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Chance-constrained Programming Model for Portfolio Selection in Uncertain Environment

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Abstract

The purpose of this paper is to solve the portfolio problem when security returns are uncertain variables. Two types of portfolio selection programming models based on uncertain measure are provided according to uncertain theory. Since the proposed optimization problems are generally difficult to solve by conventional methods, the models are converted to their crisp equivalents when the return rates are adopted some special uncertain variables such as linear uncertain variable, trapezoidal uncertain variable and normal uncertain variable. Thus the transformed models can be completed by the conventional methods. In the end of the paper, one numerical experiment is provided to illustrate the effectiveness of the method.

Keywords: Portfolio selection, Uncertain variable, Chance-constrained programming model, Optimistic value, Crisp equivalent programming model

1. Introduction

The theory of portfolio selection was initially provided by Markowitz (1952, p.77) and has been greatly developed since then. It is concerned with selecting a combination of securities among portfolios containing large number of securities to reach the goal of obtaining satisfactory investment return. In his path-break work, Markowitz proposed a principle that when making investment decision, an investor should always strike a balance between maximizing the return and minimizing the risk, i.e., the investor maximize return for a given level of risk, or one should minimize risk for a predetermined return level. More importantly, Morkowitz initially quantified investment return as the expected value of returns of securities, and risk as variance from the expected value. After Maokowitz's work, scholars have been showing great enthusiasm in portfolio management, trying different mathematical approaches to develop the theory of portfolio selection. Traditionally, returns of individual securities are assumed to be stochastic variables, and many researchers were focused on extending Markowitz's mean-variance models and on developing new mathematical approaches to solve the problems of computation. In fact, the real life decisions are usually made in the state of uncertainty. In order to deal with subjective uncertainty, Liu (2007) founded an uncertainty theory, and had it become a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity and product measure axioms. Based on the uncertain space, uncertain variables are developed to describe the uncertainty phenomena. When the return rates are neither random variables nor fuzzy ones, the return rates can be attributed to uncertain variables. In general, there are three types of stochastic programming models for optimization problems in uncertain environment. The first is expected value model (EVM), which optimizes the expected objective function subject to some expected constraints. The second chance-constrained programming (CCP) was proposed by Charnes and Cooper (1965, p.73) and developed by many scholars as means of dealing with uncertainty by specifying a confidence level at which the uncertain constraints hold. We try to do something in this area. In this paper, returns of securities are assumed to be uncertain parameters instead of stochastic ones. The portfolio will be selected according to the second type of programming models for optimization problems. When the return rates are considered as uncertain variables, the chance measure in the conventional chance-constrained programming model becomes the uncertain measure in the sense of uncertainty theory.

The rest of this paper is arranged as follows. After reviewing some necessary knowledge about uncertain variable in section 2, in section 3, one type of uncertain measure model and one type of uncertain CCP model are proposed for portfolio selection is introduced in section 3. Then section 4 discusses the special cases when the return rates are regarded as some special uncertain variables such as linear uncertain variable, trapezoidal uncertain variable and normal uncertain variable. In section 5, we provide one numerical example to demonstrate the potential application and the effectiveness of the new models. Finally, we conclude the paper in section 6.

2. Preliminaries

Let Γ be a nonempty set, and let A be a σ-algebra over Γ. Each element $\Lambda \in \mathrm{A}$ is called an event. In order to provide an axiomatic definition of uncertain measure, it is necessary to assign to each event Λ a number $M\{\Lambda\}$ which indicates the level that Λ will occur. In order to ensure that the number $M\{\Lambda\}$ has certain mathematical properties, Liu (2009) proposed the following five axioms:

Axiom 1 (Normality) $M(\Gamma)=1$;

Axiom 2 (Monotonicity) $M(\Lambda_1)\le M(\Lambda_2)$ whenever $\Lambda_1 \subseteq \Lambda_2$;

Axiom 3 (Self-duality) $M(\Lambda)+M(\Lambda^c)=1$ for every event Λ;

Axiom 4 (Countable subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$M(\textstyle\bigcup_{i=1}^{\infty}\Lambda_i)\le \sum_{i=1}^{\infty} M(\Lambda_i).$$

The following is the definition of uncertain measure.

Definition 1 (Liu (2009)). The set function is called an uncertain measure if it satisfies the normality, monotonicity, self-duality and countable subadditivity axioms.

Example 1 Let $\Gamma=\{\gamma_1,\gamma_2\}$. For this case, there are only 4events. Define

$$M\{\gamma_1\}=0.4,\ \ M\{\gamma_2\}=0.6,\quad M(\phi)=0,\ \ M(\Gamma)=1,$$

then M is an uncertain measure because it satisfies the four axioms.

