Modern Applied Science
www.ccsenet.org/mas
Vol. 4, No. 2
February 2010

New Algorithm for Listing All Permutations

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Abstract

The most challenging task dealing with permutation is when the element is large. In this paper, a new algorithm for listing down all permutations for n elements is developed based on distinct starter sets. Once the starter sets are obtained, each starter set is then cycled to obtain the first half of distinct permutations. The complete list of permutations is achieved by reversing the order of the first half of permutation. The new algorithm has advantages over the other methods due to its simplicity and easy to use.

Keywords: Permutation, Starter sets, Cycle

1. Introduction

A permutation of elements is an arrangement of those elements in some order, that is, some element is placed in the first position, another in the second position, and so on, until all elements have been placed. Aside from theoretical interest in set theory, permutations have some practical use. Permutation generation is an important problem in computer science and a great variety of solutions have been proposed in literature for sequential as well as parallel machines (Djamegni, & Tchuente, 1997). Permutations can be used to define switching networks in computer networking and parallel processing.

Various studies also have been made on generating algorithms for permutations and their restrictions for example permutation with given ups and downs (Korsh, 2001), fixed number of inversions (Effler & Ruskey, 2003), derangements (Baril & Vajnovszki, 2004), indecomposable/irreducible permutation (King, 2006), fixed number of cycles (Baril, 2007), involutions (Baronaigien, 1992), fixed-point free involutions (Walsh, 2003) and the generalization of multiset permutations (Vajnovszki, 2003) as cited by Baril (2007). There is also algorithm for linear extensions of posets (Knuth, 2002; Prusse & Ruskey, 1994) cited by Effler & Ruskey (2003).

In this study we will attempt to use an alternative approach to list all permutations. Given n elements, we try to come up with a set of n elements for a start. Then form these sets we will then generate all $n!$ permutations. To our present knowledge no study has been done using this kind of approach.

2. Prelimary Definitions and Theorems

Below are definitions and theorems needed to be understood and defined carefully before the algorithm can be developed.

Definition 1: A *starter set* is a set that is used as a basis to enumerate other permutations.

Definition 2: An *equivalence starter set* is a set that can produce the same permutation from any other starter set.

Definition 3: A reversed set is a set that is produced by reversing the order of permutation set.

Definition 4: A 2-cycle is called a *transposition*.

Theorem 1. The number of r-permutation of a set with n distinct elements is

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1).$$

Theorem 2: For integer $n > 0$, $n! = n(n-1)!$

Theorem 3: Any permutation is a product of transposition.

Theorem 4: (Steinhaus) There is a sequence of (not necessarily distinct) 2-cycles, $(a_1, b_1), \ldots, (a_n, b_n)$, where $n = n! - 1$, such that each non-trivial permutation f of $\{1, 2, \ldots, n\}$ may be expressed in the form

$$f = \prod_{i=1}^{k} (a_i, b_i)$$

for some k, $1 \le k \le N$. Furthermore, these products (for $k = 1, 2, \ldots, N$) are all distinct.

3. Derivation of Algorithm

We now focus our attention to generate permutation for $n = 2$, $n = 3$, $n = 4$ and $n = 5$. We will enumerate for these four cases as a foundation to generalize for n elements. We eliminate for $n = 1$ because it is the trivial case. Now we take a look at other different cases for example $n = 2$, $n = 3$, $n = 4$ and $n = 5$. Let S be the set of n elements.

$S = \{1, 2, 3, \ldots, n\}$.

<u>Case $n = 2$.</u>

In this case there are two permutations since 2! = 2. Let $S = \{1, 2\}$. The permutations for this case are:

{1, 2} and {2, 1}.

For the case of $n = 2$, it is enough to list down exactly one permutation for example {1, 2} and we can obtain the other permutation by reversing the order of {1, 2} which produces {2, 1}.

<u>Case $n = 3$.</u>

In this case there are six permutations since 3! = 6. Let $S = \{1, 2, 3\}$. Then the set of all permutations are:

{1, 2, 3}, {1, 3, 2}, {2, 1, 3}, {2, 3, 1}, {3, 1, 2} and {3, 2, 1}.

To see how we can enumerate permutations for these three elements, we only need one of the above permutation sets to start with. Once we have this set, the other five permutation sets can be produced easily. For instance, without loss of generality, let say the first permutation set is {1, 2, 3}. We list this set and cycle it (refer to Table 1).

If we continue to cycle {3,1,2}, we will get {1, 2, 3} which is the same as the initial set. Therefore, we stop until the third cycle. The initial permutation set before we cycle is called a *starter set*.

Referring to Table 1, by reversing the order {1, 2, 3}, we will get {3, 2, 1}. Performing the same strategy for {2,3,1} and {3,1,2} will produce {1,3,2} and {2,1,3} respectively. All these permutation sets are distinct. Table 2 shows all permutations for six elements obtained from the starter and reversed sets.

If we take a look at the permutation set for {1, 2, 3} and cycle this set, we can see that the permutation set {1, 3, 2} has already appeared in the second row form reversed set. Thus, permutation set of {1, 2, 3} is the same as {1, 3, 2}. In this construction we do not have to cycle other permutation set beside {1, 2, 3}. We called permutation set of {1, 3, 2} as an *equivalence starter set* for {1, 2, 3}.

