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# Hopf Birurcation of Lorenz-like System about Parameter $h$

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**Abstract**

The article mainly researched the Hopf bifurcation of Lorenz-like system about the coefficient of the quadratic term. When the quadratic term $h$ changes, the solution to the Lorenz-like system will become the local periodic solution. Further the stability of this periodic solution and the bifurcation direction of this periodic solution were discussed, and found when the quadratic term $h$ comes through a threshold $h_0$, the direction of hopf bifurcation and stability were given, and the result as follows. If $\alpha'(0) > 0$, when $\mu_2 > 0$, the direction of bifurcation is $h > h_0$, when $\mu_2 < 0$, the direction of bifurcation is $h < h_0$; (b) if $\alpha'(0) < 0$, we have the contrary result. That is when $\mu_2 > 0$, the direction of bifurcation is $h < h_0$; when $\mu_2 < 0$, the direction of bifurcation is $h > h_0$. If $\beta_2 < 0$, the bifurcation solution is asymptotically stable; if $\beta_2 > 0$, the bifurcation solution is not asymptotically stable. Finally employing the matlab compute the numerical periodic solution, the results fit the theory well.

**Keywords:** Lorenz-like system, stability, Hopf bifurcation

## 1. Introduction

Nonlinear science plays an important role in science research including biology, chemistry, fluid dynamics, optics and so on, but bifurcation theory is an important part in nonlinear science and many progresses have been made in the past many years. Bifurcation takes very significant role in the nonlinear system because bifurcation of the nonlinear system can lead to the chaotic behavior and induced the much more difficulty behaviors. So the research of bifurcation is the main means to understand nonlinear system well. The Lorenz-like system is very important like the Lorenz system which was found by the Lorenz in 1963 and many dynamical behaviors of this system have been discussed in the recent years. Such as the bifurcation behavior about the one order term, chaos and so on. This article mainly study the Hopf bifurcation (Lü JH, Zhou TS, Chen G..2002, Li TC, Chen G. 2004, Lü JH, Chen G. 2002, Changpin Li, Guanrong Chen. 2003)of the Lorenz-like system (Liu chongxin, Liu Ling,Liu tao, Li Peng, 2006), with precise numerical simulation. Firstly, this paper introduces some known results, and also introduces some relevant preparation including the elementary concepts. Secondly, this paper mainly employed the very important method of central manifold theory and theory of normal form to conduct a detailed discussion of Hopf bifurcation of the Lorenz-like system. Finally the precise numerical simulation was introduced to verify the theory analysis under the help of matlab. And the results of numerical simulation fit the theoretical analysis well.

## 2. Main results

Lorenz-like system is a 3-D automatics system, it satisfies the following equation,

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = bx - lxz \\ \dot{z} = -cz + hx^2 + ky^2 \end{cases} \quad (1)$$

Thereom 1 System (1) has three equilibriums if only if $ac(b^2+l^2(h+k)^2)\neq 0$, and $bcl(h+k)>0$, $S_0=(0,0,0)$, $S_+=(\sqrt{\frac{bc}{l(h+k)}},\sqrt{\frac{bc}{l(h+k)}},\frac{b}{l})$, $S_-=(-\sqrt{\frac{bc}{l(h+k)}},-\sqrt{\frac{bc}{l(h+k)}},\frac{b}{l})$.

Next, we consider the stability of equilibrium $S_+$ and $S_-$, under the condition $ac(b^2+l^2(h+k)^2)\neq 0$ and $bcl(h+k)>0$. As system (1) has the symmetry properties, $S_+$ and $S_-$ have the same stability. For the equilibrium $S_+$, we can transform system (1) into the following form by $x=X+\sqrt{\frac{bc}{l(h+k)}}$, $y=Y+\sqrt{\frac{bc}{l(h+k)}}$, $z=Z+\frac{b}{l}$,

$$\begin{cases}\dot{X}=a(Y-X)\\ \dot{Y}=-l\sqrt{\frac{bc}{l(h+k)}}Z-lXZ\\ \dot{Z}=2h\sqrt{\frac{bc}{l(h+k)}}X+2k\sqrt{\frac{bc}{l(h+k)}}Y-cZ+hX^2+kY^2\end{cases} \quad (2)$$