Definition 2 (Liu (2009)). Let Γ be a nonempty set, A a σ-algebra over Γ, and M an uncertain measure. Then the triplet (Γ,A,M) is called an uncertain space.

The product uncertain measure is defined as follows.

Axiom 5 (Liu (2009)). (Product Measure Axiom) Let Γ_k be nonempty sets on which M_k are uncertain measures, $k=1,2,\cdots,n$, respectively. Then the product uncertain measure on Γ is

$$M\{\Lambda\}=\begin{cases}\sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda}\ \min\limits_{1\le k\le n} M_k\{\Lambda_k\}, & \text{if } \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda}\ \min\limits_{1\le k\le n} M_k\{\Lambda_k\}>0.5,\\ 1-\sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c}\ \min\limits_{1\le k\le n} M_k\{\Lambda_k\}, & \text{if } \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c}\ \min\limits_{1\le k\le n} M_k\{\Lambda_k\}>0.5.\end{cases}$$

For each event $\Lambda\in\mathrm{A}$, denoted by $M=M_1\wedge M_2\wedge\cdots\wedge M_n$.

Definition 3 (Liu (2009)). An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma,\mathrm{A},\mathrm{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi\in\mathrm{B}\}=\{\gamma\in\Gamma \mid \xi(\gamma)\in\mathrm{B}\}$$

is an event.

A random variable can be characterized by a probability density function and a fuzzy variable may be described by a membership function, uncertain variable can be characterized by identification function.

Definition 4 (Liu (2009)). An uncertain variable ξ is said to have a first identification function λ if

(1) $\lambda(x)$ is a nonnegative function on R such that

$$\sup_{x\ne y}\lambda(x)+\lambda(y)=1;$$

(2) For any set B of real numbers, we have

$$M\{\xi\in\mathrm{B}\}=\begin{cases}\sup\limits_{x\in\mathrm{B}}\lambda(x), & if\ \sup\limits_{x\in\mathrm{B}}\lambda(x)<0.5,\\ 1-\sup\limits_{x\in\mathrm{B}^C}\lambda(x), & \text{if } \sup\limits_{x\in\mathrm{B}^C}\lambda(x)\ge 0.5.\end{cases}$$

Definition 5 (Liu (2009)). The uncertainty distribution $\Phi: R\to[0,1]$ of an uncertain variable ξ is defined by

$\Phi(x)=M\{\xi\le x\}$.

Definition 6 (Liu (2009)). Let ξ be an uncertain variable, and $\alpha \in (0,1]$. Then

$\xi_{\sup}(\alpha) = \sup\{r \mid M\{\xi \geq r\} \geq \alpha\}$

is called the $\alpha-$optimistic value to ξ, and

$\xi_{\inf}(\alpha) = \inf\{r \mid M\{\xi \leq r\} \geq \alpha\}$

is called the $\alpha-$pessimistic value to ξ.

3. Uncertain chance-constrained programming models

Since the optimal investment return may not be obtained in the total viewpoints, it is natural that people would like to accept the compromise optimal value in some constrained conditions. However, at a given confidence level which is considered as the safety margin, the objective must be achieved. If the investor requires that the maximal investment return should be obtained at uncertain measure not less than a predetermined confidence level, then the selection idea can be modeled as follows.

Let x_i denote the investment proportions in security i, to reflect the uncertainty of the return rates ξ_i for the ith security, $i = 1,2,\cdots,n$, respectively, let us express them in terms of uncertain variable. If the investor wants to maximize the chance of the total investment return no less than R at the confidence level α, where R is the predetermined total return and α is the predetermined confidence level, then the model can be formulated as

$$\begin{array}{l} \max\limits_{x} \max\limits_{\bar{f}} \bar{f} \\ \text{subject to} \\ \quad M\{x^T\xi \geq \bar{f}\} \geq \alpha \\ \quad M\{x^T\xi \leq r\} \leq \beta \\ \quad \mathrm{x}_1 + x_2 + \cdots + x_n = 1 \\ \quad x_i \geq 0, i = 1,2,\cdots,n \end{array} \tag{1}$$

where $\max \bar{f}$ is the α return which means the maximal investment return the investor can obtain at confidence level α, here it is actually the $\alpha-$optimistic value to the total return rate $x^T\xi$, and r is the minimum return that the investor can accept satisfying $M\{x^T\xi \leq r\} \leq \beta$ in which $x^T\xi \leq r$ means the investment risk. It is obvious that the optimal solution of the model (1) is the optimal portfolio the investor should select.