From discussion above, we can see that we need several starter set. We also need to eliminate the equivalence starter. As the number of elements increases, the equivalence sets become complex to determine. We can employ Theorem 3 and Theorem 4 to produce an equivalence starter. For example the starter set {1, 2, 3} can have {1, 2}, {2, 3} and {3, 1} 2-cycle. From these 2-cylce we can write {1, 3}, {3, 2}, {2, 1} to produce an *equivalence starter set* {1, 3, 2}. The permutation sets {1, 2, 3} and {1, 3, 2} produce the same permutation sets.

<u>Case $n = 4$.</u>

In this construction we follow the same algorithm for case $n = 3$. Let $S = \{1, 2, 3, 4\}$.

Step 1: Find the starter set.

Without loss of generality, fix element 1, then we can have {1, 2, 3, 4}.

Step2: Cycle the starter set and then reverse the order (refer to Table 3):

Step 3: Eliminate the equivalence starter set .

For example, {1, 4, 3, 2} produce the same permutation sets as {1, 2, 3, 4}. Therefore the equivalence set must be eliminated (refer to Table 3 and Table 4)

Step 4: Find the other starter sets.

If we fix one element, we have 3! to permute other elements for starter sets. Since we have two permutation sets in step 1 we have four more permutation sets once we fix one element. The possible permutations are {1, 3, 4, 2}, {1, 2, 4, 3}, {1, 4, 2, 3}, {1, 3, 2, 4}. If we look at these permutations and apply Theorem 3 and Theorem 4: {1, 3, 4, 2} is the same as {1, 2, 4, 3}. While {1, 4, 2, 3} is the same as {1, 3, 2, 4}. Without loss of generality we pick {1, 3, 4, 2} and {1, 4, 2, 3} as the other two starter sets.

Step 5: Cycle these three starter sets to produce all 4! = 24 permutations.

The starter set {1, 2, 3, 4} will produce {1, 2, 3, 4}, {2, 3, 4, 1}, {3, 4, 1, 2}, {4, 1, 2, 3}, {4, 3, 2, 1}, {1, 4, 3, 2}, {2, 1, 4, 3} and {3, 2, 1, 4} permutation sets (refer to Table 3).

The starter set {1, 3, 4, 2}will produce {1, 3, 4, 2}, {3, 4, 2, 1}, {4, 2, 1, 3}, {2, 1, 3, 4}, {2, 4, 3, 1}, {1, 2, 4, 3}, {3, 1, 2, 4} and {4, 3, 1, 2} permutation sets (refer to Table 5)

The starter set {1, 4, 2, 3} will produce {1, 4, 2, 3}, {4, 2, 3, 1}, {2, 3, 1, 4}, {3, 1, 4, 2}, {3, 2, 4, 1}, {1, 3, 2, 4}, {4, 1, 3, 2} and {2, 4, 1, 3} permutation sets.

From these three starters we can produce another 24 distinct permutations for four elements. We use the same algorithm to develop all permutation for five elements.

Case n = 5

Step 1: Find the first starter set and equivalence set

{1, 2, 3, 4, 5} = {1, 5, 4, 3, 2}

Step 2: Find other permutations for starter sets and equivalence starter sets.

We need to find other eleven starter sets. The followings are *starter sets* and its *equivalence starter sets*.

{1, 2, 3, 5, 4} = {1, 4, 5, 3, 2},
{1, 2, 4, 3, 5} = {1, 5, 3, 2, 4},
{1, 2, 4, 5, 3} = {1, 3, 5, 2, 4},
{1, 2, 5, 3, 4} = {1, 4, 3, 5, 2},
{1, 2, 5, 4, 3} = {1, 3, 4, 5, 2},
{1, 3, 4, 2, 5} = {1, 5, 2, 4, 3},
{1, 3, 2, 4, 5} = {1, 5, 4, 2, 3},
{1, 3, 2, 5, 4} = {1, 4, 5, 2, 3},
{1, 3, 5, 2, 4} = {1, 4, 2, 5, 3},
{1, 4, 3, 2, 5} = {1, 5, 2, 3, 4},
{1, 4, 2, 3, 5} = {1, 5, 3, 2, 4}.

Step 3: Cycle each starter set and find the reverse for each cycle to get 5! distinct permutations (refer to Table 7).

60 distinct permutations from twelve starter sets. By reversing the order of these 60 permutations we will produce another 60 distinct permutations. Thus, we have 120 distinct permutations for 5 elements. From the above discussion and construction we now need to establish several theorems.

Theorem 5: There exist $\frac{(n-1)!}{2}$ starter sets for each n elements for listing all n! permutations.

Proof:

There n! permutation sets, and each starter set generate n permutations and has an equivalence starter set. Thus we have

$$\frac{n!}{2n} = \frac{n(n-1)!}{2n} = \frac{(n-1)!}{2}$$

starter sets to start with for any n elements.

Theorem 6: There exist n cycles for each starter sets of n elements.

