+\frac{t}{T}(Ax)(T)\le 1\cdot\tag{2.5}$$

The above indicates $A(\Omega)\in\Omega$,i.e. A is a self mapping in Ω.

(2)Without loss generality, suppose $x_1(t),x_2(t)\in\Omega$,and when $t\ge T$, from equation(2.3), we have

$$\begin{aligned}&\left|(Ax_1)(t)-(Ax_2)(t)\right|\le\sum_{i=1}^{l}p_i(t)e^{k\tau_i(t)}\left|x_1(t-\tau_i(t))-x_2(t-\tau_i(t))\right|\\&+\frac{e^{kt}}{c(n-1)!}\sum_{j=1}^{r}\int_t^{\infty}(s-t)^{n-1}Q_j(s)\\&\left|f_j(s,ce^{-k\sigma_j(s)}x_1(\sigma_j(s)))-f_j(s,ce^{-k\sigma_j(s)}x_2(\sigma_j(s)))\right|ds\\&\le(\sum_{i=1}^{l}p_i(t)e^{k\tau_i(t)}+\frac{Le^{kt}}{(n-1)!}\sum_{j=1}^{r}\int_t^{\infty}(s-t)^{n-1}Q_j(s)e^{-k\sigma_j(s)}ds)\left\|x_1-x_2\right\|\\&\le\alpha\left\|x_1-x_2\right\|.\end{aligned}\tag{2.6}$$

When $t_0\le t\le T$,we have

$$\begin{aligned}&\left|(Ax_1)(t)-(Ax_2)(t)\right|\le\frac{t}{T}\left|(Ax_1)(T)-(Ax_2)(T)\right|\\&\le\alpha\left\|x_1-x_2\right\|.\end{aligned}\tag{2.7}$$

The above indicates that $(Ax)(t)$ is a compressive mapping. Therefore it exists fixed point $x^*(t)\in\Omega$ satisfying $(Ax^*)(t)=x*(t)$, namely

$$x^*(t)=\begin{cases}\sum_{i=1}^{l}p_i(t)e^{k\tau_i(t)}x^*(t-\tau_i(t))+\dfrac{e^{kt}}{c(n-1)!}\sum_{j=1}^{r}\int_t^{\infty}(s-t)^{n-1}Q_j(s)\\ \qquad f_j(s,ce^{-k\sigma_j(s)}x^*(\sigma_j(s)))ds,t\ge T;\\ 1-\dfrac{t}{T}+\dfrac{t}{T}(Ax^*)(T),t_0\le t\le T.\end{cases}\tag{2.8}$$

Taking $x(t)=cx^*(t)e^{-kt}$ $(t>t_0)$, from (2.8) we have

$$x(t)=\sum_{i=1}^{l}p_i(t)ce^{-k(t-\tau_i(t))}x^*(t-\tau_i(t))$$
$$+\frac{1}{(n-1)!}\sum_{j=1}^{r}\int_t^{\infty}(s-t)^{n-1}Q_j(s)f_j(s,ce^{-k\sigma_j(s)}x^*(\sigma_j(s)))ds \tag{2.9}$$
$$=\sum_{i=1}^{l}p_i(t)x(t-\tau_i(t))$$
$$+\frac{1}{(n-1)!}\sum_{j=1}^{r}\int_t^{\infty}(s-t)^{n-1}Q_j(s)f_j(s,x(\sigma_j(s)))ds,(t\ge T).$$

Derivate equation (2.9) n times, equation (1.1) is gotten. So $x(t)=cx^*(t)e^{-kt}$ is the eventually positive solution of equation(1.1)and $0<x(t)\le ce^{-kt}$. The proof is completed.

Corollary 2.1: If conditions (A_1),(A_2), (A_3) are satisfied and inequalities

$$\sum_{i=1}^{l}p_i(t)\le c,0<c<1, \tag{2.10}$$

$$\int_t^{\infty}s^{n-1}Q_j(s)ds<\infty, j=1,2\cdots r \tag{2.11}$$

hold,then the equation(1.1)exists eventually positive solution.

Proof: From(2.10),(2.11), we have

$$\limsup_{t\to\infty}\sum_{i=1}^{l}p_i(t)\le c<1, \tag{2.12}$$

$$\lim_{t\to\infty}\sum_{j=1}^{r}\int_t^{\infty}(s-t)^{n-1}Q_j(s)ds=0\cdot \tag{2.13}$$

Then when K=0, we have

$$\limsup_{t\to\infty}\left[\sum_{i=1}^{l}p_i(t)e^{k\tau_i(t)}+\frac{Le^{kt}}{(n-1)!}\sum_{j=1}^{r}\int_t^{\infty}(s-t)^{n-1}Q_j(s)e^{-k\sigma_j(s)}ds\right]$$
$$=\limsup_{t\to\infty}\left[\sum_{i=1}^{l}p_i(t)+\frac{L}{(n-1)!}\sum_{j=1}^{r}\int_t^{\infty}(s-t)^{n-1}Q_j(s)ds\right]$$
$$<1$$

From theorem 2.1, it is obtained that the equation(1.1)exists eventually positive solution.