If the investor gives the investment return one expects first, then one's objective should be to maximize the chance to obtain this return level subject to the same constraint conditions with model (1). To express the idea in mathematical formulation, we have the following model

$$\begin{array}{l} \max\limits_{x} M\{x^T\xi \geq R\} \\ \text{subject to} \\ \quad M\{x^T\xi \leq r\} \leq \beta \\ .\ \ x_1 + x_2 + \cdots + x_n = 1 \\ \quad x_i \geq 0, \ \ i = 1,2,\cdots,n \end{array} \tag{2}$$

where R is the preset return level that the investor is satisfied with, $x = (x_1, x_2, \cdots, x_n)^T$ is the decision variable and $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$.

4. Special cases

The models (1) and (2) represent ill-posed problems, since these models include uncertain parameters. In the first phase, uncertain programming model is transformed to a usual mathematical model managing the uncertainties based on various interpretations of the problem. In the second phase, the transformed mathematical model is solved by an optimal technique. The obtained solution is optimal or efficient to the transformed mathematical model, however, it is not always reasonable to the original uncertain model. Thus, in the third phase, the optimality or efficiency of the solution can be examined. If the solution is improper, the uncertain model is rebuilt to a mathematical model based on the improved interpretation and the same procedure is iterated.

Theorem 1 Let ξ be an uncertain variable with regular uncertainty distribution Φ. Then its $\alpha-$optimistic value and

$\alpha-$pessimistic value are $\xi_{\sup}(\alpha)=\Phi^{-1}(1-\alpha),\ \xi_{\inf}(\alpha)=\Phi^{-1}(\alpha)$.

In model (1), $\max \bar{f}$ is actually the $\alpha-$optimistic value to the total return rate $x^T\xi$ as referred to in the former section. In accordance with theorem 1, model (1) can be transformed into the following formulation

$$\begin{aligned}&\max_x \Phi^{-1}(1-\alpha)\\&\text{subject to}\\&\quad \Phi(r)\le\beta\\&\quad x_1+x_2+\cdots+x_n=1\\&\quad x_i\ge 0, i=1,2,\cdots,n\end{aligned}\tag{3}$$

Since the self-duality of the uncertain measure, in model (2), we have

$$M\{x^T\xi\ge R\}=1-\Phi(R).$$

Since maximizing the function $1-\Phi(R)$ is equivalent to minimizing function $\Phi(R)$.Thus the model (2) has the following form

$$\begin{aligned}&\min_x \Phi(R)\\&\text{subject to}\\&\quad \Phi(r)\le\beta\\&\quad x_1+x_2+\cdots+x_n=1\\&\quad x_i\ge 0, i=1,2,\cdots,n\end{aligned}\tag{4}$$

In model (3) and (4), the function Φ represents the uncertainty distribution of the total return rate $x^T\xi$.

Next we will establish some important results when the return rate ξ_i adopts some special uncertain variables.

4.1 Models for triangular uncertain return

By a triangular uncertain variable we mean the uncertain variable fully determined by the triplet (a,b,c) of crisp numbers with $a<b<c$, whose first identification function is

$$\lambda(x)=\begin{cases}\dfrac{x-a}{2(b-a)}, & \text{if } a\le x\le b\\[2mm] \dfrac{x-c}{2(b-c)}, & \text{if } b\le x\le c\\[2mm] 0, & \text{elsewhere.}\end{cases}$$

For simplicity, we write $\xi=(a,b,c)$. Since $\Phi(x)=M\{\xi\le x\}$, It follows by definition 4 in section 2 that the uncertainty distribution of ξ is

$$\Phi(x)=\begin{cases}0, & x\le a\\[1mm] \dfrac{x-a}{2(b-a)}, & a<x\le b\\[2mm] \dfrac{x+c-2b}{2(c-b)}, & b<x\le c\\[2mm] 1, & x>c.\end{cases}$$