Proof:

Let $S = \{1, 2, 3, \ldots, n\}$. Without loss of generality let $\{1, 2, 3, \ldots, n\}$ be the starter set. Then we have

$$\begin{array}{ccccc} 1 & 2 & 3 & \ldots & n \\ 2 & 3 & 4 & \ldots & 1 \\ 3 & 4 & 5 & \ldots & 2 \\ . & . & . & \ldots & \ldots \\ . & . & . & \ldots & \ldots \\ . & . & . & \ldots & \ldots \\ n & 1 & 2 & \ldots & n-1 \end{array}$$

which produce n cycle.

Theorem 7: There exists $n!$ permutations from $\frac{(n-1)!}{2}$ starter sets.

Proof:

Each starter set will produce $2n$ permutations (n permutation from starter set and n permutation from reversed set). Also, for n elements, we have $\frac{(n-1)!}{2}$ starter sets. Thus the number permutations for n elements is

$$2n\left(\frac{n-1}{2}\right)! = n!$$

In general, the new algorithm for finding permutation for any n elements is as follows:

Step 1: Fix one element

Step 2: Append the remaining ($n-1$) elements at random order

Step 3: List this set of starter

Step 4: Group the elements above into 2-cycle to determine the equivalence starter set and eliminate this set.

Step 5: Repeat step 2-5 to produce all $\frac{(n-1)!}{2}$ distinct starters.

Step 6: Cycle each starter to get $n!/2$ permutations

Step 7: Reverse the order of each $n!/2$ permutation to get another $n!/2$ permutations

Step 9: List all $n!$ permutations

4. Discussion and Conclusion

We have developed a new algorithm for listing all permutations for n elements based on distinct starter sets. The new algorithm has advantages due to its simplicity and ease of use if compared to the existing ones. This algorithm, therefore, is highly recommended to be employed in determining different orders of the elements of a set.

This study focuses only on the development of the algorithm. Since the number of permutations grows rapidly as the number of elements of the set increase, computer source codes should be written to implement the algorithm. In addition to that, the comparison between the new algorithm with the existing ones in terms of computational time and complexity needs also to be studied. Moreover, since the permutations are heavily computational as the number of elements gets larger, a parallel implementation of this algorithm should also be developed so that the computation can be performed faster.

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Table 1. Cycle of permutation set {1, 2, 3}

1	2	3
2	3	1
3	1	2

Table 2. Permutation sets from starter and reversed sets when $n = 3$

1	2	3
2	3	1
3	1	2

Permutation sets from the starter set

3	2	1
1	3	2
2	1	3

Permutation sets from the reversed set

Table 3. Permutation from the starter set {1,2,3,4}and its reversed set

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

Permutation sets from the starter set

4	3	2	1
1	4	3	2
2	1	4	3
3	2	1	4

Permutation sets from the reversed set

Table 4. Permutation of the equivalence set for the starter set {1,2,3,4}

1	4	3	2
4	3	2	1
3	2	1	4
2	1	4	3

2	3	4	1
1	2	3	4
4	1	2	3
3	4	1	2

Table 5. Permutation from the starter set {1,3,4,2}and its reversed set

1	3	4	2
3	4	2	1
4	2	1	3
2	1	3	4

Permutation sets from the starter set

2	4	3	1
1	2	4	3
3	1	2	4
4	3	1	2

Permutation sets from the reversed set

Table 6. Permutation from the starter set {1,4,3,2}and its reversed set

1	4	2	3
4	2	3	1
2	3	1	4
3	1	4	2

Permutation sets from the starter set

3	2	4	1
1	3	2	4
4	1	3	2
2	4	1	3

Permutation sets from the reversed set

Table 7. The cycle for each starter sets of *n* = 5.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

1	2	3	5	4
2	3	5	4	1
3	5	4	1	2
5	4	1	2	3
4	1	2	3	5

1	2	4	3	5
2	4	3	5	1
4	3	5	1	2
3	5	1	2	4
5	1	2	4	3

1	2	4	5	3
2	4	5	3	1
4	5	3	1	2
5	3	1	2	4
3	1	2	4	5

1	2	5	3	4
2	5	3	4	1
5	3	4	1	2
3	4	1	2	5
4	1	2	5	3

1	2	5	4	3
2	5	4	3	1
5	4	3	1	2
4	3	1	2	5
3	1	2	5	4

1	3	4	2	5
3	4	2	5	1
4	2	5	1	3
2	5	1	3	4
5	1	3	4	2

1	3	2	4	5
3	2	4	5	1
2	4	5	1	3
4	5	1	3	2
5	1	3	2	4

1	3	2	5	4
3	2	5	4	1
2	5	4	1	3
5	4	1	3	2
4	1	3	2	5

1	3	5	2	4
3	5	2	4	1
5	2	4	1	3
2	4	1	3	5
4	1	3	5	2

1	4	3	2	5
4	3	2	5	1
3	2	5	1	4
2	5	1	4	3
5	1	4	3	2

1	4	2	3	5
4	2	3	5	1
2	3	5	1	4
3	5	1	4	2
5	1	4	2	3