So we consider the stability of system (2) at the equilibrium $(0,0,0)$.The linearization part of (2) is $A_+=\begin{pmatrix}-a & a & 0\\ 0 & 0 & -lx_0\\ 2hx_0 & 2kx_0 & -c\end{pmatrix}$ ,where $x_0=\sqrt{\frac{bc}{l(h+k)}}$ .So the characteristic polynomial of $A_+$ is,

$$\lambda^3+(a+c)\lambda^2+(ac+2kbc/(h+k))\lambda+2abc=0 \quad (3)$$

Thereom 2 If $ac(b^2+l^2(h+k)^2)\neq 0$, $bcl(h+k)>0$, $a+c>0$, $abc>0$

$2b-a-c\neq 0$ ,and $h_0=\frac{k(a^2+2bc+ac)}{a(2b-a-c)}$, equation (3) has a pair of conjugate pure imaginary root $\pm i\sqrt{\frac{2abc}{a+c}}$ and a negative root $-(a+c)$.

Proof: if $h_0=\frac{k(a^2+2bc+ac)}{a(2b-a-c)}$, $(a+c)(ac+2kbc/(h+k))=2abc$ ,the characteristic equation (3)can be changed into $(\lambda^2+\frac{2abc}{a+c})(\lambda+a+c)=0$. So we can get roots of equation (3), $\lambda_{1,2}=\pm i\sqrt{\frac{2abc}{a+c}}$, $\lambda_3=-(a+c)$.

By differentiating (3) with respect to $h$, it is found that

$$\lambda'(h)=-\frac{2kbc\lambda(h)}{(3\lambda^2+2(a+c)\lambda+ac+2kbc/(h+k))(h+k)^2} \quad (4)$$

When $h=h_0=\frac{k(a^2+2bc+ac)}{a(2b-a-c)}$, we can obtain $\mathrm{Re}(\lambda'(h_0))=-\frac{kbc(a+c)}{(h_0+k)^2((a+c)^2+d^2)}$,

$\omega'(0)=\mathrm{Im}(\lambda'(h_0))=\frac{kbcd}{(h_0+k)^2((a+c)^2+d^2)}$. Combing Theorem 2 and the definition of Hopf bifurcation(Li Jin-bing, Feng Bei-ye. 1995) , we have the following result.

Thereom 3 The system(1)exist a hopf bifurcation at the equilibrium $S_+$ when the parameter $h$ come though $h_0$. That $h_0$ is the value of the bifurcation.

Next we consider the direction of bifurcation and the stability of bifurcating solution. when $h=h_0$, the eigenvalue of $A_+(h_0)$ were $\lambda_1=di$ , $\lambda_2=-di$ , $\lambda_3=-(a+c)$ and the corresponding eigenvectors were

$\alpha_1 + i\alpha_2$ , $\lambda_3 = \alpha_3$ respectively, where $\alpha_1 = \begin{pmatrix} 1 & 1 & \frac{d^2}{alx_0} \end{pmatrix}^T$ ,

$\alpha_2 = \begin{pmatrix} 0 & \frac{d}{a} & -\frac{d}{lx_0} \end{pmatrix}^T$ , $\alpha_3 = \begin{pmatrix} -\frac{alx_0}{c(a+c)} & \frac{lx_0}{a+c} & 1 \end{pmatrix}^T$ . Let $P_0 = \begin{pmatrix} 1 & 0 & -\frac{alx_0}{c(a+c)} \\ 1 & -\frac{d}{a} & \frac{lx_0}{a+c} \\ \frac{d^2}{alx_0} & \frac{d}{lx_0} & 1 \end{pmatrix}$ , $m = -\det(P_0) = d\frac{(a+c)^2 + d^2}{ac(a+c)}$ ,

then $P_0^- = \frac{1}{m}\begin{pmatrix} \frac{d}{a}+\frac{d}{a+c} & \frac{ad}{c(a+c)} & \frac{dlx_0}{c(a+c)} \\ 1-\frac{d^2}{a(a+c)} & -1-\frac{d^2}{c(a+c)} & \frac{lx_0}{c} \\ -\frac{a^2d+d^3}{a^2lx_0} & \frac{d}{lx_0} & \frac{d}{a} \end{pmatrix}$ .Using linear transformation $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = P_0\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ , system(3)changed into the following form:

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix} = \begin{pmatrix} 0 & -d & 0 \\ d & 0 & 0 \\ 0 & 0 & -a-c \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \tag{5}$$

Where $P = \frac{1}{m}(\frac{ad}{c(a+c)}f_2 + \frac{dlx_0}{c(a+c)}f_3) = \frac{a^2}{((a+c)^2+d^2)}f_2 + \frac{alx_0}{((a+c)^2+d^2)}f_3$

$Q = \frac{1}{m}[-(1+\frac{d^2}{c(a+c)})f_2 + \frac{lx_0}{c}f_3]$ ; $R = \frac{d}{mlx_0}f_2 + \frac{d}{ma}f_3$ ; $f_2 = -l(u_1 - \frac{alx_0}{c(a+c)}u_3)(\frac{d^2}{alx_0}u_1 + \frac{d}{lx_0}u_2 + u_3)$

$$f_3 = h(u_1 - \frac{alx_0}{c(a+c)}u_3)^2 + k(u_1 - \frac{d}{a}u_2 + \frac{lx_0}{a+c}u_3)^2 \tag{6}$$

Compute the following terms(when $h = h_0$ , $(u_1, u_2, u_3) = (0,0,0)$ ) (7)

$$g_{11} = \frac{1}{4}[\frac{\partial^2 P}{\partial u_1^2} + \frac{\partial^2 P}{\partial u_2^2} + i(\frac{\partial^2 Q}{\partial u_1^2} + \frac{\partial^2 Q}{\partial u_2^2})] \tag{8}$$

$$g_{02} = \frac{1}{4}[\frac{\partial^2 P}{\partial u_1^2} - \frac{\partial^2 P}{\partial u_2^2} - 2\frac{\partial^2 Q}{\partial u_1 \partial u_2} + i(\frac{\partial^2 Q}{\partial u_1^2} - \frac{\partial^2 Q}{\partial u_2^2} + 2\frac{\partial^2 P}{\partial u_1 \partial u_2})] \tag{9}$$

$$g_{20} = \frac{1}{4}[\frac{\partial^2 P}{\partial u_1^2} - \frac{\partial^2 P}{\partial u_2^2} + 2\frac{\partial^2 Q}{\partial u_1 \partial u_2} + i(\frac{\partial^2 Q}{\partial u_1^2} - \frac{\partial^2 Q}{\partial u_2^2} - 2\frac{\partial^2 P}{\partial u_1 \partial u_2})] \tag{10}$$

$$G_{21} = \frac{1}{8}[\frac{\partial^3 P}{\partial u_1^3} + \frac{\partial^3 P}{\partial u_1 \partial^2 u_2} + \frac{\partial^3 Q}{\partial u_1^2 \partial u_2} + \frac{\partial^3 Q}{\partial u_2^3} + i(\frac{\partial^3 Q}{\partial u_1^3} + \frac{\partial^3 Q}{\partial u_1 \partial^2 u_2} - \frac{\partial^3 P}{\partial u_1^2 \partial u_2} - \frac{\partial^3 P}{\partial u_2^3})] \tag{11}$$

As $n = 3 > 2$ , we need to computer the following terms,

$$h_{11} = \frac{1}{4}[\frac{\partial^2 R}{\partial u_1^2} + \frac{\partial^2 R}{\partial u_2^2}] = \frac{dkbc}{ma^2(a+c)} \quad h_{20} = \frac{1}{4}[\frac{\partial^2 R}{\partial u_1^2} - \frac{\partial^2 R}{\partial u_2^2} - 2i\frac{\partial^2 R}{\partial u_1 \partial u_2}] \tag{12}$$

And from the $\lambda_3\omega_{11} = -h_{11}$ ; $(\lambda_3 - 2di)\omega_{20} = -h_{20}$ , we get $\omega_{11} = \frac{dkbc}{ma^2(a+c)^2}$ ,

$$\omega_{20}=\frac{1}{(a+c)^2+d^2}[\frac{dkb(2c^2-a^2-2bc+ac)}{ma^2(2b-a-c)}+\frac{4dkb(a^2-c^2+2bc)}{ma(a+c)(2b-a-c)} +i(\frac{2kb(a^2-c^2+2bc)}{ma(2b-a-c)}-\frac{4kcb^2(2c^2-a^2-2bc+ac)}{ma(2b-a-c)(a+c)^2})] \quad (13)$$