If the return rates are all triangular uncertain variables, Let ξ_i be (a_i,b_i,c_i), where $a_i<b_i<c_i, i=1,2,\cdots,n$. It follows from operational law of triangular uncertain variables that $x^T\xi=(\sum_{i=1}^n x_ia_i,\sum_{i=1}^n x_ib_i,\sum_{i=1}^n x_ic_i,)$ which is also a triangular uncertain variable. Then the uncertainty distribution of $\sum_{i=1}^n x_i\xi_i$ is the following

$$\Phi(x)=\begin{cases}0, & x\le \sum_{i=1}^{n} x_i a_i\\ \dfrac{x-\sum_{i=1}^{n} x_i a_i}{2(\sum_{i=1}^{n} x_i b_i-\sum_{i=1}^{n} x_i a_i)}, & \sum_{i=1}^{n} x_i a_i < x\le \sum_{i=1}^{n} x_i b_i\\ \dfrac{x+\sum_{i=1}^{n} x_i c_i-2\sum_{i=1}^{n} x_i b_i}{2(\sum_{i=1}^{n} x_i c_i-\sum_{i=1}^{n} x_i b_i)}, & \sum_{i=1}^{n} x_i b_i < x\le \sum_{i=1}^{n} x_i c_i\\ 1, & x>\sum_{i=1}^{n} x_i c_i.\end{cases}$$

For any $\alpha\in(0,1]$, it is easily to verify that the function $\Phi^{-1}(1-\alpha)$ is

$$\Phi^{-1}(1-\alpha)=\begin{cases}2\alpha\sum_{i=1}^{n} x_i b_i+(1-2\alpha)\sum_{i=1}^{n} x_i c_i, & \text{if } \alpha\in(0,0.5]\\ (2\alpha-1)\sum_{i=1}^{n} x_i a_i+(2-2\alpha)\sum_{i=1}^{n} x_i b_i, & \text{if } \alpha\in(0.5,1].\end{cases}$$

When the returns are all triangular uncertain variables, the models (3) and (4) are linear programming models which can be solved by conventional methods easily.

4.2 Models for linear uncertain return

An uncertain variable ξ is called linear if it has a linear uncertainty distribution

$$\Phi(x)=\begin{cases}0, & \text{if } x<a\\ (x-a)/(b-a), & \text{if } a\le x\le b\\ 1. & \text{if } x>b\end{cases}$$

denoted by $L(a,b)$ where a and b are real number with c

Theorem 2 Let ξ be a linear uncertain variable $L(a,b)$. Then its $\alpha-$optimistic value is

$$\xi_{\sup}(\alpha)=\alpha a+(1-\alpha)b.$$

If the return rate ξ_i of the ith security is linear uncertain variable $\xi_i=L(a_i,b_i)$ with $a_i<b_i, i=1,2,\cdots,n$, then

$$x^T\xi=L(\sum_{i=1}^{n} x_i a_i, \sum_{i=1}^{n} x_i b_i).$$

Corollary 1 Suppose that the return rate ξ_i of the ith security is linear uncertain variable $\xi_i=L(a_i,b_i)$ with $a_i<b_i, i=1,2,\cdots,n$, then the $\alpha-$optimistic value of the total return $\sum_{i=1}^{n} x_i\xi_i$ is

$$\xi_{\sup}(\alpha)=\alpha\sum_{i=1}^{n} x_i a_i+(1-\alpha)\sum_{i=1}^{n} x_i b_i=\sum_{i=1}^{n} x_i[\alpha a_i+(1-\alpha)b_i].$$

Thus the models (3) and (4) can be converted into the following models

$$\begin{aligned}&\max_x \sum_{i=1}^{n} x_i[\alpha a_i+(1-\alpha)b_i]\\ &\text{subject to}\\ &\quad \Phi(r)\le\beta\\ &\quad x_1+x_2+\cdots+x_n=1\\ &\quad x_i\ge 0, i=1,2,\cdots,n\end{aligned} \tag{5}$$

and

$$\begin{aligned}&\min_x \frac{R-\sum_{i=1}^{n} x_i a_i}{\sum_{i=1}^{n} x_i b_i-\sum_{i=1}^{n} x_i a_i}\\ &\text{subject to}\\ &\quad \Phi(r)\le\beta\\ &\quad x_1+x_2+\cdots+x_n=1\\ &\quad x_i\ge 0, i=1,2,\cdots,n\end{aligned} \tag{6}$$

which can be solved by the traditional methods.

4.3 Models for normal uncertain return

An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x)=(1+\exp(\frac{\pi(e-x)}{\sqrt{3}\sigma}))^{-1}, x\in R,$$

denoted by $N(e,\sigma)$ where e and σ are real number with $\sigma > 0$.