Corollary 2.2 When conditions (A_1),(A_2), (A_3) hold ,if positive constants k,p_i,τ_i,σ_i satisfy $p_1(t)\le p_i,Q_j(t)\le Q_j$, $\tau_i(t)\le\tau_{i,}\sigma_j(t)\ge t-\sigma_j$, $(i=1,2\cdots l;j=1,2\cdots r)$,and the inequality

$$k^n(1-\sum_{i=1}^{l}p_ie^{k\tau_i})>L\sum_{j=1}^{r}Q_je^{k\sigma_j} \tag{2.14}$$

hold,then equation(1.1)exists eventually positive solution.

Proof From condition (2.14)in Corollary 2.1,

$$\limsup_{t\to\infty}\left[\sum_{i=1}^{l}p_i(t)e^{k\tau_i(t)}+\frac{Le^{kt}}{(n-1)!}\sum_{j=1}^{r}\int_t^{\infty}(s-t)^{n-1}Q_j(s)e^{-k\sigma_j(s)}ds\right]$$
$$\le\limsup_{t\to\infty}\left[\sum_{i=1}^{l}p_ie^{k\tau_i}+\frac{Le^{kt}}{(n-1)!}\sum_{j=1}^{r}\int_t^{\infty}(s-t)^{n-1}Q_je^{-k(s-\sigma_j)}ds\right]$$
$$=\sum_{i=1}^{l}p_ie^{k\tau_i}+\frac{L}{k^n}\sum_{j=1}^{r}Q_je^{k\sigma_j}<1.$$

Then equation (1.1) exists eventually positive solution by theorem 2.1.

In equation(1.1),when $f_j(t,x(\sigma_j(t)))=x(\sigma_j(t))$,equation

$$[x(t)-\sum_{i=1}^{l}p_i(t)x(t-\tau_i)]^{(n)}+(-1)^{n+1}\sum_{j=1}^{r}Q_j(t)x(\sigma_j(t))=0\cdot \tag{2.15}$$

is gotten.

Corollary 2.3 When condition (A_1) is satisfied, and if there exists constant $k \geq 0$, the inequality

$$\limsup_{t\to\infty}\left[\sum_{i=1}^{l} p_i(t)e^{k\tau_i(t)} + \frac{e^{kt}}{(n-1)!}\sum_{j=1}^{r}\int_t^{\infty}(s-t)^{n-1}Q_j(s)e^{-k\sigma_j(s)}ds\right] < 1$$

holds,then equation(2.15)exists eventually positive solution.

Example: Suppose $k > 0$ and $\frac{d^n}{d^n x}\left[x(t) - \frac{\sin^2 t}{e^k(1+\sin^2 t)}x(t-1)\right] + (-1)^{n+1}\left(a(t)x(t) + b(t)x(t-1)\right) = 0$. When $t \geq t_0$, $a(t) + e^k b(t) < \frac{k^n}{1+\sin^2 t}, a(t) > 0, b(t) > 0$, whether the equation does have the eventually positive solution?

Solution: Since $0 \leq p(t) = \frac{\sin^2 t}{e^k(1+\sin^2 t)} \leq e^{-k} < 1$, from theorem 1.1,we have

$$\begin{aligned}
&\limsup_{t\to\infty}\left[\sum_{i=1}^{l} p_i(t)e^{k\tau_i(t)} + \frac{e^{kt}}{(n-1)!}\sum_{j=1}^{r}\int_t^{\infty}(s-t)^{n-1}Q_j(s)e^{-k\sigma_j(s)}ds\right] \\
&= \limsup_{t\to\infty}\left[p(t)e^k + \frac{e^{kt}}{(n-1)!}\int_t^{\infty}(s-t)^{n-1}[a(s)e^{-ks} + b(s)e^{-k(s-1)}]ds\right] \\
&< \limsup_{t\to\infty}\left[\frac{\sin^2 t}{1+\sin^2 t} + \frac{k^n}{(1+\sin^2 t)(n-1)!}\int_t^{\infty}(s-t)^{n-1}e^{-k(s-t)}ds\right] \\
&\leq \limsup_{t\to\infty}\left[\frac{\sin^2 t}{1+\sin^2 t} + \frac{1}{1+\sin^2 t}\right] = 1,
\end{aligned}$$

Which indicates this equation has eventually positive solutions.

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