then $G_{101}=\frac{1}{2}[\frac{\partial^2 P}{\partial u_1\partial u_2}-\frac{\partial^2 Q}{\partial u_2\partial u_3}+i(\frac{\partial^2 Q}{\partial u_1\partial u_3}+\frac{\partial^2 P}{\partial u_2\partial u_3})]$, $G_{110}=\frac{1}{2}[\frac{\partial^2 P}{\partial u_1\partial u_2}+\frac{\partial^2 Q}{\partial u_2\partial u_3}+i(\frac{\partial^2 Q}{\partial u_1\partial u_3}-\frac{\partial^2 P}{\partial u_2\partial u_3})]$ (14)

so combining (5)-(14),we can get the value of $g_{21}=G_{21}+(2G_{110}\omega_{11}+G_{101}\omega_{20})$, and further we can get the value of $C_1(0)=\frac{i}{2d}[g_{20}g_{11}-2|g_{11}|^2-\frac{1}{3}|g_{02}|^2]+\frac{g_{21}}{2}$ .Finally according to the following expression $\mu_2=-\frac{\mathrm{Re}\,C_1(0)}{\alpha'(0)}$ ; $\beta_2=2\,\mathrm{Re}\,C_1(0)$

$\tau_2=-\frac{\mathrm{Im}\,C_1(0)+\mu_2\omega'(0)}{d}$, we have the following result.

Thereom 4 (a) if $\alpha'(0)>0$, when $\mu_2>0$, the direction of bifurcation is $h>h_0$,

when $\mu_2<0$, the direction of bifurcation is $h<h_0$ ;(b)if $\alpha'(0)<0$,we have the contrary result. if $\beta_2<0$, the bifurcation solution is asymptotically stable, if $\beta_2>0$, the bifurcation solution is not asymtotically stable.

**3. Numerical stimulation**

We have numerical analysis with $a=10, b=40, c=2.5, l=1, k=2$ .we can get the bifurcation value is $h_0=\frac{26}{27}$ .Then we use Matlab to computer the solution figure as follows near the $h_0$ [see.Fig1]. Further, we get value of $\mu_2=0.0237, \tau_2=0.0137$,

$\beta_2=0.0426$ , $\alpha'(0)=-0.9004<0$ . So the direction of bifurcation is $h>h_0$ , and this solution is unstable. $g_{11}=0.1306+1.2178\mathrm{i}$, $\mathrm{g}_{02}=-0.5184\text{-}0.1721\mathrm{i}$; $\mathrm{g}_{20}=-0.3960+1.4458\mathrm{i}$,

$\omega_{11}=0.0056$; $\omega_{20}=0.0100+0.0146\mathrm{i}$ . Finally we can get the period,

$\mathrm{T}=0.4967+0.2871(\mathrm{h}-0.9630)+\mathrm{O}((\mathrm{h}-0.9630)^2)$, and the periodic solution has the following expression,

$$\begin{pmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{pmatrix}=\begin{pmatrix} 5.8095 \\ 5.8095 \\ 40 \end{pmatrix}+\begin{pmatrix} 1 & 0 & -1.8590 \\ 1 & -1.2649 & 0.4648 \\ 2.7541 & 2.1773 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Where $u_1=\mathrm{Re}(W), u_2=\mathrm{Im}(W)$, $u_3=0.0056|W|^2+\mathrm{Re}((0.0100+0.0146\mathrm{i})W^2)$

$$W=6.4957\sqrt{h-0.9630}e^{2i(t+\varphi)\pi/T}+0.5560(h-0.9630).$$

$$\times[(0.1721-0.5184i)e^{-4i(t+\varphi)\pi/T}+(4.3374+1.1880i)e^{4i(t+\varphi)\pi/T}-7.3065+0.7838i]$$

Using the matlab to give the following figure (see Fig 2).

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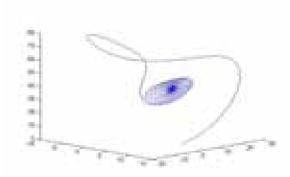

Figure1. bifurcaiton direction of bifurcating solution

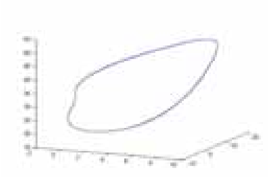

Figure 2. the bifurcating periodic solution