Theorem 3 Let ξ be a normal uncertain variable $N(e,\sigma)$. Then its $\alpha-$optimistic value is

$$\xi_{\sup}(\alpha) = e - \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

Assumed that the return rate of ith security is normally distributed with parameters e_i and $\sigma_i > 0, i = 1, 2, \cdots, n$ and $x_i, i = 1, 2, \cdots, n$ are the nonnegative decision variables. Then according to the operational laws of normal uncertain variable, we have

$$x^T\xi = N(\textstyle\sum_{i=1}^n x_i e_i, \sum_{i=1}^n x_i\sigma_i),$$

Corollary 2 Let $x_1, x_2, \cdots, x_n$ be nonnegative decision variables and $\xi_1, \xi_2, \cdots, \xi_n$ are independently uncertain variables with expected values $e_1, e_2, \cdots, e_n$ and variance $\sigma_1{}^2, \sigma_2{}^2, \cdots, \sigma_n{}^2$, respectively. Then for any $\alpha \in (0,1]$,

$$\xi_{\sup}(\alpha) = \textstyle\sum_{i=1}^n x_i(e_i - \frac{\sqrt{3}\sum_{i=1}^n \sigma_i}{\pi}\ln\frac{\alpha}{1-\alpha}). \text{ c}$$

Thus the models (3) and (4) can be converted into the following models

$$\begin{aligned}
&\max_x \textstyle\sum_{i=1}^n x_i(e_i - \frac{\sqrt{3}\sum_{i=1}^n \sigma_i}{\pi}\ln\frac{\alpha}{1-\alpha})\\
&\text{subject to}\\
&\quad \Phi(r) \le \beta\\
&\quad x_1 + x_2 + \cdots + x_n = 1\\
&\quad x_i \ge 0, i = 1,2,\cdots,n
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
&\min_x (1 + \exp(\frac{\pi(\sum_{i=1}^n x_i e_i - R)}{\sqrt{3}\sum_{i=1}^n x_i\sigma_i}))^{-1}\\
&\text{subject to}\\
&\quad \Phi(r) \le \beta\\
&\quad x_1 + x_2 + \cdots + x_n = 1\\
&\quad x_i \ge 0, i = 1,2,\cdots,n
\end{aligned} \tag{8}$$

which can be solved by the traditional methods.

5. Numerical Examples

Example 2 Assume that there are 6 securities. Among them, returns of the six securities are all triangular uncertain variables $\xi_i = (a_i, b_i, c_i),\ i = 1, 2, 3, 4, 5, 6$. The data set is given in Table 1.

Then the total return is

$$\textstyle\sum_{i=1}^6 x_i\xi_i = (a, b, c)$$

where $a = -0.2x_1 - 0.1x_2 - 0.3x_3 - 0.2x_4 - 0.4x_5 - 0.4x_6$, $b = 2x_1 + 1.8x_2 + 2x_3 + 1.6x_4 + 2.1x_5 + 1.8x_6$ and $c = 2.3x_1 + 2.5x_2 + 2.9x_3 + 2.4x_4 + 2.9x_5 + 3x_6$.

Suppose that the investor accepts 0.9 as the safe confidence level, and requires the investment return be maximized at uncertain measure not less than this level and $r = 1.0, \beta = 0.3$, then the model (3) is given as follows:

$$\begin{aligned}
&\max_x 0.24x_1 + 0.28x_2 + 0.16x_3 + 0.16x_4 + 0.1x_5 + 0.04x_6\\
&\text{subject to}\\
&\quad 1.12x_1 + 1.04x_2 + 1.08x_3 + 0.88x_4 + 1.1x_5 + 0.92x_6 \ge 1.0\\
&\quad x_1 + x_2 + \cdots + x_6 = 1\\
&\quad x_i \ge 0, i = 1,2,\cdots,6
\end{aligned}$$

By use of Matlab 7.0 on PC, we gain the maximum investment return at uncertain measure not less than 0.85, the investor should assign his money according to the Table 2. The corresponding maximum return is 0.2125.

6. Conclusions

In this paper, we have considered uncertain chance-constrained portfolio selection problems involving uncertain returns and propose two different models. To describe uncertain events, we provide the portfolio selection models based on uncertainty measures. The uncertain programming problems are converted into equivalent deterministic programming problems using identification function of uncertain variables. One numerical example is shown for better illustration of our models. In future, we will apply there general uncertain portfolio selection problems to other asset allocation problem, multi-period problems and combinational optimization models and so on. The proposed models can also be extended to complex portfolio selection models considering higher moments.

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Table 1. Uncertain returns of 6 securities (units per stock)

Security i	Uncertain return
1	(-0.2, 2.0, 2.3)
2	(-0.1, 1.8, 2.5)
3	(-0.3, 2.0, 2.9)
4	(-0.2, 1.6, 2.4)
5	(-0.4, 2.1, 2.9)
6	(-0.4, 1.8, 3.0)

Table 2. Allocation of money to 6 securities

	Securities i					
	1	2	3	4	5	6
Allocation of money	0.0000	0.4375	0.5625	0.0000	0.0000	0.0000